## ABSTRACT

ELSINGER, JASON ROBERT. Classification of Orbifold Modules under an Automorphism of Order Two. (Under the direction of Bojko Bakalov.)

Two-dimensional conformal field theory is important in physics as it plays a crucial role in string theory. A *vertex algebra* is essentially the same as a chiral algebra in conformal field theory. Vertex algebras arose naturally in the representation theory of infinite dimensional *Lie algebras* and were first axiomatized in 1986. Both string theory in physics and *monstrous moonshine* in mathematics played crucial roles in the development of the theory.

In this thesis, we investigate the representation theory of the fixed point vertex subalgebra $V_Q^\sigma$ of the *lattice* vertex algebra $V_Q$ associated to an arbitrary positive definite even lattice $Q$ under an automorphism of order two. It is a fundamental problem in the theory of orbifolds to classify the irreducible modules and the main result uses explicitly a number of previous results in classifying the irreducible orbifold modules. We also give explicit constructions of all orbifold modules corresponding to the *simply-laced simple* Lie algebras with a *Dynkin* diagram automorphism of order two.

Classification of Orbifold Modules under an Automorphism of Order Two

by
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# DEDICATION

To my family and many dear friends, for their infinite support.

## BIOGRAPHY

Jason Robert Elsinger was born in Patchogue, New York on August 8th 1987. He has always enjoyed mathematics and the rules of logic. In high school he found a passion for both mathematics and the teaching thereof. He got accepted to Bloomsburg University in pursuit of teaching at the college level, majoring in mathematics and physics. He then got accepted to North Carolina State University as a doctoral student in mathematics. During his years as a graduate student, he gained much experience as an instructor teaching 7 courses and is now a faculty member in the mathematics department at Spring Hill College in Mobile, AL.

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# Chapter 1

# Introduction

In 1968, Victor Kac and Robert Moody independently discovered a new class of Lie algebras now called *Kac-Moody algebras*, which are infinite dimensional analogs of finite dimensional simple Lie algebras. A special type of these algebras, called *affine* Lie algebras, has rich and beautiful structures. It turns out their representation theory has many applications in both mathematics and physics, particulary in proving the moonshine conjectures.

The monstrous moonshine began as an astonishing set of conjectures relating the largest sporadic finite simple group, the Monster group, to the theory of modular functions in number theory. However, the connections between the Monster group, number theory, and other fields is still not fully understood. For more details, see the introduction to [**LL**].

From the study of the moonshine conjectures, a new kind of algebra of operators was emerging, called *vertex operator algebras*, based on the operator product expansion in quantum field theory. These operators were introduced in the early days of string theory in order to describe certain kinds of physical interactions. Vertex algebras arose naturally in the representation theory of affine Kac-Moody Lie algebras and were first axiomatized by Richard Borcherds in 1986.

Let $Q$ be an integral lattice, i.e., a free abelian group equipped with a $\mathbb{Z}$-valued symmetric bilinear form $(\cdot|\cdot)$. Then one can construct an associated *vertex algebra* called a *lattice* vertex algebra and denoted $V_Q$ [**B**; **K1**; **LL**]. Any automorphism $\sigma$ of $Q$ can be naturally lifted to an automorphism of $V_Q$ but the order may double [**BK**].

For a vertex algebra $V$ and a finite group of automorphisms $\Gamma$ of $V$, the subalgebra $V^\Gamma$ of $\Gamma$-invariant elements in $V$ is an *orbifold vertex algebra*. Geometrically, an orbifold can be viewed as a generalization of a manifold by considering the orbits of an action of a finite group on the manifold.

Now each simply-laced simple Lie algebra can be associated with an *even* integral lattice. The *twisted modules* for vertex algebras associated to an even integral lattice have been considered in [**FLM**; **L**; **KP**; **D1**]. In particular, [**D1**; **AD**; **DN**] considers the specific cases when $\sigma = 1$ and $\sigma = -1$. For a root lattice of a simply-laced Lie algebra of finite type, the lattice vertex algebra gives a representation of the associated affine Kac-Moody algebra at level one (see Theorem 3.3.10). It has also been shown in [**BK**] that for an even integral lattice $Q$, the irreducible $\sigma$-twisted $V_Q$-modules are in one-to-one correspondence with the space $(Q^*/Q)^\sigma$ of $\sigma$-invariant elements in $Q^*/Q$.

It is an open question as to whether every orbifold module can be realized as a restriction of a twisted module. We present a full classification of the $V_Q^\sigma$-modules corresponding to an even positive definite integral lattice $Q$ and an automorphism $\sigma$ of the lattice of order two. Other examples of orbifolds and general properties of orbifold theories have been studied in [**DVVV**; **DLM2**; **KT**].

In Chapters 2 and 3, we review necessary background material as well as the results used throughout this thesis. Previous knowledge of Lie algebras is not assumed. Chapter 2 discusses the definition and notion of a Lie algebra and its representations. Some particular examples are singled out as they will be pertinent in Chapter 5. Chapter 3 discusses the definition and notion of a vertex algebra and its twisted and untwisted representations. We also describe the explicit construction of twisted modules over lattice vertex algebras.

Chapter 4 contains the main result. We first determine the structure of the orbifold of a lattice vertex algebra with an automorphism of order two. It turns out that a suitable sublattice is sufficent to describe the structure. We then construct the orbifold modules of twisted and untwisted type. More specifically, we use results from [**AD**; **DN**; **FHL**; **A1**] to obtain the following result:

*Let $Q$ be an even positive definite integral lattice, $V_Q$ the corresponding lattice vertex algebra and let $\sigma$ be an automorphism of $Q$ of order two. Then each irreducible $V_Q^\sigma$-module is a submodule of a twisted $V_Q$-module.*

In Chapter 5, instances of the main theorem are worked out explicitly for each root lattice corresponding to the simply-laced simple Lie algebras with a Dynkin diagram automorphism of order two. Note that a Dynkin diagram automorphism is an example of an outer automorphism, i.e., not an element of the Weyl group. We also show how the orbifold modues of twisted type are constructed using [**BK**]. At the end of each example, we present a correspondence between the two constructions.

# Chapter 2

# Lie Algebras and Root Systems

In this chapter, we review classical and affine (Kac-Moody) Lie algebras, root systems, and their connections with lattices. Unless otherwise stated, all algebras will be over the complex numbers $\mathbb{C}$. We also introduce certain examples used throughout this thesis as well as review some representation theory. For more details concerning the theory of Lie algebras, see [**H**] for finite dimensional Lie algebras, and [**K1**] for infinite dimensional Lie algebras.

## 2.1 Lie Algebras

In this section, we give the basic notions and definitions. To give a bit of history, the study of Lie algebras arose from the studies of Sophus Lie in the late nineteenth century. He admired Galois's work on the symmetries of algebraic equations and wanted to do a similar study for the solutions to differential equations. He did this by considering all the (infinite) solutions together and viewed how one morphed into another as initial parameters changed. This led him to groups of "continuous transformations", where one operation could gradually transform into another. He was later led to the concept of "finite, continuous groups" - now called *Lie groups*.

An indispensable tool for studying Lie groups is its associated *Lie algebra*, which can briefly be described as the tangent space of a Lie group at its identity. Lie algebras contain a natural product, called the *bracket*, which is neither commutative nor associative. It turns out that much information about a Lie group can be determined from its Lie algebra. From this realization came the abstract study of Lie algebras, where the definition can be given axiomatically and independent of Lie groups.

**Definition 2.1.1** *A* Lie algebra *is a vector space* $\mathfrak{g}$ *over* $\mathbb{C}$ *together with a map* $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ *such that for* $\alpha \in \mathbb{C}$ *and* $x, y, z \in \mathfrak{g}$*,* *i)* $[\alpha x + y, z] = \alpha[x, z] + [y, z]$,

*ii)* $[x, x] = 0$,

*iii)* $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$.

Note that the first two axioms imply the bracket is bilinear and the second axiom implies the bracket is skew-symmetric (*i.e. that* $[x, y] = -[y, x]$). The third axiom is called the *Jacobi identity* and is crucial to the structure of Lie algebras.

**Example 2.1.2** *For a vector space $V$ with an associative product, one can define a Lie algebra structure via the* commutator bracket $[v, w] = v \cdot w - w \cdot v$*, where $v, w \in V$.*

**Example 2.1.3** *Let $V$ be a complex vector space. The set of all linear endomorphisms,* $\operatorname{End} V$*, is an associative algebra under function composition. The Lie algebra obtained from* $\operatorname{End} V$ *endowed with the commutator bracket is denoted $\mathfrak{gl}(V)$.*

As with most algebraic structures, the notions of subalgebras and structure preserving maps are defined in the natural way.

**Definition 2.1.4** *A* (Lie) subalgebra *of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A* Lie algebra homomorphism *is a linear map $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$. An* isomorphism *is a bijective homomorphism.*

**Example 2.1.5** *Consider the vector space $\mathfrak{gl}_n = \mathfrak{gl}(n, \mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$. This space naturally becomes a Lie algebra under the commutator bracket. This Lie algebra has a subalgebra of traceless matrices, denoted $\mathfrak{sl}_n = \mathfrak{sl}(n, \mathbb{C}) = \{A \in \mathfrak{gl}_n | tr(A) = 0\}$. This subspace is a subalgebra since for $A, B \in \mathfrak{sl}_n$,*

$$tr([A, B]) = tr(AB) - tr(BA) = tr(AB) - tr(AB) = 0.$$

*A basis for $\mathfrak{gl}_n$ consists of the matrices $E_{ij}$, having a 1 in the $(i, j)$ position and zero elsewhere. The bracket in $\mathfrak{gl}_n$ can then be given as*

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj},$$

*where $\delta_{ij}$ is the kronecker delta. Note that the bracket on the basis elements extends to the whole algebra via bilinearity. The Lie algebra $\mathfrak{gl}_n$ is called the* general linear Lie algebra *and $\mathfrak{sl}_n$ is called the* special linear Lie algebra *and also denoted as $A_{n-1}$.*

We also have the notion of "irreducible" Lie algebras, where attention most often is restricted.

**Definition 2.1.6** *An* ideal *of a Lie algebra* $\mathfrak{g}$ *is a subalgebra* $\mathfrak{i} \subset \mathfrak{g}$ *such that* $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$. *A* simple Lie algebra *is a nonabelian Lie aglebra for which the only ideals are itself and the trivial ideal* $\{0\}$.

**Example 2.1.7** *Let* $I_n$ *denote the identity matrix in* $\mathfrak{gl}_n$. *Then* $\mathbb{C}I_n$ *is an ideal of* $\mathfrak{gl}_n$ *so that* $\mathfrak{gl}_n$ *is not simple. It can be shown that* $\mathfrak{sl}_n$ *is a simple Lie algebra for every* $n$.

The classification of finite dimensional simple Lie algebras over $\mathbb{C}$ is beautifully described in terms of associated connected digraphs called *Dynkin diagrams*. The classification can be found in many texts, notabaly [**H**]. Examples of the main result are computed for certain families in this classification and are presented in chapter 5.

Again as with most algebraic structures, we also have notions of modules and representations.

**Definition 2.1.8** *Let* $\mathfrak{g}$ *be a Lie algebra and* $V$ *be a vector space. Then* $V$ *is a* $\mathfrak{g}$-module *if there is a bilinear map* $(\cdot, \cdot) : \mathfrak{g} \times V \longrightarrow V$, *denoted* $(g, v) = g \cdot v$, *such that* $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ *for all* $x, y \in \mathfrak{g}$ *and* $v \in V$.

**Remark 2.1.9** *Given a* $\mathfrak{g}$*-module* $V$, *each* $g \in \mathfrak{g}$ *yields a linear map:* $\phi(g)(v) = g \cdot v$, *for all* $v \in V$. *It is easy to see* $\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ *is a homomorphism, called a* representation *of* $\mathfrak{g}$ *on* $V$. *Conversely, each representation* $\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ *corresponds to a* $\mathfrak{g}$*-module* $V$ *with action defined by* $g \cdot v = \phi(g)(v)$.

**Example 2.1.10** *For a Lie algebra* $\mathfrak{g}$ *and an element* $x \in \mathfrak{g}$, *define a linear map* $\mathrm{ad}_x$ *on* $\mathfrak{g}$ *given by* $\mathrm{ad}_x(y) = [x, y]$. *The map* $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ *given by* $\mathrm{ad}(x) = \mathrm{ad}_x$ *is called the* adjoint mapping *and forms a Lie algebra homomorphism, due to the Jacobi identity. Hence* $\mathfrak{g}$ *is a module over itself and the mapping* ad *is also called the* adjoint representation.

**Definition 2.1.11** *A* $\mathfrak{g}$*-module* $V$ *is* completely reducible *if* $V$ *can be written as a direct sum of irreducible* $\mathfrak{g}$*-modules.*

## 2.2 Root Systems

Let $V$ be a complex vector space and $T$ be a linear operator on $V$. The operator $T$ is called *nilpotent* if $T^N = 0$ for some positive integer $N$ and called *semisimple* if its minimal polynomial has distinct roots. Note that over the algebraically closed field $\mathbb{C}$, the condition of $T$ being semisimple is equivalent to $T$ being diagonalizable.

This thesis will mainly be concerned with simple and *semisimple* Lie algebras, i.e., Lie algebras that can be written as a direct sum of simple ideals.

**Definition 2.2.1** *For a Lie algebra $\mathfrak{g}$ not consisting entirely of nilpotent elements, a* toral subalgebra *of $\mathfrak{g}$ is a subalgebra generated by semisimple elements.*

**Remark 2.2.2** *Let* $\operatorname{Der}\mathfrak{g}$ *be the set of all* derivations *of the Lie algebra $\mathfrak{g}$, i.e., the set of linear maps $\delta$ that satisfy*

$$\delta[x.y] = [\delta x, y] + [x, \delta y],$$

*for all $x, y \in \mathfrak{g}$. It is well known for a semisimple Lie algebra $\mathfrak{g}$, that $\operatorname{ad}\mathfrak{g} = \operatorname{Der}\mathfrak{g}$ and that the map* ad *is injective. This implies that each $x \in \mathfrak{g}$ can be uniquely expressed in the form $x = s+n$, where $s, n \in \mathfrak{g}$ with $[s, n] = 0$ and $\operatorname{ad}_s$ is semisimple, $\operatorname{ad}_n$ is nilpotent (see* [**H**]*). The elements $s$ and $n$ are called the* semisimple *and* nilpotent parts *of $x$, respectively, and the decomposition $x = s + n$ is the (abstract)* Jordan-Chevalley decomposition. *Hence a (semi)simple Lie algebra must contain at least one semisimple element.*

It is known that toral subalgebras are abelian. Let $\mathfrak{g}$ be a semisimple Lie algebra and fix a *maximal* toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e., one that is not properly contained in any other. Then since $\mathfrak{h}$ is abelian, the set of maps $\operatorname{ad}_{\mathfrak{g}}\mathfrak{h} = \{\operatorname{ad}_h \,|\, h \in \mathfrak{h}\}$ is a commuting family of semisimple endomorphisms of $\mathfrak{g}$. It is a standard result of linear algebra that the set $\operatorname{ad}_{\mathfrak{g}}\mathfrak{h}$ is then simultaneously diagonalizable. Thus $\mathfrak{g}$ can be written as a direct sum of eigenspaces

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} |\, [h, x] = \alpha(h)x \;\forall\; h \in \mathfrak{h}\}, \tag{2.1}$$

where $\alpha$ ranges over the dual space $\mathfrak{h}^*$. It can be shown that $\mathfrak{g}_0$ is precicely $\mathfrak{h}$ (see [**H**]). Then we obtain the following important decomposition of any semisimple Lie algebra.

**Definition 2.2.3** *Let $\mathfrak{g}$ be a semisimple Lie algebra with maximal toral subalgebra $\mathfrak{h}$. The* root space decomposition *of $\mathfrak{g}$ is*

$$\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

*where $\Phi = \{\alpha \in \mathfrak{h}^* \backslash \{0\} |\, \mathfrak{g}_\alpha \neq 0\}$ is the set of* roots *of $\mathfrak{g}$ relative to $\mathfrak{h}$.*

In the following proposition, we give more precise information about the structure of the root space decomposition. The details of the proofs can be found in many standard texts.

**Proposition 2.2.4** *Let $\mathfrak{g}$ be a semisimple Lie algebra with maximal toral subalgebra $\mathfrak{h}$. Then the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha\in\Phi} \mathfrak{g}_\alpha$ has the following properties:*

*i)* *the set of roots* $\Phi$ *spans the dual space* $\mathfrak{h}^*$,

*ii)* *for each* $\alpha \in \Phi$, *the subpaces* $\mathfrak{g}_\alpha$ *and* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *are one dimensional,*

*iii)* *for each* $\alpha \in \Phi$ *and nonzero* $x_\alpha \in \mathfrak{g}_\alpha$, *there exists* $y_\alpha \in \mathfrak{g}_{-\alpha}$ *such that* $x_\alpha$, $y_\alpha$, $h_\alpha = [x_\alpha, y_\alpha]$ *span a three dimensional simple subalgebra of* $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}_2$.

*iv)* *for each root* $\alpha \in \Phi$, *the only scalar multiples of* $\alpha$ *which are also roots are* $\pm\alpha$,

*v)* *if* $\alpha, \beta, \alpha + \beta \in \Phi$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$,

*vi)* $\mathfrak{g}$ *is generated (as a Lie algebra) by the root spaces* $\mathfrak{g}_\alpha$.

**Example 2.2.5** *We write the root space decomposition for* $\mathfrak{sl}_3$. *Recall that* $\mathfrak{sl}_3$ *is an 8 dimensional vector space with the basis* $E_{ij}$ *for* $i \neq j$ *and the diagonal elements* $h_i = E_{ii} - E_{i+1,i+1}$. *The maps* $\mathrm{ad}_{h_1}$ *and* $\mathrm{ad}_{h_2}$ *on the basis elements are as follows:*

$$\begin{array}{rcrcrcr}
[h_1, E_{12}] & = & 2E_{12} & \qquad & [h_2, E_{12}] & = & -E_{12} \\
[h_1, E_{13}] & = & E_{13} & & [h_2, E_{13}] & = & E_{13} \\
[h_1, E_{23}] & = & -E_{23} & & [h_2, E_{23}] & = & 2E_{23} \\
[h_1, E_{21}] & = & -2E_{21} & & [h_2, E_{21}] & = & E_{21} \\
[h_1, E_{31}] & = & -E_{31} & & [h_2, E_{31}] & = & -E_{31} \\
[h_1, E_{32}] & = & E_{32} & & [h_2, E_{32}] & = & -2E_{32}
\end{array}$$

*Reading the eigenvalues of* $\mathrm{ad}_{h_1}$ *and* $\mathrm{ad}_{h_2}$ *gives the roots of* $\mathfrak{sl}_3$. *The following table presents the values of each root on* $h_1$ *and* $h_2$.

| | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ |
|---|---|---|---|---|---|---|
| $h_1$ | 2 | 1 | −1 | −2 | −1 | 1 |
| $h_2$ | −1 | 1 | 2 | 1 | −1 | −2 |

*Notice that* $\alpha_1 = -\alpha_4$, $\alpha_2 = -\alpha_5$, *and* $\alpha_3 = -\alpha_6$. *Hence the set of roots are* $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$ *and the root space decomposition is*

$$\mathfrak{sl}_3 = \mathbb{C}E_{12} \oplus \mathbb{C}E_{13} \oplus \mathbb{C}E_{23} \oplus \mathrm{span}_{\mathbb{C}}\{h_1, h_2\} \oplus \mathbb{C}E_{21} \oplus \mathbb{C}E_{31} \oplus \mathbb{C}E_{32},$$

*where the maximal toral subalgebra is* $\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{h_1, h_2\}$.

The set of roots also has a rich structure among themselves and can be studied in an abstract setting, where the roots are viewed as vectors in a *Euclidean space*. Recall a Euclidean space is a vector space $E$ with a positive definite, symmetric, bilinear form $(\cdot|\cdot)$ (an example being the dot product among vectors). For $\alpha, \beta \in E$, the *reflection* of $\beta$ about $\alpha$ is given by $\sigma_\alpha(\beta) = \beta - \dfrac{2(\beta|\alpha)}{(\alpha|\alpha)}\alpha$. Geometrically, each reflection $\sigma_\alpha$ in $E$ is an invertible linear transformation leaving pointwise fixed the *hyperplane* $P_\alpha = \{\beta \in E \,|\, (\beta|\alpha) = 0\}$.

**Definition 2.2.6** *A subset $\Phi$ of the Euclidean space $E$ is called a* root system *in $E$ if:*

*i) $\Phi$ is finite, $E = \operatorname{span}\{\Phi\}$ and $0 \notin \Phi$,*

*ii) for $\alpha \in \Phi$, the only multiples of $\alpha$ in $\Phi$ are $\pm\alpha$,*

*iii) for $\alpha \in \Phi$, the reflection $\sigma_\alpha$ leaves $\Phi$ invariant,*

*iv) for $\alpha, \beta \in \Phi$, the expression $\dfrac{2(\beta|\alpha)}{(\alpha|\alpha)}$ is an integer.*

The form $(\cdot|\cdot)$ on the set of roots can be given in terms of the *Killing form* on Lie algebras: $\kappa(x, y) = \operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y)$. Therefore, for a semisimple Lie algebra $\mathfrak{g}$ and maximal toral subalgebra $\mathfrak{h}$, we have a correspondence between pairs $(\mathfrak{g}, \mathfrak{h})$ and pairs $(\Phi, E)$. A classification of root systems then corresponds to a classification of semisimple Lie algebras. This axiomatic approach to root systems actually has the advantage of providing results which apply simultaneously to Lie algebras, Lie groups, and linear algebraic groups.

**Remark 2.2.7** *The root space decomposition of a semisimple Lie algebra is unique in the following sense:*

*two semisimple Lie algebras having the same root system are isomorphic, and*

*all maximal toral subalgebras of a semisimple Lie algebra are conjugate.*

*Thus a semisimple Lie algebra is uniquely determined (up to isomorphism) by its root system relative to any maximal toral subalgebra.*

One nice property about root systems is the existence of a special type of basis, called a *base*.

**Definition 2.2.8** *A subset $\Delta$ of $\Phi$ is a* base *if:*

*i)* $\Delta$ *is a basis of* $E$,

*ii)* *each root* $\beta$ *can be written as* $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$, *where the* $k_\alpha$*'s are all nonnegative or all nonpositive integers.*

The definition of a base fails to garauntee that such a basis exists, but it can be shown that all root systems indeed have a base (see [**H**]).

**Example 2.2.9** *In Example 2.2.5, the roots for* $\mathfrak{sl}_3$ *were computed to be* $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$, *where* $\alpha_2 = \alpha_1 + \alpha_3$. *Thus a base of* $\mathfrak{sl}_3$ *is* $\Delta = \{\alpha_1, \alpha_3\}$.

The classification of the simple finite dimensional Lie algebras is determined from the classification of (irreducible) root systems and forms a beautiful theory. The details of the classification are given in many texts, and here we only give the necessary data for this thesis–that concerning the *simply laced* simple Lie algebras. These Lie algebras are labeled as $A_n (n \geq 1)$, $D_n (n \geq 4)$ and (the exceptional types) $E_6, E_7, E_8$. We now give a brief description of the construction of the families $A_n$ and $D_n$. The description of the exceptional type $E_6$ will be given in chapter 5.

$A_n = \mathfrak{sl}_{n+1} (n \geq 1)$:

Define the elements $\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(\mathrm{diag}(a_1, \ldots, a_{n+1})) = a_i$. Let $I$ be the $\mathbb{Z}$-span of the $\varepsilon_i$'s and let $E$ be the $n$ dimensional subspace of $\mathbb{R}^{n+1}$ orthogonal to the vector $\varepsilon_1 + \cdots + \varepsilon_{n+1}$. Take $\Phi = \{\alpha \in I \cap E \,|\, (\alpha|\alpha) = 2\} = \{\varepsilon_i - \varepsilon_j \,|\, i \neq j\}$. One can check that $\Phi$ forms a root system in $E$. A base for this root system is given by $\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\}$: clearly the $\alpha_i$'s are independent and $\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $i < j$.

$D_n (n \geq 4)$:

Let $E = \mathbb{R}^n$ and take $\Phi = \{\alpha \in I \,|\, (\alpha|\alpha) = 2\} = \{\pm(\varepsilon_i \pm \varepsilon_j) \,|\, i \neq j\}$. One can show that $\Phi$ forms a root system in $E$. A base for this root system is given by the $n$ independent vectors $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$.

## 2.3 Lattices

The construction of the main object of interest in this thesis requires the use of *lattices*. We give the necessary definitions and briefly describe important examples used later in the constructions.

**Definition 2.3.1** *A (rational)* lattice *of rank $n \in \mathbb{N}$ is a rank $n$ free abelian group $L$ equipped with a $\mathbb{Q}$-valued symmetric $\mathbb{Z}$-bilinear form*

$$(\cdot|\cdot) : L \times L \longrightarrow \mathbb{Q}.$$

The lattices used in this thesis require several nice properties we now define.

**Definition 2.3.2** *Let $L$ be a lattice and $\alpha, \beta \in L$.*

*i)* $L$ *is* nondegenerate *if its form $(\cdot|\cdot)$ is nondegenerate in the sense that $(\alpha|L) = 0$ implies $\alpha = 0$.*

*ii)* $L$ *is* even *if $(\alpha|\alpha) \in 2\mathbb{Z}$ for all $\alpha$.*

*iii)* $L$ *is* positive definite *if $(\alpha|\alpha) > 0$ for all $\alpha \in L\backslash\{0\}$.*

*iv)* $L$ *is* integral *if $(\alpha|\beta) \in \mathbb{Z}$ for all $\alpha, \beta$.*

**Remark 2.3.3** *A lattice may be equivalently defined as the $\mathbb{Z}$-span of a basis of a finite dimensional rational vector space equipped with a symmetric bilinear form. A lattice isomorphism is also called an* isometry*.*

Let $\{\alpha_1, \ldots, \alpha_n\}$ be a $\mathbb{Z}$-basis of a lattice $L$. Then an equivalent way of determining whether $L$ is nondegenerate amounts to the condition

$$\det((\alpha_i|\alpha_j))_{i,j=1}^n \neq 0. \tag{2.2}$$

For a field $\mathbb{E}$ of characteristic zero, the lattice $L$ can be embedded in the $\mathbb{E}$-vector space $L_{\mathbb{E}} = L \otimes_{\mathbb{Z}} \mathbb{E}$ and the form on $L$ can be extended to the symmetric $\mathbb{E}$-bilinear form

$$(\cdot|\cdot) : L_{\mathbb{E}} \times L_{\mathbb{E}} \longrightarrow \mathbb{E}.$$

Then $L$ is positive definite if and only if the real vector space $L_{\mathbb{R}}$ is a Euclidean space. It can also be shown that an even lattice is automatically integral.

The *dual* of a lattice $L$ is the set

$$L^* = \{\alpha \in L_{\mathbb{Q}} \mid (\alpha|L) \subset \mathbb{Z}\}. \tag{2.3}$$

As long as the lattice $L$ has full rank, the dual $L^*$ will also be a lattice. Equivalently, the dual $L^*$ forms a lattice if and only if $L$ is nondegenerate. Note that $L$ is integral if and only if $L \subset L^*$. A lattice $L$ is also called *self-dual* if $L^* = L$. For a nondegenerate lattice $L$ with basis $\{\alpha_1, \dots, \alpha_n\}$, the dual lattice $L^*$ has as a basis the *dual basis* $\{\alpha_1^*, \dots, \alpha_n^*\}$, i.e., a basis for which $(\alpha_i^*|\alpha_j) = \delta_{ij}$.

Some important examples that are used later involve lattices corresponding to the root system of the simply-laced simple Lie algebras. Let $\Delta$ be a base for the root system of a simply-laced simple Lie algebra $\mathfrak{g}$. The lattice

$$Q = \mathbb{Z}\Delta = \left\{\sum n_i\alpha_i \,|\, n_i \in \mathbb{Z},\, \alpha_i \in \Delta\right\} \tag{2.4}$$

generated by $\Delta$ is the *root lattice* of $\mathfrak{g}$. Its dual

$$P = Q^* = \{\alpha \in \mathfrak{h} \,|\, (\alpha|Q) \subset \mathbb{Z}\} \tag{2.5}$$

is the *weight lattice*.

Let $Q_X$ be the root lattice corresponding to the simple Lie algebra $X$. Then the following are the root lattices corresponding to $A_n$ and $D_n$:

$$Q_{A_n} = \left\{\sum_{i=1}^{n+1} m_i\varepsilon_i \,\middle|\, m_i \in \mathbb{Z},\, \sum_i m_i = 0\right\}, \tag{2.6}$$

$$Q_{D_n} = \left\{\sum_{i=1}^{n} m_i\varepsilon_i \,\middle|\, m_i \in \mathbb{Z},\, \sum_i m_i \in 2\mathbb{Z}\right\}. \tag{2.7}$$

The root lattice for $E_6$ will be described in chapter 5. The root lattice $Q_{E_8}$ corresponding to $E_8$ is self-dual and the only even self-dual lattice among the root lattices of simple Lie algebras, even allowing for possible rescaling of the lattices (see [**FLM**]).

## 2.4 Affine Lie Algebras

There are two equivalent ways of defining the (untwisted) *affine Kac-Moody Lie algebras*. One can give the notion of a *generalized Cartan matrix*, and define the affine algebras to be one of three possible classes of such matrices. Alternatively, one can describe an affine algebra as the central extention of the *loop algebra* of a simple Lie algebra of finite type (the ones given by the classification above). We will use this latter definition as it is more explicit and suggestive of further constructions.

**Definition 2.4.1** *Let $\mathfrak{g}$ be a Lie algebra. The algebra $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \mathfrak{g}[t, t^{-1}]$ is called the* loop algebra *of $\mathfrak{g}$.*

The loop algebra can be described as the set of Laurent polynomials with coefficients in $\mathfrak{g}$. Elements in $\mathfrak{g}[t, t^{-1}]$ are sums of ones of the form $a \otimes t^n$ ($a \in \mathfrak{g}, n \in \mathbb{Z}$) and this is abbreviated as $a_n$. The loop algebra indeed forms a Lie algebra under the bracket $[a_n, b_m] = [a, b]_{n+m}$.

It is well known that the loop algebra $\tilde{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ has a one-dimensional central extension

$$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K, \tag{2.8}$$

and this set also forms a Lie algebra with bracket defined by

$$[a_n, b_m] = [a, b]_{n+m} + n\delta_{n,-m}(a|b)K, \tag{2.9}$$

$$[\hat{\mathfrak{g}}, K] = 0, \tag{2.10}$$

where $(\cdot|\cdot)$ is a symmetric invariant bilinear form on $\mathfrak{g}$.

**Remark 2.4.2** *It is necessary for the form $(\cdot|\cdot)$ to be symmetric, invariant and bilinear in order for $\hat{\mathfrak{g}}$ to form a Lie algebra. The construction for $\hat{\mathfrak{g}}$ is called the* affinization *of $\mathfrak{g}$.*

We thus obtain the following definition of affine algebra [**K1**].

**Definition 2.4.3** *Let $\mathfrak{g}$ be a Lie algebra with symmetric invariant bilinear form $(\cdot|\cdot)$. Then the Lie algebra*

$$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K,$$

*with bracket defined by*

$$[a_n, b_m] = [a, b]_{n+m} + n\delta_{n,-m}(a|b)K,$$

$$[\hat{\mathfrak{g}}, K] = 0,$$

*is called the* affine Kac-Moody Lie algebra *associated with $\mathfrak{g}$ and $(\cdot|\cdot)$.*

When $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, we shall consider $\tilde{\mathfrak{h}}$ and $\hat{\mathfrak{h}}$ as subalgebras of $\tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}$ in the obvious way. We will also use an analog of affinization by "twisting" by an *involution* of $\mathfrak{g}$. Recall an involution is an automorphism $\sigma$ for which $\sigma^2 = 1$.

Let $\sigma$ be an involution of $\mathfrak{g}$ which is also an isometry with respect to the form $(\cdot|\cdot)$, i.e., satisfying the condition

$$(\sigma x|\sigma y) = (x|y). \tag{2.11}$$

For $i \in \mathbb{Z}/2\mathbb{Z}$, set

$$\mathfrak{g}_{(i)} = \{x \in \mathfrak{g} \,|\, \sigma x = (-1)^i x\}. \tag{2.12}$$

Then we have the following decomposition:

$$\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}, \tag{2.13}$$

$$[\mathfrak{g}_{(0)}, \mathfrak{g}_{(0)}] \subset \mathfrak{g}_{(0)}, \quad [\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}] \subset \mathfrak{g}_{(1)}, \quad [\mathfrak{g}_{(1)}, \mathfrak{g}_{(1)}] \subset \mathfrak{g}_{(0)}, \tag{2.14}$$

$$\big(\mathfrak{g}_{(0)} | \mathfrak{g}_{(1)}\big) = 0. \tag{2.15}$$

Consider the algebra $\mathbb{C}[t^{1/2}, t^{-1/2}]$ of Laurent polynomials in the indeterminate $t^{1/2}$ whose square is $t$, and form the algebra

$$\mathfrak{i} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{1/2}, t^{-1/2}] \oplus \mathbb{C}K. \tag{2.16}$$

Formulas (2.9) and (2.10) make $\mathfrak{i}$ into a Lie algebra. Let $\theta$ be the involution of $\mathbb{C}[t^{1/2}, t^{-1/2}]$ given by $\theta(t^{1/2}) = -t^{1/2}$ and denote by $\sigma$ the automorphism of $\mathfrak{i}$ determined by

$$\sigma(K) = K,$$
$$\sigma(x \otimes f) = \sigma(x) \otimes \theta(f),$$

for $x \in \mathfrak{g}$ and $f \in \mathbb{C}[t^{1/2}, t^{-1/2}]$.

**Definition 2.4.4** *The subalgebra*

$$\begin{aligned} \hat{\mathfrak{g}}[\sigma] &= \{x \in \mathfrak{i} \,|\, \sigma x = x\} \\ &= \big(\mathfrak{g}_{(0)} \otimes \mathbb{C}[t, t^{-1}]\big) \oplus \Big(\mathfrak{g}_{(1)} \otimes t^{1/2}\mathbb{C}[t, t^{-1}]\Big) \oplus \mathbb{C}K \end{aligned}$$

*is the* twisted affine algebra *associated with* $\mathfrak{g}$ *and* $(\cdot|\cdot)$.

Two particular Lie algebras will be useful later on, those being the (infinite dimensional) *Heisenberg algebra* and the *Virasoro algebra*. We breifly define each and give some pertinent details.

**Definition 2.4.5** *A Lie algebra* $\mathfrak{l}$ *is a* Heisenberg Lie algebra *if the center of* $\mathfrak{l}$ *is equal to* $[\mathfrak{l}, \mathfrak{l}]$, *and is one dimensional.*

We will be concerned with the Heisenberg algebra $\mathfrak{s}$ with basis $\{a_n, K \,|\, n \in \mathbb{Z}\}$ and commutation relations

$$[a_m, a_n] = m\delta_{m,-n}K, \quad [K, a_m] = 0. \tag{2.17}$$

The algebra $\mathfrak{s}$ is also referred to as the *oscillator algebra* and has a representation on the space of polynomials in infinitely many variables $B = \mathbb{C}[x_1, x_2, \ldots]$ given by:

$$a_n = \frac{\partial}{\partial x_n}, \quad a_{-n} = n x_n, \quad (n > 0)$$
$$a_0 = 0, \quad K = 1.$$

It is easily shown that $B$ is indeed a representation of $\mathfrak{s}$, called the *Bosonic Fock space*.

The $\mathfrak{s}$-module $B$ is also graded via $\dim x_n = n$, i.e., $B = \bigoplus_{n=0}^{\infty} B_n$ with $\dim B_n = p(n)$, where $p(n)$ is the partition function. The algebra $\mathfrak{s}$ also has the following triangular decomposition:

$$\mathfrak{s} = \operatorname{span}\{a_n\}_{n<0} \oplus \operatorname{span}\{a_0, K\} \oplus \operatorname{span}\{a_n\}_{n>0}.$$

As infinite matrices, the elements $a_n$, with $n < 0$, are lower triangular and called *creation operators*, and the elements $a_n$, with $n > 0$, are upper triangular and called *annihilation operators*. It can further be shown that the Bosonic Fock space $B$ is an irreducible $\mathfrak{s}$-module and that every such representation is isomorphic to $B$.

**Definition 2.4.6** *The* Virasoro algebra *is a Lie algebra with basis $\{L_n, C \mid n \in \mathbb{Z}\}$ and commutation relations*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}C, \tag{2.18}$$

*and $[L_n, C] = 0$ for all $n$.*

The Virasoro algebra can be constructed in a few different ways. One way is to realize the Virasoro operators from the Heisenberg operators:

$$L_0 = \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty} a_{-k}a_k, \tag{2.19}$$

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_{n-k}a_k, \quad n \neq 0, \tag{2.20}$$

$$C = 1. \tag{2.21}$$

The Virasoro operators commute with the Heisenberg operators via $[L_n, a_m] = -m a_{m+n}$. From this relation, the commutator (2.18) can then be verified.

Another way is to recognize the Virasoro operators as a central extension of the derivations of the algebra $\mathbb{C}[t,t^{-1}]$. This Lie algebra is given by

$$\operatorname{Der}\mathbb{C}[t,t^{-1}] = \mathbb{C}[t,t^{-1}]\frac{d}{dt},$$

the set of polynomial vector fields on the circle, and has as a basis the set

$$\left\{ d_m = -t^{m+1}\frac{d}{dt} \,\middle|\, m \in \mathbb{Z} \right\}.$$

It can be shown that these basis elements satisfy the commutation relation

$$[d_m, d_n] = (m-n)d_{m+n}.$$

Then the Virasoro algebra is a central extension of the Lie algebra $\operatorname{Der}\mathbb{C}[t,t^{-1}]$. Suppose in the central extension $\operatorname{Der}\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}C$ we have

$$[d_m, d_n] = (m-n)d_{m+n} + \alpha(m,n)C,$$

where $C$ is the central element. Then it can be shown that $\alpha(m,n) = \delta_{m,-n}\dfrac{m^3-m}{12}$ and the choice of the factor $\dfrac{1}{12}$ comes from physics.

# Chapter 3

# Vertex Algebras

In this chapter we introduce the notion of a *vertex operator algebra* and present some pertinent examples. The structure of a vertex algebra was first introduced by Richard Borcherds in 1986 [**B**] and have since been realized as having deep roots and applications in both mathematics and physics. Roughly speaking, the elements of these algebras are types of "vertex operators", which were introduced when string theory was first coming about. These operators were meant to describe certain kinds of physical interactions within the context of string theory.

After some time, it was further realized that the theory of vertex operator algebras could be used to describe a remarkably beautiful mathematical entity called the *Monster*, the largest sporadic finite simple group [**FLM**]. It is a symmetry group of a special structure, the *Griess algebra* of dimension 196883. It has been proved that the Monster is actually the full automorphism group of the Griess algebra.

Vertex algebras have also been recognized as being closely related to two-dimensional quantum field theory [**BPZ**]. The application of vertex algebras in this thesis will be the representation theory of infinite dimensional Lie algebras, which is also the area out of which these objects were born. For more details on vertex operator algebras and their representations, as well as information concerning their history, the reader is referred to [**FHL**; **FLM**; **K2**; **LL**].

## 3.1 Definitions and Examples

Let $V$ be a vector space and $z$ be a formal variable. We denote by $V[[z,z^{-1}]]$ the vector space of *formal Laurent series* in $z$ with coefficients in $V$:

$$V[[z,z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \,\middle|\, v_n \in V \right\}. \tag{3.1}$$

**Remark 3.1.1** *In vertex algebra theory, this vector space is often taken as* $\mathrm{End}\, V$*, the endomorphisms of the vector space* $V$*, and the formal series is written as* $\sum_{n\in\mathbb{Z}} v_n z^{-n-1}$*, where* $v_n \in \mathrm{End}\, V$ *is parametrized by the element* $v \in V$ *and* $n \in \mathbb{Z}$*. Important such formal series will be called "vertex operators".*

The space $V[[z,z^{-1}]]$ contains a number of subspaces which become useful in the theory:

$$V[z,z^{-1}] = \left\{ \sum_{n=-m}^{k} v_n z^n \,\middle|\, m,k \geq 0, v_n \in V \right\}, \quad \text{(formal Laurent polynomials)} \tag{3.2}$$

$$V[[z]] = \left\{ \sum_{n\in\mathbb{N}} v_n z^n \,\middle|\, v_n \in V \right\}, \quad \text{(formal power series)} \tag{3.3}$$

$$V((z)) = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \,\middle|\, v_n \in V, v_n = 0 \text{ for } n \ll 0 \right\}. \quad \text{(truncated Laurent series)} \tag{3.4}$$

The notation $v_n = 0$ for $n \ll 0$ means that there exists some integer $N < 0$ such that $v_n = 0$ for all $n < N$, i.e., that $v_n$ is zero for $n$ sufficiently negative.

**Remark 3.1.2** *A formal sum or product of formal series of operators on a vector space is understood to exist if and only if the coefficient of any monomial in the formal sum or product acts as a finite sum of operators when applied to any fixed, but arbitrary, vector in the space. Hence infinite sums of operators are allowed, but only under this restrictive condition.*

In general, we cannot always multiply formal series. An example of a nonexistent product in $\mathbb{C}[[z,z^{-1}]]$ is

$$\left( \sum_{n\geq 0} z^n \right) \left( \sum_{n\leq 0} z^n \right).$$

**Definition 3.1.3** *A formal series of the form*

$$\sum_{m,n,\ldots\in\mathbb{Z}} a_{m,n,\ldots} z^m w^n \ldots,$$

*where* $a_{m,n,\ldots}$ *are elements of a vector space* $V$*, is a* formal distribution *in the indeterminates* $z, w, \ldots$ *with values in* $V$*.*

Now consider the affine algebra $\hat{\mathfrak{g}}$. For $a \in \mathfrak{g}$, the formal distribution

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

is called the *current* associated to $a \in \mathfrak{g}$. For $a, b \in \mathfrak{g}$, the corresponding formal distributions $a(z)$ and $b(w)$ can be used to write the commutation relations among all $a_m$ and $b_n$.

In order to write the commutation relation between two formal distributions, we define a formal distribution in $z$ and $w$ with values in $\mathbb{C}$:

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1} \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]], \tag{3.5}$$

called the *formal delta function*. Then using this distribution, it can be shown that

$$[a(z), b(w)] = [a, b](w)\delta(z-w) + (a|b)K\partial_w\delta(z-w). \tag{3.6}$$

The bracket $[a_m, b_n]$ is then determined by computing the coefficient of $z^{-m-1}w^{-n-1}$.

The delta function has additional properties that characterize an important axiom of vertex algebras.

**Proposition 3.1.4** *The following are some properties of the delta function:*

*i)* $\delta(z-w) = \delta(w-z)$,

*ii)* $\partial_z\delta(z-w) = -\partial_w\delta(z-w)$,

*iii)* $(z-w)^{j+1}\partial_w^j\delta(z-w) = 0, \quad j \geq 0.$

For the affine algebra $\hat{\mathfrak{g}}$, we therefore obtain

$$(z-w)^2[a(z), b(w)] = 0.$$

Whenever the bracket between two distributions is in the null space of the operator of multiplication by $(z-w)^N$, for $N$ sufficiently large, we say the distributions are *mutually local.* The idea of locality is important for many calculations in vertex algebras and is also a central axiom of the definition.

**Definition 3.1.5** *A* vertex algebra *is a vector space $V$ endowed with a vector $|0\rangle$ (called the* vacuum vector*), an endomorphism $T$ (called the* infinitesimal translation operator*), and a linear map*

$$\begin{aligned} Y(\cdot,z):V &\longrightarrow (\mathrm{End}\,V)((z)) &(3.7)\\ a &\mapsto Y(a,z)=\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}, \quad a_{(n)}\in \mathrm{End}\,V &(3.8)\end{aligned}$$

*such that $a_{(n)}v=0$ for $n\gg 0$ and $v\in V$ and the following axioms hold for all $a,b\in V$:*

*i)* **(translation covariance):** $[T,Y(a,z)]=\partial Y(a,z)$,

*ii)* **(vacuum):** $T|0\rangle=0,\ Y(|0\rangle,z)=I_V,\ Y(a,z)|0\rangle|_{z=0}=a$,

*iii)* **(locality):** $(z-w)^N[Y(a,z),Y(b,z)]=0$ *for* $N\gg 0$.

**Remark 3.1.6** *A formal distribution*

$$\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}\in(\mathrm{End}\,V)[[z,z^{-1}]]$$

*is called a* field *if $a_{(n)}v=0$ for $n\gg 0$ and $v\in V$. The elements of a vertex algebra are called* states*; the linear map $Y$ is called the* state-field correspondence *and the coefficients $a_{(n)}$ are called* modes.

An important notion that causes many products among formal distributions to be well defined is that of *normal ordering.*

**Definition 3.1.7** *For two fields $a(z)$ and $b(z)$, their* normally ordered product *is*

$$:a(z)b(z):=a(z)_+b(z)+b(z)a(z)_-, \tag{3.9}$$

*where* $a(z)_+=\sum_{n<0}a_{(n)}z^{-n-1}$ *and* $a(z)_-=\sum_{n\geq 0}a_{(n)}z^{-n-1}$.

We now give two important examples of vertex algebras, those corresponding to the Heisenberg algebra and Virasoro algebra (cf. Section 2.4).

**Example 3.1.8** *Recall the Heisenberg algebra* $\mathfrak{s}$ *with basis* $\{a_n, K \mid n \in \mathbb{Z}\}$ *and bracket given by* (2.17)*. Consider the* $\mathfrak{s}$*-valued formal distribution*

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}.$$

*Then it is straightforward to check that the commutator is given in terms of the delta function by the formula*

$$[a(z), a(w)] = \partial_w \delta(z-w)K.$$

*From this commutator, it is clear that* $a(z)$ *is local with respect to itself. Here we have that* $(a|a) = 1$ *and the (even) formal distribution* $a(z)$ *is called a* free boson*.*

**Example 3.1.9** *The* Virasoro field *can also be written in terms of the free boson. Define* $L(z)$ *to be the field*

$$L(z) = \frac{1}{2} : a(z)^2 :, \tag{3.10}$$

*where* $a(z)$ *is the free boson. It is an exercise to show that the commutator is given by*

$$[L(z), L(w)] = \partial L(w)\delta(z-w) + 2L(w)\partial_w\delta(z-w) + \frac{1}{12}C\partial_w^3\delta(z-w). \tag{3.11}$$

*This commutator relation is equivalent to the bracket given in* (2.18)*. It is also clear that the field* $L(z)$ *is local with respect to itself.*

Most important examples of vertex algebras used in applications contain a vector whose corresponding field is the Virasoro field $L(z)$.

**Definition 3.1.10** *A vector* $\nu$ *in a vertex algebra* $V$ *is a* conformal vector *if its corresponding field* $Y(\nu, z)$ *is a Virasoro field, i.e, satisfies* (3.11)*. Such a vertex algebra is called a* conformal vertex algebra*.*

In Lie algebra theory, most interesting algebras are either finite dimensional or, more generally, $\mathbb{Z}$-graded with finite-dimensional homogeneous subspaces. Correspondingly, we are mostly interested in vertex algebras that have similar properties, those vertex algebras called *vertex operator algebras.*

**Definition 3.1.11** *A* vertex operator algebra *is a $\mathbb{Z}$-graded vector space*

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)},$$

*such that*

$$\begin{aligned} \dim V_{(n)} < \infty, &\quad n \in \mathbb{Z}, \\ V_{(n)} = 0, &\quad n \ll 0, \end{aligned}$$

*that is equipped with a vertex algebra structure $(V, T, Y, |0\rangle)$ and a conformal vector $\nu$ of weight 2 ($\nu \in V_{(2)}$) whose corresponding field satisfies the Virasoro algebra relations, where*

$$Y(\nu, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-1}$$

*and the central element $C$ acts as a scalar (the* central charge*). In addition,*

$$L_0 v = nv, \quad (n \in \mathbb{Z})$$

*where* $\mathrm{wt}\, v = n$ *for $v \in V_{(n)}$, and finally,*

$$L_{-1} = T. \tag{3.12}$$

Notice that the $L_0$-eigenspace decomposition of $V$ coincides with its grading.

**Definition 3.1.12** *A* vertex operator algebra of CFT type *is a vertex operator algebra for which $V_0 = \mathbb{C}|0\rangle$ and $V_n = \{0\}$ for $n < 0$.*

**Remark 3.1.13** *A vertex algebra can equivalently be defined in terms of (a partial vacuum axiom and) the* Borcherds identity*: $\forall\, a, b, c \in V$ and $k, m, n \in \mathbb{Z}$,*

$$\sum_{j=0}^{\infty} \binom{m}{j} \left(a_{(n+j)} b\right)_{(m+k-j)} c \;=\; \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left(a_{(m+n-j)} b_{(k+j)} c - (-1)^n b_{(n+k-j)} a_{(m+j)} c\right).$$

*Some important instances of this formula are useful for calculation:*

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)} \qquad (\textit{commutator formula}) \tag{3.13}$$

$$(a_{(-1)}b)_{(n)} = \sum_{j<0} a_{(j)}b_{(n-j-1)} + \sum_{j\geq 0} b_{(n-j-1)}a_{(j)} \qquad (-1\textit{st product identity}) \tag{3.14}$$

The last definition we present is the notion of product-preserving maps between vertex algebras.

**Definition 3.1.14** *Let $V_1$ and $V_2$ be vertex algebras and $\phi : V_1 \longrightarrow V_2$ be a linear map such that $(v, w \in V_1)$*

$$\phi(|0\rangle) = |0\rangle, \tag{3.15}$$

$$Y(\phi(v), z)\phi(w) = \phi(Y(v, z)w). \tag{3.16}$$

*Then $\phi$ is called a* vertex algebra homomorphism*.*

## 3.2 Lattice Vertex Algebras

Our main object of study is a certain class of vertex algebras called *lattice vertex algebras*. We give the general construction of such algebras and present the associated fields. These were the algebras introduced in Borcherds' original paper [B].

Let $Q$ be an even lattice equipped with symmetric nondegenerate bilinear form $(\cdot|\cdot) : Q \times Q \longrightarrow \mathbb{Z}$. We denote by $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ the corresponding complex vector space considered as an abelian Lie algebra, with the bilinear form extended to it. The bosonic Fock space (cf. Section 2.4) for the Heisenberg algebra $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ can also be written as the irreducible (highest weight) representation

$$M := \mathrm{Ind}_{\mathfrak{h}[t]\oplus\mathbb{C}K}^{\hat{\mathfrak{h}}} \mathbb{C} \cong S(\mathfrak{h}[t^{-1}]t^{-1})$$

with highest weight 1 on which $K = 1$.

**Remark 3.2.1** *In other works, this space is denoted $M(1)$, but here we will write $M$ for brevity.*

Following [**FK**; **B**], we consider a 2-cocycle $\varepsilon\colon Q \times Q \longrightarrow \{\pm 1\}$ such that

$$\varepsilon(\alpha, \alpha) = (-1)^{|\alpha|^2(|\alpha|^2+1)/2}\,, \quad |\alpha|^2 := (\alpha|\alpha)\,, \qquad \alpha \in Q\,, \tag{3.17}$$

and the associative algebra $\mathbb{C}_\varepsilon[Q]$ with basis $\{e^\alpha\}_{\alpha\in Q}$ and multiplication

$$e^\alpha e^\beta = \varepsilon(\alpha,\beta)e^{\alpha+\beta}\,. \tag{3.18}$$

Such a 2-cocycle $\varepsilon$ is unique up to equivalence and can be chosen to be bimultiplicative. Then we have

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)+|\alpha|^2|\beta|^2}\,, \qquad \alpha,\beta\in Q\,. \tag{3.19}$$

**Definition 3.2.2** *The* lattice vertex algebra [**B**; **FLM**; **K2**; **FB**; **LL**] *is the tensor product*

$$V_Q = M\otimes \mathbb{C}_\varepsilon[Q], \tag{3.20}$$

*where the vacuum vector is* $|0\rangle\otimes e^0$*.*

**Remark 3.2.3** *When* $V$ *is a superalgebra, the parity of all vectors in* $M\otimes e^\alpha$ *is* $|\alpha|^2 \mod 2\mathbb{Z}$*.*

We let the Heisenberg algebra act on $V_Q$ by

$$a_n e^\beta = \delta_{n,0}(a|\beta)e^\beta\,, \quad n\geq 0\,, \qquad a\in\mathfrak{h}\,,\ a_n = at^n\,. \tag{3.21}$$

The state-field correspondence on $V_Q$ is uniquely determined by the generating fields:

$$Y(a_{-1}|0\rangle,z) = \textstyle\sum_{n\in\mathbb{Z}} a_n\, z^{-n-1}\,, \qquad a\in\mathfrak{h}\,, \tag{3.22}$$

$$Y(e^\alpha,z) = e^\alpha z^{\alpha_0}\exp\Big(\textstyle\sum_{n<0}\alpha_n\frac{z^{-n}}{-n}\Big)\exp\Big(\textstyle\sum_{n>0}\alpha_n\frac{z^{-n}}{-n}\Big)\,, \tag{3.23}$$

where $z^{\alpha_0}e^\beta = z^{(\alpha|\beta)}e^\beta$.

Notice that $M\subset V_Q$ is a vertex subalgebra, which we call the *Heisenberg vertex algebra*. The map $\mathfrak{h}\longrightarrow M$ given by $a\mapsto a_{-1}|0\rangle$ is injective. From now on, we will slightly abuse the notation and identify $a\in\mathfrak{h}$ with $a_{-1}|0\rangle\in M$; then $a_{(n)} = a_n$ for all $n\in\mathbb{Z}$.

When the lattice $Q$ is even and positive definite, the lattice vertex algebra $V_Q$ has the structure of a vertex operator algebra. Let $\{a^i\}$ and $\{b^i\}$ be dual bases of $\mathfrak{h}$ (i.e. $(a^i|b^j)=\delta_{ij}$). Then the conformal vector is given by

$$\nu = \frac{1}{2}\sum_i a^i_{-1}b^i_{-1}|0\rangle. \tag{3.24}$$

The central charge is the *rank* of the lattice $Q$, the fields $h(z)$ ($h\in\mathfrak{h}$) have conformal weight 1, and the fields $Y(e^\alpha,z)$ have conformal weight $\frac{1}{2}(\alpha|\alpha)$. Note that the lattice being even and positive definite is necessary for the proper grading of a vertex operator algebra.

**Example 3.2.4** *When the lattice $Q$ can be written as an orthogonal direct sum, $Q = L_1 \oplus L_2$, the corresponding lattice vertex algebra is given by the tensor product:*

$$V_Q = V_{L_1} \otimes V_{L_2}. \tag{3.25}$$

## 3.3 Twisted Modules

Given a lattice vertex algebra $V_Q$ and an automorphism $\sigma$ of $V_Q$, we will construct a set of *$\sigma$-twisted modules*. We first define three notions of (untwisted) modules over vertex operator algebras and also define the terms rational and regular (see [**ABD**]).

**Definition 3.3.1** *A* weak module *of a vertex operator algebra $V$ is a vector space $M$ endowed with a linear map $Y^M(\cdot, z)\cdot \colon V \otimes M \longrightarrow M(\!(z)\!)$ (cf. (3.4), (3.8)) such that*

*i)* **(truncation):** $v_{(n)}w = 0$ *for* $n \gg 0$*, where* $v \in V$ *and* $w \in M$*,*

*ii)* **(vacuum):** $Y^M(|0\rangle, z) = I_M$*,*

*iii) the Borcherds identity (cf. Remark 3.1.13) holds for* $a, b \in V$*,* $c \in M$*.*

**Remark 3.3.2** *Of the three types of vertex operator algebra modules, only weak modules have no grading assumptions. The structure of weak modules coincides with the structure of modules over vertex algebras.*

**Definition 3.3.3** *An* admissible module *of a vertex operator algebra $V$ is a weak module $M$ which carries a $\mathbb{Z}_+$-grading*

$$M = \bigoplus_{n \in \mathbb{Z}_+} M(n) \tag{3.26}$$

*such that if $v \in V_{(k)}$, then $v_{(m)}M(n) \subset M(n + k - m - 1)$.*

**Definition 3.3.4** *An* ordinary module *of a vertex operator algebra $V$ is a weak module $M$ which carries a $\mathbb{C}$-grading*

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda \tag{3.27}$$

*such that*

*i)* $\dim M_\lambda < \infty$*,*

*ii)* $M_{\lambda - n} = 0$ *for fixed* $\lambda$ *and* $n \gg 0$*,* *iii)* $L_0 w = \lambda w = \mathrm{wt}(w) w$ *for* $w \in M_\lambda$.

Note that an ordinary module has a grading that matches the $L_0$ action of the Virasoro representation. It turns out that the finite dimensionality of graded pieces in ordinary modules is a strong condition, so that ordinary modules are also admissible. Hence we have the following inclusions of modules:

$$\{\text{ordinary modules}\} \subseteq \{\text{admissible modules}\} \subseteq \{\text{weak modules}\}.$$

**Definition 3.3.5** *A vertex operator algebra is* rational *if every admissible module is a direct sum of simple admissible modules.*

In other words, a vertex operator algebra is rational if there is complete reducibility of admissible modules. It is proved in [**DLM2**] that for rational vertex operator algebras with a certain finiteness condition, there are only finitely many simple admissible modules up to isomorphism and any simple admissible module is an ordinary module. The strongest form of complete reducibility is when weak modules can be realized in terms of ordinary modules.

**Definition 3.3.6** *A vertex operator algebra is* regular *if every weak module is a direct sum of simple ordinary modules.*

Thus for regular vertex operator algebras, every simple weak module is an ordinary module. Now let $\sigma$ be an automorphism of $V$ of a finite order $r$. Then $\sigma$ is diagonalizable. The notion of a twisted vertex algebra representation was introduced in [**FFR**; **D2**; **L**]. The main difference is that the image of the above map $Y^M$ is allowed to have nonintegral rational powers of $z$.

**Definition 3.3.7** *A* $\sigma$-twisted module *of a vertex algebra* $V$ *is a vector space* $M$ *endowed with a linear map*

$$a \mapsto Y^M(a,z) = \sum_{n \in \frac{1}{r}\mathbb{Z}} a^M_{(n)} z^{-n-1}, \tag{3.28}$$

*where* $a^M_{(n)} \in \operatorname{End} M$*, such that the following axioms hold for all* $a, b, c \in V$*:*

*i)* **(vacuum):** $Y^M(|0\rangle, z) = I_V$,

*ii)* **(covariance):** $Y^M(\sigma a, z) = Y^M(a, e^{2\pi \mathrm{i}} z)$,

*iii)* *the Borcherds identity* (3.1.13) *is satisfied by the modes, provided that* $a$ *is an eigenvector of* $\sigma$.

More precisely, the linear map $Y^M$ satisfies

$$Y^M(a,z) = \sum_{n \in p+\mathbb{Z}} a^M_{(n)}\, z^{-n-1}\,, \qquad \text{if} \quad \sigma a = e^{-2\pi \mathrm{i} p} a\,,\ p \in \frac{1}{r}\mathbb{Z}\,. \tag{3.29}$$

**Remark 3.3.8** *The notion of a twisted representation axiomatizes the properties of the so-called "twisted vertex operators"* [**L**]*, which were used in the construction of the "moonshine module" vertex algebra in* [**FLM**] *in the study of the Monster group.*

When restricted to the $\sigma$-invariant subalgebra $V^\sigma \subset V$, a $\sigma$-twisted representation for $V$ becomes untwisted for $V^\sigma$. This subalgebra will be the main object of study.

**Definition 3.3.9** *The subalgebra $V^\sigma \subset V$ of $\sigma$-invariant elements of $V$ is called the* orbifold (see for example [**DVVV**; **KT**; **DLM2**]).

The following theorem is due to Frenkel and Kac (see [**FK**; **K1**; **K2**]) and relates modules of root lattice vertex algebras to modules over affine Kac-Moody algebras at level 1.

**Theorem 3.3.10** (Frenkel-Kac Construction)
*Let $\mathfrak{g}$ be a simply-laced finite dimensional simple Lie algebra, and $Q$ be its root lattice. Then the untwisted representations of the lattice vertex algebra $V_Q$ provide a construction of level the* 1 *representations of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ in the homogeneous realization.*

The next result provides a rigorous interpretation of the *operator product expansion* in conformal field theory in the case of twisted modules.

**Proposition 3.3.11 ([BM])** *Let $V$ be a vertex algebra, $\sigma$ an automorphism of $V$, and $M$ a $\sigma$-twisted $V$-module. Then*

$$\frac{1}{k!}\partial_z^k\Big((z-w)^N\, Y^M(a,z)Y^M(b,w)c\Big)\Big|_{z=w} = Y^M(a_{(N-1-k)}b,w)c \tag{3.30}$$

*for all $a,b \in V$, $c \in M$, $k \geq 0$, and sufficiently large $N$.*

### 3.3.1 Twisted Heisenberg Algebra

Let $\mathfrak{h}$ be a finite dimensional vector space equipped with a symmetric nondegenerate bilinear form $(\cdot|\cdot)$, as in Section 2.4. Then we have the Heisenberg algebra $\hat{\mathfrak{h}}$ and its highest weight representation (the Fock space $M$) which has the structure of a vertex algebra. Every automorphism $\sigma$ of $\mathfrak{h}$ preserving the bilinear form induces automorphisms of $\hat{\mathfrak{h}}$ and $M$, which will be denoted again as $\sigma$. As before, assume that $\sigma$ has finite order $r$. The action of $\sigma$ can be extended to $\mathfrak{h}[t^{1/r}, t^{-1/r}] \oplus \mathbb{C}K$ by letting

$$\sigma(at^m) = \sigma(a)e^{2\pi \mathrm{i} m} t^m\,, \quad \sigma(K) = K\,, \qquad a \in \mathfrak{h}\,,\ m \in \frac{1}{r}\mathbb{Z}\,.$$

The *$\sigma$-twisted Heisenberg algebra* $\hat{\mathfrak{h}}_\sigma$ is defined as the set of all $\sigma$-invariant elements (see [**KP**; **L**]). In other words, $\hat{\mathfrak{h}}_\sigma$ is spanned over $\mathbb{C}$ by $K$ and the elements $a_m = at^m$ such that $\sigma a = e^{-2\pi \mathrm{i} m}a$. This is a Lie algebra with bracket (cf. (2.17))

$$[a_m, b_n] = m\delta_{m,-n}(a|b)K\,, \qquad a, b \in \mathfrak{h}\,,\ \ m, n \in \frac{1}{r}\mathbb{Z}\,.$$

Let $\hat{\mathfrak{h}}_\sigma^{\geq}$ (respectively $\hat{\mathfrak{h}}_\sigma^{<}$) be the subalgebra of $\hat{\mathfrak{h}}_\sigma$ spanned by all elements $a_m$ with $m \geq 0$ (respectively $m < 0$). The elements of $\hat{\mathfrak{h}}_\sigma^{\geq}$ are the annihilation operators, and the elements of $\hat{\mathfrak{h}}_\sigma^{<}$ are the creation operators.

The *$\sigma$-twisted Fock space* is defined as

$$M(\sigma) := \mathrm{Ind}_{\hat{\mathfrak{h}}_\sigma^{\geq} \oplus \mathbb{C}K}^{\hat{\mathfrak{h}}_\sigma} \mathbb{C} \cong S(\hat{\mathfrak{h}}_\sigma^{<})\,, \tag{3.31}$$

where $\hat{\mathfrak{h}}_\sigma^{\geq}$ acts on $\mathbb{C}$ trivially and $K$ acts as the identity operator. $M(\sigma)$ is an irreducible highest weight representation of $\hat{\mathfrak{h}}_\sigma$ and has the structure of a $\sigma$-twisted representation of the vertex algebra $M$ (see [**FLM**; **FFR**; **DL2**]). This structure can be described as follows. We let $Y(|0\rangle, z)$ be the identity operator and

$$Y(a, z) = \sum_{n \in p + \mathbb{Z}} a_n\, z^{-n-1}\,, \qquad a \in \mathfrak{h}\,,\ \ \sigma a = e^{-2\pi \mathrm{i} p} a\,, \tag{3.32}$$

where $p \in \frac{1}{r}\mathbb{Z}$ (cf. (3.29)), and we extend $Y$ to all $a \in \mathfrak{h}$ by linearity.

The action of $Y$ on other elements of $M$ is then determined by applying several times the product formula (3.30). More explicitly, $M$ is spanned by elements of the form $a^1_{m_1} \cdots a^k_{m_k}|0\rangle$, where $a^j \in \mathfrak{h}$, and we have:

$$Y(a^1_{m_1} \cdots a^k_{m_k}|0\rangle, z)c = \prod_{j=1}^{k} \partial_{z_j}^{(N-1-m_j)} \Bigl( \prod_{j=1}^{k} (z_j - z)^N\, Y(a^1, z_1) \cdots Y(a^k, z_k)c \Bigr)\Big|_{z_1 = \cdots = z_k = z}$$

for all $c \in M(\sigma)$ and sufficiently large $N$. In the above formula, we use the divided-power notation $\partial^{(n)} := \partial^n / n!$.

### 3.3.2 Twisted Representations of Lattice Vertex Algebras

Now let $Q$ be a positive definite even lattice and $\sigma$ be an automorphism of the lattice $Q$ of finite order $r$ such that

$$(\sigma\alpha|\sigma\beta) = (\alpha|\beta)\,, \qquad \alpha, \beta \in Q\,. \tag{3.33}$$

The uniqueness of the cocycle $\varepsilon$ and (3.33), (3.19) imply that

$$\eta(\alpha+\beta)\varepsilon(\sigma\alpha, \sigma\beta) = \eta(\alpha)\eta(\beta)\varepsilon(\alpha, \beta) \tag{3.34}$$

for some function $\eta\colon Q \longrightarrow \{\pm 1\}$, and

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}. \tag{3.35}$$

**Lemma 3.3.12** *Let $L$ be a sublattice of $Q$ such that $\varepsilon(\sigma\alpha, \sigma\beta) = \varepsilon(\alpha, \beta)$ for $\alpha, \beta \in L$. Then there exists a function $\eta\colon Q \longrightarrow \{\pm 1\}$ satisfying* (3.34) *and $\eta(\alpha) = 1$ for all $\alpha \in L$.*

**Proof** First observe that, by (3.17) and (3.33), (3.34) for $\alpha = \beta$, we have $\eta(2\alpha) = 1$ for all $\alpha \in Q$. Since, by bimultiplicativity, $\varepsilon(2\alpha, \beta) = 1$, we obtain that $\eta(2\alpha + \beta) = \eta(\beta)$ for all $\alpha, \beta$. Therefore, $\eta$ is defined on $Q/2Q$. If $\alpha_1, \dots, \alpha_\ell$ is any $\mathbb{Z}$-basis for $Q$, we can set all $\eta(\alpha_i) = 1$ and then $\eta$ is uniquely extended to the whole $Q$ by (3.34). We can pick a $\mathbb{Z}$-basis for $Q$ so that $d_1\alpha_1, \dots, d_m\alpha_m$ is a $\mathbb{Z}$-basis for $L$, where $m \le \ell$ and $d_i \in \mathbb{Z}$. Then the extension of $\eta$ to $Q$ will satisfy $\eta(\alpha) = 1$ for all $\alpha \in L$.

In particular, $\eta$ can be chosen such that

$$\eta(\alpha) = 1\,, \qquad \alpha \in Q \cap \mathfrak{h}_0\,, \tag{3.36}$$

where $\mathfrak{h}_0$ denotes the subspace of $\mathfrak{h}$ consisting of vectors fixed under $\sigma$. The automorphism $\sigma$ of $Q$ can be lifted to an automorphism of the lattice vertex algebra $V_Q$ by setting

$$\sigma(a_n) = \sigma(a)_n\,, \quad \sigma(e^\alpha) = \eta(\alpha)e^{\sigma\alpha}\,, \qquad a \in \mathfrak{h}\,,\ \alpha \in Q\,. \tag{3.37}$$

We now recall the construction of irreducible $\sigma$-twisted $V_Q$-modules (see [**KP**; **D2**; **BK1**; **L**]). Introduce the group $G = \mathbb{C}^\times \times \exp\mathfrak{h}_0 \times Q$ consisting of elements $c\,e^h U_\alpha$ ($c \in \mathbb{C}^\times$, $h \in \mathfrak{h}_0$,

$\alpha \in Q$) with multiplication

$$e^h e^{h'} = e^{h+h'}\,, \tag{3.38}$$

$$e^h U_\alpha e^{-h} = e^{(h|\alpha)} U_\alpha\,, \tag{3.39}$$

$$U_\alpha U_\beta = \varepsilon(\alpha,\beta) B_{\alpha,\beta}^{-1} U_{\alpha+\beta}\,, \tag{3.40}$$

where

$$B_{\alpha,\beta} = r^{-(\alpha|\beta)} \prod_{k=1}^{r-1} \bigl(1 - e^{2\pi \mathrm{i} k/r}\bigr)^{(\sigma^k\alpha|\beta)}\,. \tag{3.41}$$

Note that $B_{-\alpha,\beta} = B_{\alpha,\beta}^{-1} = B_{\alpha,-\beta}$ and that the elements

$$\eta(\alpha) U_{\sigma\alpha}^{-1} U_\alpha e^{2\pi\mathrm{i}(b_\alpha + \pi_0\alpha)}\,, \qquad \alpha \in Q \tag{3.42}$$

are central in $G$, where

$$b_\alpha = \frac{|\pi_0\alpha|^2 - |\alpha|^2}{2} \tag{3.43}$$

and $\pi_0$ is the projection of $\mathfrak{h}$ onto $\mathfrak{h}_0$. Let $G_\sigma$ be the factor of $G$ over the central subgroup

$$N_\sigma = \{\eta(\alpha) U_{\sigma\alpha}^{-1} U_\alpha e^{2\pi\mathrm{i}(b_\alpha + \pi_0\alpha)} \,|\, \alpha \in Q\}. \tag{3.44}$$

Then the $\sigma$-twisted $V_Q$-modules are in correspondence with representations of $G_\sigma$ (see [**BK1**], Proposition 4.2). The center of $G_\sigma$ is given by

$$Z(G_\sigma) \simeq \mathbb{C}^\times \times (Q^*/Q)^\sigma, \tag{3.45}$$

$$U_{(1-\sigma)\lambda} \leftrightarrow \lambda. \tag{3.46}$$

Let $\Omega$ be an irreducible representation of $G$, on which all elements (3.42) act as the identity. Such representations are parameterized by the set $(Q^*/Q)^\sigma$ of $\sigma$-invariants in $Q^*/Q$, i.e., by elements $\lambda + Q$ such that $\lambda \in Q^*$ and $(1-\sigma)\lambda \in Q$ (see [**BK1**], Proposition 4.4).

The action of the group algebra $\exp \mathfrak{h}_0$ on $\Omega$ is semisimple:

$$\Omega = \bigoplus_{\mu \in \pi_0(Q^*)} \Omega_\mu\,, \tag{3.47}$$

where

$$\Omega_\mu = \{v \in \Omega \,|\, e^h v = e^{(h|\mu)} v \;\text{ for }\; h \in \mathfrak{h}_0\}\,. \tag{3.48}$$

Then $M(\sigma) \otimes \Omega$ is an irreducible $\sigma$-twisted $V_Q$-module with an action defined as follows. We define $Y(a, z)$ for $a \in \mathfrak{h}$ as before (see (3.32)), and for $\alpha \in Q$ we let

$$Y(e^\alpha, z) = :\exp\left(\sum_{n \in \frac{1}{r}\mathbb{Z}\setminus\{0\}} \alpha_n \frac{z^{-n}}{-n}\right): \otimes\, U_\alpha z^{b_\alpha + \pi_0\alpha}. \tag{3.49}$$

The action of $z^{\pi_0\alpha}$ is given by $z^{\pi_0\alpha}v = z^{(\pi_0\alpha|\mu)}v$ for $v \in \Omega_\mu$, and $(\pi_0\alpha|\mu) \in \frac{1}{r}\mathbb{Z}$. The action of $Y$ on all of $V_Q$ can be obtained by applying the product formula (3.30). By Theorem 4.2 in [**BK1**], every irreducible $\sigma$-twisted $V_Q$-module is obtained in this way, and every $\sigma$-twisted $V_Q$-module is a direct sum of irreducible ones.

**Example 3.3.13** *We present the special case when $\sigma = -1$. The results will be needed in chapter 4.*

*Here $\mathfrak{h}_0 = 0$ so that $G = \mathbb{C}^* \times Q$ and the only relation is* (3.40). *Since $r = 2$, we have*

$$B_{\alpha,\beta} = 4^{-(\alpha|\beta)}, \tag{3.50}$$

$$b_\alpha = -\frac{1}{2}|\alpha|^2, \tag{3.51}$$

*for any $\alpha, \beta \in Q$. Since $\varepsilon(\sigma\alpha, \sigma\beta) = \varepsilon(\alpha, \beta)$ we can set $\eta = 1$. Hence the central subgroup $N_\sigma$ consists of elements of the form*

$$U_{-\alpha}^{-1} U_\alpha,$$

*and the commutator is given by*

$$C_{\alpha,\beta} = U_\alpha U_\beta U_\alpha^{-1} U_\beta^{-1} \tag{3.52}$$

$$= (-1)^{(\alpha|\beta)}. \tag{3.53}$$

*The irreducible representations of $G$ are parameterized by the set of elements $\lambda + Q$ such that $\lambda \in Q^*$ and $(1-\sigma)\lambda = 2\lambda \in Q$. Thus the center of $G_\sigma$ is given by*

$$Z(G_\sigma) = \mathbb{C}^\times \times \{U_{2\lambda} \,|\, \lambda \in Q^*, 2\lambda \in Q\}.$$

When the lattice $Q$ can be written as an orthogonal direct sum, $Q = L_1 \oplus L_2$, such that both factors $L_1$ and $L_2$ are $\sigma$-invariant, each twisted $V_Q$-module can be realized in terms of twisted modules over $V_{L_1}$ and $V_{L_2}$. The following lemma will be useful later.

**Lemma 3.3.14** *Let $Q$ be an even lattice and $\sigma$ be an automorphism of $Q$. Suppose $Q$ decomposes* *as the direct sum*

$$Q = L_1 \oplus L_2,$$

*such that* $\sigma(L_i) \subset L_i \quad (i = 1, 2)$*. In addition, set* $\sigma_i = \sigma|_{L_i}$*. Then any irreducible* $\sigma$*-twisted* $V_Q$*-module* $M$ *is the tensor product*

$$M \simeq M_1 \otimes M_2,$$

*where* $M_i$ *is an irreducible* $\sigma_i$*-twisted* $V_{L_i}$*-module.*

**Proof** Let $M_i$ be a $\sigma_i$-twisted $V_{L_i}$-module for $i = 1, 2$. Then the tensor product $M_1 \otimes M_2$ becomes a twisted module over $V_Q$ under the automorphism $\sigma = (\sigma_1, \sigma_2)$.

Conversely, for any such decomposition of $Q$, we have the following decomposition of the $\sigma$-twisted Fock space:

$$M(\sigma) = M(\sigma_1) \otimes M(\sigma_2).$$

Let $T_\chi$ be a $G_\sigma$-module with central character $\chi$. Then by (3.53), two orthogonal elements commute. Let $h \in \mathfrak{h}_0$ and $\gamma \in Q$. Then $h = h_1 + h_2$ and $\gamma = \alpha + \beta$ for some $h_1, \alpha \in L_1$ and $h_2, \beta \in L_2$. Thus for an element $e^h U_\gamma \in G$,

$$\begin{aligned} e^h U_\gamma &= e^{h_1+h_2} U_{\alpha+\beta} \\ &= e^{h_1} e^{h_2} U_\alpha U_\beta \\ &= e^{h_1} U_\alpha e^{h_2} U_\beta. \end{aligned}$$

Hence $T_\chi$ can be decomposed as

$$T_\chi = T_{\chi_1} \otimes T_{\chi_2},$$

where $\chi_i = \chi|_{L_i}$, so that

$$M(\sigma) \otimes T_\chi = (M(\sigma_1) \otimes T_{\chi_1}) \otimes (M(\sigma_2) \otimes T_{\chi_2}).$$

### 3.3.3 The case $\sigma = -1$

We now describe other work that has been done in the special case when $\sigma = -1$. We also describe the notation used in [**D1**; **DN**; **AD**].

Let $L$ be an even lattice with positive definite integral $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ and let $\hat{L}$ be the central extension of $L$ by the cyclic group $\langle -1\rangle$ of order 2:

$$1 \longrightarrow \langle -1\rangle \longrightarrow \hat{L} \longrightarrow L \longrightarrow 0.$$

The commutator map is given by $c(\alpha,\beta) = (-1)^{(\alpha|\beta)}$ for $\alpha,\beta \in L$. Let $e : L \longrightarrow \hat{L}$ be a section such that $e_0 = 1$ and let $\varepsilon : L \times L \longrightarrow \langle -1\rangle$ be the corresponding 2-cocycle that can be taken as bimultiplicative. Then for $\alpha,\beta,\gamma \in L$,

$$\begin{aligned} \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) &= (-1)^{(\alpha|\beta)} &(3.54)\\ \varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) &= \varepsilon(\beta,\gamma)\varepsilon(\alpha,\beta+\gamma) &(3.55)\\ e_\alpha e_\beta &= \varepsilon(\alpha,\beta)e_{\alpha+\beta}. &(3.56)\end{aligned}$$

Recall Dong's Theorem that the irreducible $V_L$-modules are classified by the set $L^*/L$ (see [**D1**]). Explicitly, they are given by:

$$V_{\lambda+L} = M \otimes \mathbb{C}_\varepsilon[L]e^\lambda\,, \qquad \lambda \in L^*\,. \tag{3.57}$$

Let $\theta$ be the automorphism of $\hat{L}$ used in [**DN**; **AD**] defined by $\theta(e_\alpha) = e_{-\alpha}$ and $\theta(-1) = -1$. Set $K = \{a^{-1}\theta(a) \,|\, a \in \hat{L}\}$. Also define $V_L^T = M(\theta) \otimes T$ (cf. (3.31)) for any $\hat{L}/K$-module $T$ such that $-1$ acts as the scalar $-1$. Then $V_L^T$ forms a $\sigma$-twisted $V_L$-module. The map $\theta$ acts on $V_L^T$ by

$$\theta\left(h^1_{(-n_1)}\cdots h^k_{(-n_k)}t\right) = (-1)^k h^1_{(-n_1)}\cdots h^k_{(-n_k)}t$$

for $h^i \in \mathfrak{h}$, $n_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ and $t \in T$. The eigenspaces for $\theta$ are denoted $V_L^{T,\pm}$. More explicitly, the central character is given by

$$\chi_\mu(e_{2\lambda}) = (-1)^{(2\lambda|\mu)}, \tag{3.58}$$

for some $\mu \in (2L^* \cap L)^*$. Any irreducible $\sigma$-twisted $V_L$-module is isomorphic to $V_L^{T_\chi}$ and the eigenspaces $V_L^{T_\chi,\pm}$ are irreducible $V_L^+$-modules. Furthermore, any irreducible $V_L^+$-module of twisted type is isomorphic to one of $V_L^{T_\chi,\pm}$.

**Theorem 3.3.15** [**DN**; **AD**] *Let $L$ be a positive definite even lattice. Then any irreducible admissible $V_L^+$-module is isomorphic to one of the following:*

$$V_L^\pm, \quad V_{\lambda+L}^\pm \quad (\lambda \in L^*,\, 2\lambda \in L), \quad V_{\lambda+L} \quad (\lambda \in L^*,\, 2\lambda \notin L), \quad V_L^{T_\chi,\pm},$$

*for any irreducible $\hat{L}/K$-module $T_\chi$ with central character $\chi$.*

The correspondence with the notation presented in Section 3.3.2 is as follows. We have $\hat{L} = G$, where $a = U_\alpha$ and $\theta = \sigma|_L$. Then

$$K = \{U_\alpha^{-1} U_{-\alpha} \mid \alpha \in Q\} = N_\sigma.$$

Note that the elements in $K$ are scalar multiples of elements of the form $U_{2\alpha}$ ($\alpha \in Q$). The irreducible modules over $G_\sigma = \hat{L}/K$ are classified by the central characters $\chi$ of the center $Z(G_\sigma)$ (cf. (3.45)).

It has been shown that $V_L$ and the orbifold $V_L^\sigma$ are both rational [**ABD**; **Y**] and also regular [**ABD**; **DLM**]. Thus we need only be concerned with weak modules over lattice vertex algebras.

Another tool we will need are the *intertwining operators*. To define them, we add to the list in (3.1) the space

$$V\{z\} = \left\{ \sum_{n \in \mathbb{Q}} v_{(n)} z^{-n-1} \middle| v_{(n)} \in V \right\} \tag{3.59}$$

of $V$-valued formal series involving rational powers of $z$, where $V$ is a vector space.

**Definition 3.3.16** *Let $V$ be a vertex operator algebra and let $M_1, M_2$ and $M_3$ be three $V$-modules (not necessarily distinct, and possibly equal to $V$). An* intertwining operator *of type* $\begin{pmatrix} M_3 \\ M_1 \;\; M_2 \end{pmatrix}$ *is a linear map* $\mathcal{Y} : M_1 \otimes M_2 \longrightarrow M_3\{z\}$, *or equivalently,*

$$\begin{aligned} \mathcal{Y} : M_1 &\longrightarrow \operatorname{Hom}(M_2, M_3)\{z\} \\ v &\mapsto \mathcal{Y}(v, z) = \sum_{n \in \mathbb{Q}} v_{(n)} z^{-n-1}, \quad v_{(n)} \in \operatorname{Hom}(M_2, M_3) \end{aligned}$$

*such that for $w \in M_1$ and $u \in M_2$,*

*i)* $w_{(n)} u = 0$ *for* $n \gg 0$,

*ii) the $L_{-1}$-derivative property holds (cf.* (3.12) *and Definition 3.1.5),*

*iii) Borcherds identity* (3.1.13) *holds for $a \in V, b \in M_1$ and $c \in M_2$ with $k \in \mathbb{Q}$ and $m, n \in \mathbb{Z}$:*

$$\sum_{j=0}^{\infty} \binom{m}{j} \left(a_{(n+j)} b\right)_{(m+k-j)} c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left(a_{(m+n-j)} b_{(k+j)} c - (-1)^n b_{(n+k-j)} a_{(m+j)} c\right).$$

Note that each of the terms in Borcherds identity make sense. For instance, the terms of the left hand side have the form $\left(a_{(l)} b\right)_{(s)} c$ with $l \in \mathbb{Z}$ and $s \in \mathbb{Q}$. Since $a \in V$ and $b \in M_1$, we have $a_{(l)} b \in M_1$ where $l \in \mathbb{Z}$, since $M_1$ is a $V$-module. Then $\left(a_{(l)} b\right)_{(s)} c \in M_3$ where $s$ can be rational.

**Example 3.3.17** *The map $Y(\cdot, z)$ acting on $v$ is an intertwining operator of type $\begin{pmatrix} & V & \\ V & & V \end{pmatrix}$, and $Y(\cdot, z)$ acting on any $V$-module $W$ is an intertwining operator of type $\begin{pmatrix} & W & \\ V & & W \end{pmatrix}$. These intertwining operators also satisfy the* normalization condition $Y(|0\rangle, z) = 1$.

The intertwining operators of type $\begin{pmatrix} & M_3 & \\ M_1 & & M_2 \end{pmatrix}$ form a vector space denoted $\mathcal{V}^{M_3}_{M_1 \; M_2}$.

**Definition 3.3.18** *The* fusion rule *associated with an algebra $V$ and its modules is the number*

$$N^{M_3}_{M_1 \; M_2} = \dim \mathcal{V}^{M_3}_{M_1 \; M_2} \quad (\leq \infty). \tag{3.60}$$

**Example 3.3.19** *When $V$ and the $V$-module $W$ are nonzero, the corresponding fusion rules $N^V_{VV}$ and $N^W_{VW}$ are positive.*

The fusion rules for $V_L^+$ were calculated in [**A1**; **ADL**] to be either zero or one. In order to present their theorem, we first introduce some additional notation. Let $c(\cdot,\cdot)$ be the commutator map of $\hat{L}^*$ defined by $c(\lambda,\mu) = (-1)^{(\lambda|\mu)+|\lambda|^2|\mu|^2}$. For $\lambda, \mu \in L^*$, also set

$$\pi_{\lambda,\mu} = (-1)^{(\lambda|\mu)} c(\lambda,\mu) = (-1)^{|\lambda|^2|\mu|^2}. \tag{3.61}$$

Notice that $\pi_{\lambda,2\mu} = 1$ for $\mu \in 2L^* \cap L$. Also when the lattice $\mathbb{Z}\lambda + L$ is integral, the 2-cocyle $\varepsilon$ can be defined on it using only $\pm 1$. Next for a central character $\chi$ of $\hat{L}/K$ and $\lambda \in L^*$ such that $2\lambda \in L$ set

$$c_\chi(\lambda) = (-1)^{(\lambda|2\lambda)} \varepsilon(\lambda, 2\lambda) \chi(e_{2\lambda}). \tag{3.62}$$

For any $\lambda \in L^*$ and central character $\chi$ of $\hat{L}/K$, let $\chi^{(\lambda)}$ be the central character defined by

$$\chi^{(\lambda)}(U_\alpha) = (-1)^{(\alpha|\lambda)} \chi(U_\alpha) \tag{3.63}$$

and set $T_{\chi^{(\lambda)}} = T_\chi^{(\lambda)}$. The following theorem will be needed later in order to construct the orbifold modules.

**Theorem 3.3.20** ([**ADL**], Thm 5.1) *Let $L$ be a positive definite even lattice and let $\lambda, \mu \in L^* \cap \frac{1}{2}L$ with $\pi_{\lambda,2\mu} = 1$. Then the fusion rule of type $\begin{pmatrix} & M_3 & \\ V^\epsilon_{\lambda+L} & & M_2 \end{pmatrix}$, where $M_2$ and $M_3$ are irreducible $V_L^+$-modules and $\epsilon \in \{\pm\}$, is equal to 1 if and only if the pair $(M_2, M_3)$ is one of the* *following:*

$$(V_{\mu+L}^{\epsilon_1}, V_{\lambda+\mu+L}^{\epsilon_2}), \quad \textit{where } \epsilon_2 = \epsilon_1\epsilon, \ \ \epsilon_1 \in \{\pm\}, \tag{3.64}$$

$$(V_L^{T_\chi, \epsilon_1}, V_L^{T_\chi^{(\lambda)}, \epsilon_2}), \quad \textit{where } \epsilon_2 = c_\chi(\lambda)\epsilon_1\epsilon, \ \ \epsilon_1 \in \{\pm\}. \tag{3.65}$$

*Furthermore, the fusion rules are zero otherwise.*

# Chapter 4

# General Results

Throughout this chapter, we assume $Q$ is a positive definite even lattice and that $\sigma$ is an automorphism of $Q$ of order two. We provide an explicit description of the orbifold vertex algebra $V_Q^\sigma$ and classify its irreducible representations.

## 4.1 The sublattice $\bar{Q}$

Fix the following notation:

$$\pi_\pm = \frac{1}{2}(1 \pm \sigma), \qquad \alpha_\pm = \pi_\pm(\alpha)\,, \tag{4.1}$$

$$\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q = \mathfrak{h}_+ \oplus \mathfrak{h}_-, \qquad \text{where} \quad \mathfrak{h}_\pm = \pi_\pm(\mathfrak{h})\,, \tag{4.2}$$

$$L_\pm = \mathfrak{h}_\pm \cap Q, \qquad L = L_+ \oplus L_- \subseteq Q\,, \tag{4.3}$$

$$\tilde{L}_\pm = \pi_\pm(Q), \qquad \tilde{L} = \tilde{L}_+ \oplus \tilde{L}_- \supseteq Q\,. \tag{4.4}$$

Note that $2\alpha_\pm \in L_\pm$ and $|\alpha_\pm|^2 = \frac{1}{4}|2\alpha_\pm|^2 \in \frac{1}{2}\mathbb{Z}$. In particular, the order of all elements in $\tilde{L}_\pm/L_\pm$ is either 1 or 2. It is clear that the automorphism $\sigma$ acts trivially on the quotient $Q/L$ since $\alpha - \sigma\alpha = 2\alpha_- \in L_- \subseteq L$ implies $\alpha + L = \sigma(\alpha) + L = \sigma(\alpha + L)$.

**Lemma 4.1.1** *For $\alpha \in Q$, the following are equivalent*:

*i)* $\sigma^2(e^\alpha) = e^\alpha$,

*ii)* $|\alpha_\pm|^2 \in \mathbb{Z}$,

*iii)* $(\alpha|\sigma\alpha) \in 2\mathbb{Z}$.

**Proof** Note that $4|\alpha_\pm|^2 = |\alpha \pm \sigma\alpha|^2 = 2|\alpha|^2 \pm 2(\alpha|\sigma\alpha)$, so that

$$|\alpha_\pm|^2 = \frac{1}{2}(\alpha|\sigma\alpha) \mod \mathbb{Z}.$$

This shows the equivalence between $(ii)$ and $(iii)$.

Using (3.37), we find $\sigma^2(e^\alpha) = \eta(\alpha)\eta(\sigma\alpha)e^\alpha$. On the other hand, by (3.19), (3.34) and (3.36), we have

$$\eta(\alpha)\eta(\sigma\alpha) = \varepsilon(\alpha, \sigma\alpha)\varepsilon(\sigma\alpha, \alpha) = (-1)^{(\alpha|\sigma\alpha)}.$$

This shows the equivalence between $(i)$ and $(iii)$.

From now on, we let

$$\bar{Q} = \{\alpha \in Q \,|\, (\alpha|\sigma\alpha) \in 2\mathbb{Z}\}\,. \tag{4.5}$$

**Lemma 4.1.2** *The subset $\bar{Q}$ is a sublattice of $Q$ of index 1 or 2.*

**Proof** It is clear that $\bar{Q}$ forms a sublattice. For any $\alpha, \beta \in Q$, we have

$$\begin{aligned}(\alpha - \beta|\sigma\alpha - \sigma\beta) &= (\alpha|\sigma\alpha) + (\beta|\sigma\beta) - (\alpha|\sigma\beta) - (\beta|\sigma\alpha) \\ &= (\alpha|\sigma\alpha) + (\beta|\sigma\beta) \mod 2\mathbb{Z}\,,\end{aligned}$$

since

$$(\alpha|\sigma\beta) = (\sigma\alpha|\sigma^2\beta) = (\beta|\sigma\alpha)\,.$$

Now if $\alpha, \beta \in \bar{Q}$ or $\alpha, \beta \notin \bar{Q}$, then $\alpha - \beta \in \bar{Q}$.

By definition, we have $(V_Q)^{\sigma^2} = V_{\bar{Q}}$. Then

$$V_Q^\sigma = \left((V_Q)^{\sigma^2}\right)^\sigma = V_{\bar{Q}}^\sigma\,. \tag{4.6}$$

Therefore, we may assume that $|\sigma| = 2$ on $V_Q$ and only work with the sublattice $\bar{Q}$. For simplicity, we use $Q$ instead of $\bar{Q}$ for the rest of this chapter.

## 4.2 Description of the orbifold

From [**FHL**; **LL**], we have that the tensor product $V_L \simeq V_{L_+} \otimes V_{L_-}$ forms a subalgebra of $V_Q$.

In order to obtain a precise description of $V_Q^\sigma$, we break $V_Q$ as modules over $V_L^\sigma$ in two steps. The first step is to break $V_Q$ as modules over $V_L$. This is done using the cosets

$$Q/L = \{\gamma_0 + L, \gamma_1 + L, \dots, \gamma_r + L\}, \tag{4.7}$$

where $\gamma_0 = 0$. It follows that

$$V_Q = \bigoplus_{\gamma \in Q/L} V_\gamma,$$

where each $V_\gamma$ is an irreducible $V_L$-module [**D1**]. Set $\gamma = \gamma_i + L$ and $\gamma_\pm = \gamma_{i,\pm} + L_\pm$. Then writing each $\gamma_i$ in the form $\gamma_i = \gamma_{i,+} + \gamma_{i,-}$, we arrive at the following lemma.

**Lemma 4.2.1** *For $\gamma \in L^*/L$, $V_\gamma \simeq V_{\gamma_+} \otimes V_{\gamma_-}$ as $V_L$-modules.*

Thus, for each coset representative $\gamma$ of $Q/L$, we have that

$$V_Q \simeq \bigoplus_{\gamma \in Q/L} V_{\gamma_+} \otimes V_{\gamma_-} \tag{4.8}$$

as $V_L$-modules. Since $\sigma$ acts on the module $V_{\gamma_-}$, it breaks into $\pm 1$-eigenspaces for $\sigma$ and each eigenspace is an irreducible $V_{L_-}^+$-module [**AD**].

The second step is to restrict each module $V_{\gamma_-}$ to an eigenspace for $\sigma$. We then obtain the following description of $V_Q^\sigma$.

**Proposition 4.2.2** *The orbifold can be realized as the direct sum of $V_L^\sigma$-modules*:

$$V_Q^\sigma \simeq \bigoplus_{\gamma \in Q/L} V_{\gamma_+} \otimes V_{\gamma_-}^{\eta(\gamma)}, \tag{4.9}$$

where $\eta$ is given by (3.37).

**Proof** Using (3.37) and Lemma 4.1.1, we have that $e^\gamma + \eta(\gamma)e^{\sigma\gamma} \in V_Q^\sigma$ for each $\gamma \in Q$. Now

$$e^\gamma + \eta(\gamma)e^{\sigma\gamma} = e^{\gamma_+} \otimes (e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}),$$

where $e^{\gamma_+} \in V_{\gamma_+ + L_+}$ and $e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-} \in V_{\gamma_- + L_-}^{\eta(\gamma)}$.

## 4.3 Restricting the Orbifold $V_Q^\sigma$ to $V_L^\sigma$

From the study of tensor products in [**FHL**; **LL**] and the structure of $V_Q^\sigma$ given in (4.2.2), irreducible $V_Q^\sigma$-modules are sums of tensor products of irreducible modules over the factors $V_{L_+}$

and $V_{L_-}^+$. By Dong's Theorem [**D1**], the irreducible $V_{L_+}$-modules are given as $V_{\lambda+L_+}$, where $\lambda \in L_+^*$. The irreducible $V_{L_-}^+$-modules are classified in [**AD**], and come in three types:

$$V_\mu, \quad \text{where} \quad \mu \in L_-^*/L_- \quad \text{and} \quad 2\mu \neq L_-, \tag{4.10}$$

$$V_\mu^\pm, \quad \text{where} \quad \mu \in L_-^*/L_- \quad \text{and} \quad 2\mu = L_-, \tag{4.11}$$

$$V_{L_-}^{T_\chi,\pm}, \tag{4.12}$$

where $T_\chi$ is an irreducible $G_\sigma$-module with central character $\chi$. Futhermore, each of these $V_{L_-}^+$-modules can be obtained by restricting a twisted $V_{L_-}$-module.

**Proposition 4.3.1** *Every $V_Q^\sigma$-module is a direct sum of irreducible $V_L^\sigma$-modules. In particular, $V_Q^\sigma$ has this form.*

**Proof** Consider $V_L^\sigma = V_{L_+} \otimes V_{L_-}^+ \subseteq V_Q^\sigma$. Then $V_L^\sigma$ forms a vertex subalgebra of $V_Q^\sigma$. It is shown in Theorem 3.16 of [**DLM3**] that the vertex algebra $V_{L_+}$ is regular, since $L_+$ is positive definite. It is also shown in [**ABD**; **DJL**] that the vertex algebra $V_{L_-}^+$ is regular. Since the tensor product of regular vertex algebras is again regular (Proposition 3.3 in [**DLM3**]), we have that $V_L^\sigma$ is also regular.

Since irreducible modules of $V_L^\sigma$ are tensor products of irreducible modules over the factors $V_{L_+}$ and $V_{L_-}^+$, the $V_L^\sigma$-modules

1. $V_\lambda \otimes V_\mu$, where $2\mu \neq 0$,
2. $V_\lambda \otimes V_\mu^\pm$, where $2\mu = 0$,
3. $V_\lambda \otimes V_{L_-}^{T_\chi,\pm}$,

are the irreducible ones. We refer to the orbifold modules obtained from untwisted $V_L$-modules as *orbifold modules of untwisted type* and orbifold modules obtained from twisted $V_L$-modules as *orbifold modules of twisted type.*

In order to determine the irreducible $V_Q^\sigma$-modules, we first start with a $V_L^\sigma$-module. Any $V_Q^\sigma$-module is automatically a $V_L^\sigma$-module by restriction. It follows from [**D1**; **DN**; **AD**] that the $V_L^\sigma$-modules can all be obtained by restricting twisted $V_L$-modules. We will show there exists a lifting of each twisted $V_L$-module to a twised $V_Q$-module, and describe how the $V_Q^\sigma$-module is obtained using the intertwining operators for $V_L^\sigma$. The twisted $V_Q$-module will be determined using the orbits of each field $Y(e^\gamma, z)$, for $\gamma \in Q$, on the set of irreducible $V_L^\sigma$-modules.

## 4.4 Irreducible Modules over $V_Q^\sigma$

In this section we present the main result, that the irreducible $V_Q^\sigma$-modules are submodules of twisted $V_Q$-modules. We also provide an explicit list of the irreducible $V_Q^\sigma$-modules.

**Lemma 4.4.1** *Suppose $\lambda + \mu \in Q^*$. Then for the $V_L$-module $M(\lambda,\mu) = V_{\lambda+L_+} \otimes V_{\mu+L_-}$, there is a $V_Q$-module given by*

$$M_Q(\lambda,\mu) = \bigoplus_{\gamma \in Q/L} M(\lambda+\gamma_+, \mu+\gamma_-). \tag{4.13}$$

**Proof** Consider the untwisted $V_Q$-module $V_{\lambda+\mu+Q}$. Since untwisted modules over $V_L$ have the form $V_{\lambda+L_+} \otimes V_{\mu+L_-}$, we have that

$$\begin{aligned} V_{\lambda+\mu+Q} &= \bigoplus_{\gamma\in Q/L} V_{\gamma+\lambda+\mu} \\ &= \bigoplus_{\gamma\in Q/L} V_{\gamma_++\lambda} \otimes V_{\gamma_-+\mu} \\ &= \bigoplus_{\gamma\in Q/L} M\left(\lambda+\gamma_+, \mu+\gamma_-\right) \end{aligned}$$

as a direct sum of irreducible $V_L$-modules.

From the proof of Lemma 4.4.1, each $V_Q$-module is obtained from the set of $V_L$-modules whose arguments are closed under addition modulo $L$.

**Theorem 4.4.2** *Let $Q$ be an even positive definite lattice for which $(\alpha|\sigma\alpha)$ is even for all $\alpha \in Q$, $V_Q$ the corresponding lattice vertex algebra, and let $\sigma$ be an automorphism of $Q$ of order two. Then each irreducible $V_Q^\sigma$-module is isomorphic to one of the following*:

$$\bigoplus_{\gamma\in Q/L} V_{\gamma_++\lambda} \otimes V_{\gamma_-+\mu}, \quad \text{where} \quad \lambda \in L_+^*/L_+,\ \mu \in L_-^*/L_- \quad \text{and} \quad 2\mu \neq 0, \tag{4.14}$$

$$\bigoplus_{\gamma\in Q/L} V_{\gamma_++\lambda} \otimes V_{\gamma_-+\mu}^{\epsilon\eta(\gamma)}, \quad \text{where} \quad \lambda \in L_+^*/L_+,\ \mu \in L_-^*/L_-,\ 2\mu = 0, \text{ and } \epsilon \in \{\pm\}, \tag{4.15}$$

$$\bigoplus_{\gamma\in Q/L} V_{\gamma_++\lambda} \otimes V_{L_-}^{T_\chi^{(\gamma_-)},\epsilon_\gamma}, \quad \text{where} \quad \lambda \in L_+^*/L_+,\ \epsilon_\gamma = \epsilon\eta(\gamma)c_\chi(\gamma_-), \quad \epsilon \in \{\pm\}. \tag{4.16}$$

**Proof** Let $W$ be an irreducible $V_Q^\sigma$-module. Then $W$ is a $V_L^\sigma$-module by restriction. By Proposition 4.3.1, we have that $W$ is a direct sum of irreducible $V_L^\sigma$-modules. Suppose $A \subseteq W$ is an irreducible $V_L^\sigma$-module and define $A^{(\gamma)}$ from $A$ as follows:

$$\begin{array}{cccc} A & V_\lambda \otimes V_\mu & V_\lambda \otimes V_\mu^\pm & V_\lambda \otimes V_{L_-}^{T_\chi,\pm} \\ A^{(\gamma)} & V_{\lambda+\gamma_+} \otimes V_{\mu+\gamma_-} & V_{\lambda+\gamma_+} \otimes V_{\mu+\gamma_-}^{\pm\eta(\gamma)} & V_{\lambda+\gamma_+} \otimes V_{L_-}^{T_\chi^{(\gamma_-)},\pm\epsilon} \end{array}$$

where $\epsilon = c_\chi(\gamma)\eta(\gamma)$. We work out separatly the untwisted and twisted types.

Let $A$ be of untwisted type so that $A$ is one of the modules $V_\lambda \otimes V_\mu$, for $2\mu \neq 0$, or $V_\lambda \otimes V_\mu^\pm$, for $2\mu = 0$. Let $B \subseteq W$ be another irreducible $V_L^\sigma$-module that is possibly of twisted type. By Proposition 4.2.2, we have that

$$V_Q^\sigma \simeq \bigoplus_{\gamma \in Q/L} V_{\gamma_+} \otimes V_{\gamma_-}^{\eta(\gamma)}, \tag{4.17}$$

where each summand is also an irreducible $V_L^\sigma$-module and is generated by the vector

$$v_\gamma = e^\gamma + \eta(\gamma)e^{\sigma\gamma} = e^{\gamma_+} \otimes (e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}).$$

By restricting the field $Y(v_\gamma, z)$ to $A$ and then projecting onto $B$, we obtain an intertwining operator of $V_L^\sigma$-modules of type $\begin{pmatrix} & B & \\ V_{\gamma+L}^{\eta(\gamma)} & & A \end{pmatrix}$. From the study of intertwining operators in [**ADL**], we have that the intertwining operator $Y(v_\gamma, z)$ can be written as the tensor product

$$Y(v_\gamma, z) = Y(e^{\gamma_+}, z) \otimes Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z),$$

where $Y(e^{\gamma_+}, z)$ is an intertwining operator of type $\begin{pmatrix} & V_{\lambda'+L_+} & \\ V_{\gamma_++L_+} & & V_{\lambda+L_+} \end{pmatrix}$ and $Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)$ is an intertwining operator of type $\begin{pmatrix} & V_{\mu'+L_-} & \\ V_{\gamma_-+L_-}^{\eta(\gamma)} & & V_{\mu+L_-} \end{pmatrix}$ or of type $\begin{pmatrix} & V_{\mu'+L_-}^{\pm\eta(\gamma)} & \\ V_{\gamma_-+L_-}^{\eta(\gamma)} & & V_{\mu+L_-}^\pm \end{pmatrix}$. From the study of intertwining operators in [**DL1**], the fusion rules for $Y(e^{\gamma_+}, z)$ are zero unless $\lambda' = \lambda + \gamma_+$. Since $\gamma_- \in L_-^* \cap \frac{1}{2}L_-$ and $|\lambda|^2 \in \mathbb{Z}$, we have that $\pi_{\gamma_-,2\lambda} = 1$ (cf. (3.61)). Hence the fusion rules for $Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)$ are zero unless $\mu' = \mu + \gamma_-$, by Theorem 3.3.20. Therefore, for $\gamma + L \in Q/L$, we have that

$$v_\gamma : A \longrightarrow A^{(\gamma)}$$

so that $B = A^{(\gamma)}$. Therefore $A \subseteq W$ implies that $\bigoplus_{\gamma \in Q/L} A^{(\gamma)} \subseteq W$. Since $W$ is irreducible, we must have that

$$W = \bigoplus_{\gamma \in Q/L} A^{(\gamma)}.$$

Now let $A = V_{\lambda+L_+} \otimes V_{L_-}^{T_\chi,\pm}$ and $B \subseteq W$ be another irreducible $V_L^\sigma$-module that is possibly of untwisted type. As with the untwisted type, the field $Y(v_\gamma, z)$ gives rise to an intertwining operator of $V_L^\sigma$-modules of type $\begin{pmatrix} & B & \\ V_{\gamma+L}^{\eta(\gamma)} & & A \end{pmatrix}$ and can be written as the tensor product

$$Y(v_\gamma, z) = Y(e^{\gamma_+}, z) \otimes Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z),$$

where $Y(e^{\gamma_+}, z)$ is an intertwining operator of type $\begin{pmatrix} & V_{\lambda'+L_+} & \\ V_{\gamma_++L_+} & & V_{\lambda+L_+} \end{pmatrix}$ and $Y(e^{\gamma_-}+\eta(\gamma)e^{-\gamma_-}, z)$ is an intertwining operator of type $\begin{pmatrix} & V_{L_-}^{T_{\chi'},\pm\epsilon} & \\ V_{\gamma_-+L_-}^{\eta(\gamma)} & & V_{L_-}^{T_\chi,\pm} \end{pmatrix}$, where $\epsilon = c_\chi(\gamma)\eta(\gamma)$. As with the untwisted type, the fusion rules for $Y(e^{\gamma_+}, z)$ are zero unless $\lambda' = \lambda + \gamma_+$. By Theorem 3.3.20, the action of $Y(e^{\gamma_-}, z)$ on $V_{L_-}^{T_\chi}$ is determined by computing $c_\chi(\gamma_-)$ (cf. (3.62)) and is zero unless $\chi' = \chi^{(\gamma_-)}$. Since the lattice $\mathbb{Z}\gamma_- + L_-$ is integral (cf. Lemma 4.1.1), the map $\varepsilon$ can be extended to this lattice with values $\pm1$. Therefore $\varepsilon(\gamma_-, 2\gamma_-) = \varepsilon(\gamma_-, \gamma_-)^2 = 1$ and (3.62) becomes $c_\chi(\gamma_-) = \chi(U_{2\gamma_-})$. Hence the eigenspace of each summand in the $V_Q^\sigma$-module may change depending on the signs of each $U_{2\gamma_-}$. Therefore, as with the untwisted case, we have that $B = A^{(\gamma)}$ so that

$$W = \bigoplus_{\gamma\in Q/L} A^{(\gamma)}.$$

**Corollary 4.4.3** *Let $Q$ be an even positive definite lattice for which $(\alpha|\sigma\alpha)$ is even for all $\alpha \in Q$, $V_Q$ the corresponding lattice vertex algebra and let $\sigma$ be an automorphism of $Q$ of order two. Then each irreducible $V_Q^\sigma$-module is a submodule of a twisted $V_Q$-module.*

**Proof** By Theorem 4.4.2, irreducible $V_Q^\sigma$-modules of untwisted type are given by (4.14) or (4.15). Let $\gamma \in Q$. Then since the nonzero fusion rules for $Y(e^{\gamma_+}, z)$ and $Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)$ are equal to 1 and the intertwining operators in [**ADL**] are given by the usual formula (3.23) up to a scalar multiple, an action of each $e^\gamma$ for $\gamma \in Q$ can be determined. Using that

$$(\gamma|\lambda+\mu) - (\sigma\gamma|\lambda+\mu) = (2\gamma_-|\lambda+\mu) = (2\gamma_-|\mu) \in \mathbb{Z},$$

we have that for some $m \in \mathbb{Z}$,

$$Y(e^\gamma + \eta(\gamma)e^{\sigma\gamma}, z)e^{\lambda+\mu} = (e^{\gamma+\lambda+\mu} + \eta(\gamma)e^{\sigma\gamma+\lambda+\mu})z^{(\gamma|\lambda+\mu)}(E(\gamma, z) + z^m E(\sigma\gamma, z)),$$

where $E(\alpha, z) = \exp\Big(\sum_{n<0} \alpha_n \frac{z^{-n}}{-n}\Big) \exp\Big(\sum_{n>0} \alpha_n \frac{z^{-n}}{-n}\Big)$ and contains only integral powers of $z$. Hence we must have that $(\lambda + \mu|\gamma) \in \mathbb{Z}$ for any representative $\gamma + L$ in $Q/L$. Then by Lemma

4.4.1, the space $M_Q(\lambda, \mu)$ given in (4.4.1) forms a $V_Q$-module and contains the irreducible $V_Q^\sigma$-module $W$. Using the fusion rules and intertwining operators in [**ADL**], $W$ is a submodule when restricted to $V_Q^\sigma$.

By Theorem 4.4.2, irreducible $V_Q^\sigma$-modules of twisted type are given by (4.16). Let $\gamma \in Q$. Then the nonzero fusion rules for $Y(e^{\gamma_+}, z)$ and $Y(e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-}, z)$ are equal to 1 and the intertwining operators in [**ADL**] are given by the usual formula (3.23) up to a scalar multiple. Since these scalars can be absorbed in $U_\gamma$, we will have (3.23) without loss of generality. Therefore an action of each $e^\gamma$ for $\gamma \in Q$ can be determined. Writing $e^\gamma = e^{\gamma_+} \otimes (e^{\gamma_-} + \eta(\gamma)e^{-\gamma_-})$, it is clear from the intertwining operators in [**ADL**] that

$$Y(e^\gamma, z) : V_\lambda \otimes V_{L_-}^{T_\chi, \pm} \longrightarrow V_{\gamma_+ + \lambda} \otimes V_{L_-}^{T_\chi^{(\gamma_-)}, \pm\epsilon},$$

where $\epsilon = c_\chi(\gamma)\eta(\gamma)$. Hence the twisted $V_Q$-module is given by

$$T(\lambda, \chi) = \bigoplus_{\gamma \in Q/L} V_{\gamma_+ + \lambda} \otimes V_{L_-}^{T_\chi^{(\gamma_-)}, \pm\epsilon}$$

and contains the irreducible $V_Q^\sigma$-module $W$. Using the fusion rules and intertwining operators in [**ADL**], $W$ is a submodule when restricted to $V_Q^\sigma$.

# Chapter 5

# Examples

In this chapter, we work out examples of the lattice $Q$ being a root lattice of type ADE, corresponding to the simply-laced simple Lie algebras discussed in Chapter 2, as well as a one-dimensional lattice. We use explicitly the classification from [**D1**; **DN**; **AD**] described in chapter 4 and the construction in [**BK**] described in Section 3.3.2 to construct the twisted $V_Q^\sigma$-modules. In each case, a correspondence between the two constructions is shown.

To use the classification from [**D1**; **DN**; **AD**], we first calculate $\bar{Q}$ and $L$. Then the twisted $V_L$-modules are found. When necessary, the intertwiners from [**ADL**] are used to construct the $V_Q^\sigma$-modules. The $V_Q^\sigma$-modules from the construction in [**BK**] are calculated using $(\bar{Q})^*/\bar{Q}$ and its $\sigma$-invariant elements.

## 5.1 One Dimensional Case

Consider the one-dimensional positive definite even lattice $Q = \mathbb{Z}\alpha$, where $(\alpha|\alpha) = 2k$ and $k > 0$. Then the only nontrivial automorphism of $Q$ preserving the form $(\cdot|\cdot)$ is $\sigma = -1$. This automorphism can be extended to $V_Q$ by letting

$$\begin{aligned}\sigma(h_m) &= -h_m, \\ \sigma(e^{n\alpha}) &= \eta(n\alpha)e^{-n\alpha},\end{aligned}$$

where $h \in \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q = \mathbb{C}\alpha$ and $m, n \in \mathbb{Z}$. Since $(n\alpha|\sigma(n\alpha)) = -2n^2k \in 2\mathbb{Z}$ for all $n \in \mathbb{Z}$, we have that $Q = \bar{Q}$. Here $L_+ = \{0\}$ so that

$$L_- = L = Q = \bar{Q}.$$

Hence the quotient $Q/L$ is trivial. Thus, by Proposition 4.2.2, we have that

$$V_Q^\sigma \simeq V_{L_-}^+. \tag{5.1}$$

Using the classification in [**DN**], the irreducible $V_Q^\sigma$-modules are parameterized by the set $L_-^*/L_- = Q^*/Q$. The dual lattice of $Q$ is given by $Q^* = \mathbb{Z}\dfrac{\alpha}{2k}$ so that the orbifold modules are parametrized by the set

$$Q^*/Q \quad = \quad \left\{Q, \frac{\alpha}{2k} + Q, \frac{\alpha}{k} + Q, \ldots, \frac{\alpha}{2} + Q, \ldots, \frac{(2k-1)\alpha}{2k} + Q\right\}.$$

The automorphism $\sigma$ acts on $V_Q$ and $V_{\frac{\alpha}{2}+Q}$ since $Q^* \cap \frac{1}{2}Q = \{0, \frac{\alpha}{2}\}$ and identifies the other modules in pairs since

$$\sigma\left(\frac{i}{2k}\alpha\right) = \frac{2k-i}{2k}\alpha \mod Q.$$

Thus there are $k - 1 + 2(2) = k + 3$ distinct irreducible $V_Q^\sigma$-modules of untwisted type

$$V_{\mathbb{Z}\alpha}^\pm, \quad V_{\frac{\alpha}{2}+\mathbb{Z}\alpha}^\pm, \quad V_{\frac{1}{2k}\alpha+\mathbb{Z}\alpha}, \quad \ldots, \quad V_{\frac{k-1}{2k}\alpha+\mathbb{Z}\alpha}, \tag{5.2}$$

and four distinct irreducible $V_Q^\sigma$-modules of twisted type

$$V_{\mathbb{Z}\alpha}^{T_i,\pm}, \quad i = 1, 2. \tag{5.3}$$

**Remark 5.1.1** *The $k + 7$ modules given above are as in* Theorem 5.13 *in* [**DN**].

We now construct the orbifold modules of twisted type using Section 3.3.2 (cf. Example 3.3.13). The 2-cocycle $\varepsilon$ satisfies

$$\varepsilon(\alpha, \alpha) = (-1)^{2k\binom{2k+1}{2}} = (-1)^k.$$

Thus by bimultiplicativity,

$$\begin{aligned} \varepsilon(m\alpha, n\alpha) &= (-1)^{mnk}, \\ \varepsilon(m\alpha, n\alpha)\varepsilon(n\alpha, m\alpha) &= 1. \end{aligned}$$

Since $\varepsilon(\sigma(\alpha), \sigma(\beta)) = \varepsilon(-\alpha, -\beta) = \varepsilon(\alpha, \beta)$, we can set $\eta = 1$ so that $\sigma(e^\alpha) = e^{\sigma(\alpha)} = e^{-\alpha}$. We have in this case that $\mathfrak{h}_0 = 0$ so that $G = \mathbb{C}^\times \times Q$ and $\mathfrak{h}_0^\perp = \mathfrak{h} = \mathbb{C}\alpha$. Thus

$$\begin{aligned} b_\alpha &= \frac{1}{2}\left(|\alpha_0|^2 - |\alpha|^2\right) = -k, \\ b_{m\alpha} &= -m^2 k, \quad m \in \mathbb{Z}, \\ B_{m\alpha,n\alpha} &= 4^{-(m\alpha|n\alpha)} = 16^{-mnk}. \end{aligned}$$

For elements $U_{m\alpha}$ and $U_{n\alpha}$ we have

$$\begin{aligned} U_{m\alpha}U_{n\alpha} &= \varepsilon(m\alpha, n\alpha)B^{-1}_{m\alpha,n\alpha}U_{(m+n)\alpha} \\ &= (-1)^{mnk}16^{mnk}U_{(m+n)\alpha} \\ &= (-16)^{mnk}U_{(m+n)\alpha}. \end{aligned}$$

In particular, $G$ is abelian ($C_{m\alpha,n\alpha} = 1$). Thus the irreducible representations of $G$ will be one dimensional. The elements of $G_\sigma$ must satisfy

$$U_{\sigma(\alpha)} = \eta(\alpha)U_\alpha e^{2\pi \mathrm{i} b_\alpha} = U_\alpha e^{-2\pi \mathrm{i} k} = U_\alpha$$

so that $U_{-\alpha} = U_{\sigma(\alpha)} = U_\alpha$. Applying $U_\alpha$ to both sides yields $U_\alpha^2 = U_\alpha U_{-\alpha} = (-16)^{-k}$ and therefore $U_\alpha$ has two possible actions: $U_\alpha = \pm(4\mathrm{i})^{-k}$. Other elements of $G$ are determined using induction on $m$:

$$\begin{aligned} U_{m\alpha} &= U_{\alpha+(m-1)\alpha} \\ &= (-16)^{-(m-1)k}U_\alpha U_{(m-1)\alpha} \\ &= (-16)^{-(m-1)k}\left((\pm 1)^{m-1}(-16)^{-\frac{(m-1)^2}{2}k}\right)U_\alpha \\ &= (\pm 1)^{m-1}(-16)^{-\frac{1}{2}m^2k+\frac{1}{2}k}U_\alpha \\ &= (\pm 1)^{m-1}(-16)^{-\frac{1}{2}m^2k+\frac{1}{2}k}\left((\pm 1)(-16)^{-\frac{1}{2}k}\right) \\ &= (\pm 1)^{m}(-16)^{-\frac{1}{2}m^2k}. \end{aligned}$$

Let $T_+$ be the $G_\sigma$-module corresponding to $U_\alpha = (4\mathrm{i})^{-k}$ and $T_-$ be the $G_\sigma$-module corresponding to $U_\alpha = -(4\mathrm{i})^{-k}$. We therefore have the two $\sigma$-twisted $V_Q$-modules $M(\sigma) \otimes T_\pm$. The automorphism $\sigma$ acts on these twisted modules via $\sigma\left(\alpha_{-n_1}\cdots\alpha_{-n_k} \otimes t\right) = (-1)^k \alpha_{-n_1}\cdots\alpha_{-n_k} \otimes t$, where $t \in T_\pm$ and $n_i \in \frac{1}{2} + \mathbb{Z}$. Each space then splits into two eigenspaces corresponding to the eigenvalues $\pm 1$. Therefore there are four total irreducible $V_Q^\sigma$-modules of twisted type:

$$M(\sigma)^s \otimes T_\pm, \qquad s \in \{\pm\},$$

which coincide with the modules from (5.3).

## 5.2 The Root Lattice $A_2$

Consider the $A_2$ simple roots $\{\alpha_1, \alpha_2\}$. The nondegenerate symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ is given by $(\alpha_1|\alpha_2) = -1$ and $(\alpha_i|\alpha_i) = 2$ for $i = 1, 2$. The associated even lattice is $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$. Consider the Dynkin diagram automorphism $\sigma : \alpha_1 \longleftrightarrow \alpha_2$. Set $\alpha = \alpha_1 + \alpha_2$ and $\beta = \alpha_1 - \alpha_2$. Then $\alpha$ and $\beta$ are eigenvectors for $\sigma$ with eigenvalues 1 and $-1$, respectively. Also, $(\alpha|\alpha) = 2, (\beta|\beta) = 6$ and $(\alpha|\beta) = 0$.

In order to determine $\bar{Q}$, we find conditions for which $\gamma = m_1\alpha_1 + m_2\alpha_2 \in Q$ satisfies $(\gamma|\sigma\gamma) \in 2\mathbb{Z}$. Since

$$\begin{aligned}(\gamma|\sigma\gamma) &= -m_1^2 - m_2^2 + 4m_1m_2 \\ &= m_1 + m_2 \mod 2\mathbb{Z},\end{aligned}$$

we have that $\bar{Q} = \{m_1\alpha_1 + m_2\alpha_2 \,|\, m_1 = m_2 \mod 2\mathbb{Z}\} = \mathbb{Z}\alpha + \mathbb{Z}\beta$. Therefore

$$L_+ = \mathbb{Z}\alpha, \tag{5.4}$$

$$L_- = \mathbb{Z}\beta, \tag{5.5}$$

$$\bar{Q} = L. \tag{5.6}$$

Note that $L$ is written as an orthogonal decomposition and the quotient $\bar{Q}/L$ is trivial. Thus by Proposition 4.2.2, we have that

$$V_Q^\sigma \simeq V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^+. \tag{5.7}$$

We can now restrict our attention to the sublattice $L$. Using the classification in [**DN**], the irreducible $V_L^\sigma$-modules are parameterized by the sets $L_\pm^*/L_\pm$. Since $L_+^*/L_+ = \mathbb{Z}\frac{\alpha}{2}/\mathbb{Z}\alpha = \{\mathbb{Z}\alpha, \frac{\alpha}{2} + \mathbb{Z}\alpha\}$, the set of irreducible $V_{L_+}$-modules is given by

$$V_{\mathbb{Z}\alpha}, \quad V_{\frac{\alpha}{2}+\mathbb{Z}\alpha}.$$

Since $(\mathbb{Z}\beta)^* = \mathbb{Z}\frac{\beta}{6}$, we have that

$$(\mathbb{Z}\beta)^*/\mathbb{Z}\beta \;=\; \left\{\mathbb{Z}\beta, \frac{\beta}{6}+\mathbb{Z}\beta, \frac{\beta}{3}+\mathbb{Z}\beta, \frac{\beta}{2}+\mathbb{Z}\beta, \frac{2\beta}{3}+\mathbb{Z}\beta, \frac{5\beta}{6}+\mathbb{Z}\beta\right\}.$$

The $\sigma$-invariant elements in $(\mathbb{Z}\beta)^*/\mathbb{Z}\beta$ are $\mathbb{Z}\beta$ and $\frac{\beta}{2}+\mathbb{Z}\beta$. The other elements are identified by $\sigma$ in pairs. Thus the set of distinct irreducible $V_{L_-}^+$-modules is parameterized by the set

$$\left\{\mathbb{Z}\beta, \frac{\beta}{6}+\mathbb{Z}\beta, \frac{\beta}{3}+\mathbb{Z}\beta, \frac{\beta}{2}+\mathbb{Z}\beta\right\}.$$

The corresponding modules of untwisted type are

$$V_{\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\beta}{6}+\mathbb{Z}\beta}, \quad V_{\frac{\beta}{3}+\mathbb{Z}\beta}$$

and the corresponding modules of twisted type are

$$V_{\mathbb{Z}\beta}^{T_i,\pm}, \quad i = 1, 2.$$

By (5.7) and [**FHL**], every irreducible $V_L^\sigma$-module is isomorphic to a tensor product of irreducible $V_{L_+}$ and $V_{L_-}^+$-modules. Thus there are a total of 20 distinct irreducible orbifold modules, given by the following list:

$$\begin{array}{ll}
V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{\pm}, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{\pm}, \\[2ex]
V_{\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}, \\[2ex]
V_{\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{6}+\mathbb{Z}\beta}, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{6}+\mathbb{Z}\beta}, \\[2ex]
V_{\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{3}+\mathbb{Z}\beta}, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\frac{\beta}{3}+\mathbb{Z}\beta}, \\[2ex]
V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{T_i,\pm}, & V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{T_i,\pm}, \quad i = 1, 2.
\end{array}$$

We now construct the orbifold modules of twisted type using Section 3.3.2. The 2-cocycle $\varepsilon$

on $L$ satisfies

$$\begin{aligned}\varepsilon(\alpha,\alpha) &= (-1)^{2\binom{2+1}{2}} = -1,\\ \varepsilon(\beta,\beta) &= (-1)^{6\binom{6+1}{2}} = -1,\\ \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) &= 1.\end{aligned}$$

Thus we can set $\varepsilon(\alpha,\beta) = 1 = \varepsilon(\beta,\alpha)$. It is clear that $\varepsilon$ is $\sigma$-invariant on $L$ so that we may take $\eta$ to be trivial, that is, $\eta(\gamma) = 1$ for all $\gamma \in L$.

The dual lattice to $L$ is given by

$$L^* = \mathbb{Z}\frac{\alpha}{2} \oplus \mathbb{Z}\frac{\beta}{6}. \tag{5.8}$$

Note that this is an orthogonal decomposition. The $\sigma$-invariant elements in $L^*/L$ are $L$, $\frac{\alpha}{2}+L$, $\frac{\beta}{2}+L$ and $\frac{\alpha}{2}+\frac{\beta}{2}+L$. The other elements are identified by $\sigma$ in pairs. The distinct irreducible untwisted and $\sigma$-twisted $V_L$-modules are then parametrized by the set

$$\left\{L, \frac{\alpha}{2}+L, \frac{\beta}{2}+L, \frac{\alpha}{2}+\frac{\beta}{2}+L, \frac{\beta}{6}+L, \frac{\beta}{3}+L, \frac{\beta}{6}+\frac{\alpha}{2}+L, \frac{\beta}{3}+\frac{\alpha}{2}+L\right\}.$$

Thus there are 12 distinct orbifold modules of untwisted type given by

$$V_L^{\pm}, \quad V^{\pm}_{\frac{\alpha}{2}+L}, \quad V^{\pm}_{\frac{\beta}{2}+L}, \quad V^{\pm}_{\frac{\alpha}{2}+\frac{\beta}{2}+L},$$

$$V_{\frac{\beta}{6}+L}, \quad V_{\frac{\beta}{3}+L}, \quad V_{\frac{\beta}{6}+\frac{\alpha}{2}+L}, \quad V_{\frac{\beta}{3}+\frac{\alpha}{2}+L}.$$

We have in this case that $\mathfrak{h}_0 = \mathbb{C}\alpha$, $G = \mathbb{C}^\times \times e^{\mathbb{C}\alpha_{(0)}} \times L$ and $\mathfrak{h}_0^\perp = \mathbb{C}\beta$. Since $G$ is abelian, the irreducible representations of $G$ are one dimensional. To determine the orbifold modules of twisted type, the quantities that will be needed are the following:

$$\begin{aligned} B_{\beta,-\beta} &= 2^{12},\\ b_\beta &= \frac{1}{2}(0-6) = -3.\end{aligned}$$

The elements in $G_\sigma$ must satisfy $U_{\sigma\gamma} = \eta(\gamma)U_\gamma e^{2\pi\mathrm{i}(b_\gamma+\gamma_0)}$. In particular, $U_{-\beta} = U_\beta e^{-6\pi\mathrm{i}} = U_\beta$ so that

$$U_\beta^2 = U_\beta U_{-\beta} = \varepsilon(\beta,-\beta)B^{-1}_{\beta,-\beta} = -2^{-12}.$$

Thus there are two possible actions of $U_\beta$,

$$U_\beta = \pm \frac{1}{64}\mathrm{i},$$

and $U_\alpha$ acts freely on each $V_L$-module. Let $U_\alpha$ act as multiplication by $q$ on the vector space $P = \mathbb{C}[q, q^{-1}]$. To determine the action of $e^{\pi \mathrm{i}\alpha_{(0)}}$, consider its commutation relation with $U_\alpha$:

$$e^{\pi \mathrm{i}\alpha_{(0)}} U_\alpha e^{-\pi \mathrm{i}\alpha_{(0)}} = e^{\pi \mathrm{i}(\alpha|\alpha)} U_\alpha = U_\alpha,$$

i.e., $e^{\pi \mathrm{i}\alpha_{(0)}} q e^{-\pi \mathrm{i}\alpha_{(0)}} = q$. Thus $e^{\pi \mathrm{i}\alpha_{(0)}} q^n = q^n e^{\pi \mathrm{i}\alpha_{(0)}}(1)$. Since $e^{2\pi \mathrm{i}\alpha_{(0)}} = 1$, we must have that $e^{\pi \mathrm{i}\alpha_{(0)}}(1) = \pm 1$ so that $e^{\pi \mathrm{i}\alpha_{(0)}} q^n = \pm q^n$. Thus on the space $P$ we have

$$\begin{aligned} e^{\pi \mathrm{i}\alpha_{(0)}} &= \pm 1, && (5.9)\\ U_\alpha &= q, && (5.10)\\ U_\beta &= \pm \frac{1}{64}\mathrm{i}, && (5.11) \end{aligned}$$

where the signs in (5.9) and (5.11) are independent. The automorphism $\sigma$ acts on each of these modules. To see why, we calculate $\sigma(U_\beta \cdot 1)$ in two different ways. Since

$$\begin{aligned} \sigma(U_\beta \cdot 1) &= U_{-\beta}\sigma(1) = U_\beta \sigma(1),\\ \sigma(U_\beta \cdot 1) &= \sigma(C \cdot 1) = C\sigma(1), \end{aligned}$$

we have that $\sigma(1)$ lies in the same module as 1. The action of $\sigma$ is then given by

$$\sigma(U_\alpha^n \cdot 1) = U_\alpha^n \cdot 1 = q^n.$$

Thus the automorphism $\sigma$ acts as the identity on all modules. Denote these four modules as

$$P_s, \quad \text{where} \quad s = (s_1, s_2), s_i \in \{\pm\} \tag{5.12}$$

and $s_1$ is the sign in (5.9), $s_2$ is the sign in (5.11). The entire $\sigma$-twisted $V_Q$-module is then $M(\sigma) \otimes P_s$. Since $M(\sigma)$ itself decomposes into $\pm 1$-eigenspaces of $\sigma$, there are a total of 8 orbifold modules of twisted type:

$$M(\sigma)^{\pm} \otimes P_s, \quad s = (s_1, s_2), s_i \in \{\pm\}.$$

In addition, we have the following correspondence:

$$\begin{aligned} M(\sigma)^{\pm} \otimes P_{(+,+)} &\simeq V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}, \\ M(\sigma)^{\pm} \otimes P_{(-,+)} &\simeq V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}, \\ M(\sigma)^{\pm} \otimes P_{(+,-)} &\simeq V_{\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{T_2,\pm}, \\ M(\sigma)^{\pm} \otimes P_{(-,-)} &\simeq V_{\frac{\alpha}{2}+\mathbb{Z}\alpha} \otimes V_{\mathbb{Z}\beta}^{T_2,\pm}. \end{aligned}$$

## 5.3 The Root Lattice $A_3$

Consider the $A_3$ simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$. The nondegenerate symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ is given by $(\alpha_1|\alpha_2) = -1$, $(\alpha_1|\alpha_3) = 0$, $(\alpha_2|\alpha_3) = -1$ and $(\alpha_i|\alpha_i) = 2$ for $i = 1, 2, 3$. The associated even lattice is $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$. Consider the Dynkin diagram automorphism $\sigma : \alpha_1 \longleftrightarrow \alpha_3$ and $\alpha_2 \longleftrightarrow \alpha_2$. Set $\alpha = \alpha_1 + \alpha_3$, and $\beta = \alpha_1 - \alpha_3$. Then $\alpha$ and $\beta$ are eigenvectors for $\sigma$ with eigenvalues 1 and $-1$, respectively. Also, $(\alpha|\alpha) = 4 = (\beta|\beta)$ and $(\alpha|\beta) = 0$. Since $(\alpha_1|\alpha_3) = 0$ and $(\alpha_2|\alpha_2) = 2$, we have that $\alpha_1, \alpha_2, \alpha_3 \in \bar{Q}$. Therefore

$$\begin{aligned} Q &= \bar{Q}, && (5.13) \\ L_+ &= \mathbb{Z}\alpha + \mathbb{Z}\alpha_2, && (5.14) \\ L_- &= \mathbb{Z}\beta, && (5.15) \\ Q/L &= \{L, \alpha_1 + L\}. && (5.16) \end{aligned}$$

Hence by Proposition 4.2.2, we have that

$$V_Q^{\sigma} \simeq \left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{+}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{+}\right) \tag{5.17}$$

since $(\alpha_1)_+ = \frac{\alpha}{2}$ and $(\alpha_1)_- = \frac{\beta}{2}$.

In order to compute the $V_L^{\sigma}$-modules, we first determine $L_+^*/L_+$ using the Gram matrix for $L_+$. Ordering the basis of $L_+$ as $\{\alpha, \alpha_2\}$, the Gram matrix for $L_+$ is given by

$$G = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$$

with inverse

$$G^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Thus a basis for $L_+^*$ is given by $\left\{\frac{\alpha+\alpha_2}{2}, \frac{\alpha}{2}+\alpha_2\right\}$ and therefore

$$L_+^*/L_+ = \left\{L_+, \frac{\alpha}{2}+L_+, \frac{\alpha+\alpha_2}{2}+L_+, \frac{\alpha_2}{2}+L_+\right\}.$$

Since $(\mathbb{Z}\beta)^* = \mathbb{Z}\frac{\beta}{4}$ we have that

$$(\mathbb{Z}\beta)^*/\mathbb{Z}\beta \quad = \quad \left\{\mathbb{Z}\beta, \frac{\beta}{4}+\mathbb{Z}\beta, \frac{\beta}{2}+\mathbb{Z}\beta, \frac{3\beta}{4}+\mathbb{Z}\beta\right\}.$$

Since $\sigma\left(\frac{\beta}{4}\right) = \frac{3\beta}{4} \mod \mathbb{Z}\beta$, the automorphism $\sigma$ identifies two of the corresponding modules so that the set of distinct irreducible $V_{\mathbb{Z}\beta}^+$-modules is parameterized by the set

$$\left\{\mathbb{Z}\beta, \frac{\beta}{4}+\mathbb{Z}\beta, \frac{\beta}{2}+\mathbb{Z}\beta\right\},$$

with corresponding modules of untwisted type

$$V_{\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\beta}{4}+\mathbb{Z}\beta},$$

and corresponding modules of twisted type

$$V_{\mathbb{Z}\beta}^{T_i,\pm}, \quad i=1,2.$$

From the proof of Corollary 4.4.3 and (5.16), the irreducible $V_L^\sigma$-modules are obtained from elements $\lambda \in L_+^*$ and $\mu \in (\mathbb{Z}\beta)^*$ such that $(\lambda+\mu|\alpha_1) \in \mathbb{Z}$. Since $(\frac{\alpha_2}{2}|\alpha_1) = -\frac{1}{2}$, $(\frac{\alpha}{2}|\alpha_1) = 1$ and $(\frac{\beta}{4}|\alpha_1) = \frac{1}{2}$, the irreducible $V_L^\sigma$-modules are one of the following:

$$V_{L_+} \otimes V_{\mathbb{Z}\beta}^{\pm}, \quad V_{L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\alpha_2}{2}+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta},$$

$$V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\alpha_2}{2}+\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta},$$

$$V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_i,\pm}, \quad V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_i,\pm}, \quad i=1,2.$$

We show below that $\eta$ can be taken as trivial. Thus by Theorem 4.4.2, each irreducible $V_Q^\sigma$-module of untwisted type is isomorphic to one of the following:

$$\left(V_{L_+}\otimes V_{\mathbb{Z}\beta}^{\pm}\right)\bigoplus\left(V_{\frac{\alpha}{2}+L_+}\otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}\right),\quad \left(V_{\frac{\alpha_2}{2}+L_+}\otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\right)\bigoplus\left(V_{\frac{\alpha_2}{2}+\frac{\alpha}{2}+L_+}\otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\right),$$
$$\left(V_{L_+}\otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}\right)\bigoplus\left(V_{\frac{\alpha}{2}+L_+}\otimes V_{\mathbb{Z}\beta}^{\pm}\right).$$

The irreducible $V_Q^\sigma$-modules of twisted type are obtained using the fusion rules for $V_{\mathbb{Z}\beta}^+$ with $M_1=V_{\frac{\beta}{2}+\mathbb{Z}\beta}^+$. Note that each irreducible character $\chi: 2L_-^*\cap L_-\longrightarrow\{\pm1\}$ for which $\chi(e_{-\alpha})=\chi(e_\alpha)$ can be written as

$$\chi(e_\alpha)=(-1)^{(\alpha|\mu)}$$

for some $\mu\in(2L_-^*\cap L_-)^*$. Thus

$$\begin{aligned}\chi_\mu^{(\beta/2)}(e_\beta) &= (-1)^{(\beta|\frac{\beta}{2})}(-1)^{(\beta|\mu)} &(5.18)\\ &= (-1)^{(\beta|\mu)} &(5.19)\\ &= \chi_\mu(e_\beta) &(5.20)\end{aligned}$$

so that the module $V_{\frac{\beta}{2}+\mathbb{Z}\beta}^+$ sends $V_{\mathbb{Z}\beta}^{T_i}$ to $V_{\mathbb{Z}\beta}^{T_i}$. Since $c_\chi\left(\frac{\beta}{2}\right)=\chi(e_\beta)$, we have that

$$c_i\left(\frac{\beta}{2}\right)=(-1)^{i-1}\qquad(i=1,2)\tag{5.21}$$

corresponding to the twisted modules $V_{\mathbb{Z}\beta}^{T_i}$. So the eigenspaces for $V_{\mathbb{Z}\beta}^{T_1}$ remain the same in each summand of the orbifold module but will switch for $V_{\mathbb{Z}\beta}^{T_2}$. Hence by Theorem 4.4.2, each irreducible $V_Q^\sigma$-module of twisted type is isomorphic to one of the following:

$$\left(V_{L_+}\otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\right)\bigoplus\left(V_{\frac{\alpha}{2}+L_+}\otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\right),\quad\left(V_{L_+}\otimes V_{\mathbb{Z}\beta}^{T_2,\pm}\right)\bigoplus\left(V_{\frac{\alpha}{2}+L_+}\otimes V_{\mathbb{Z}\beta}^{T_2,\mp}\right).$$

We now construct the orbifold modules of twisted type using Section 3.3.2. The 2-cocycle $\varepsilon$ satisfies the following:

$$\begin{aligned}\varepsilon(\alpha_i,\alpha_i) &= (-1)^{2\left(\frac{2+1}{2}\right)}=-1,\\ \varepsilon(\alpha_1,\alpha_2)\varepsilon(\alpha_2,\alpha_1) &= (-1)^{-1}=-1,\\ \varepsilon(\alpha_1,\alpha_3)\varepsilon(\alpha_3,\alpha_1) &= (-1)^0=1,\\ \varepsilon(\alpha_2,\alpha_3)\varepsilon(\alpha_3,\alpha_2) &= (-1)^{-1}=-1.\end{aligned}$$

Set $\varepsilon$ to be the following on the basis:

$$\begin{array}{ll} \varepsilon(\alpha_1,\alpha_2)=1, & \varepsilon(\alpha_2,\alpha_1)=-1, \\ \varepsilon(\alpha_1,\alpha_3)=1, & \varepsilon(\alpha_3,\alpha_1)=1, \\ \varepsilon(\alpha_2,\alpha_3)=-1, & \varepsilon(\alpha_3,\alpha_2)=1. \end{array}$$

Then using bimultiplicativity, we have $\varepsilon(\alpha,\alpha)=\varepsilon(\beta,\beta)=\varepsilon(\alpha,\beta)=\varepsilon(\beta,\alpha)=1$. With these notions, we have that $\varepsilon(\sigma\gamma_1,\sigma\gamma_2)=\varepsilon(\gamma_1,\gamma_2)$ for all $\gamma_1,\gamma_2\in Q$ so that we can take $\eta$ to be trivial, that is, $\eta(\gamma)=1$ for all $\gamma\in Q$.

The dual lattice to $A_3$ is spanned by the elements

$$\lambda_1=\frac{1}{4}(3\alpha_1+2\alpha_2+\alpha_3),\quad \lambda_2=\frac{1}{4}(2\alpha_1+4\alpha_2+2\alpha_3),\quad \lambda_3=\frac{1}{4}(\alpha_1+2\alpha_2+3\alpha_3).$$

The only $\sigma$-invariant elements in $Q^*/Q$ are $Q$ and $\lambda_2+Q$ and $\sigma$ identifies the other two corresponding modules since $\sigma(\lambda_1)=\lambda_3$. Thus there are 5 distinct orbifold modules of untwisted type:

$$V_Q^{\pm},\quad V_{\lambda_1+Q},\quad V_{\lambda_2+Q}^{\pm}.$$

For the orbifold modules of twisted type, the quantities that will be needed are the following:

$$\begin{array}{rcl} B_{\beta,-\beta} &=& 2^8, \\ B_{\alpha_1,\alpha_3} &=& 4, \\ B_{\alpha_1,-\alpha_3} &=& \frac{1}{4}, \\ B_{\alpha_3,-\alpha_3} &=& 4, \\ b_\beta &=& \frac{1}{2}(0-4)=-2, \\ b_\alpha &=& \frac{1}{2}(4-4)=0, \\ b_{\alpha_3} &=& \frac{1}{2}\,(1-2)=-\frac{1}{2}, \\ C_{\gamma,\theta} &=& (-1)^{(\gamma|\theta)}. \end{array}$$

The elements in $G_\sigma$ must satisfy $U_{\sigma\gamma}=U_\gamma e^{2\pi \mathrm{i}(b_\gamma+\gamma_0)}$. In particular, $U_{-\beta}=U_\beta e^{-4\pi\mathrm{i}}=U_\beta$ so that $U_\beta^2=U_\beta U_{-\beta}=\varepsilon(\beta,-\beta)B_{\beta,-\beta}^{-1}=2^{-8}$. Thus there are two possible actions of $U_\beta$:

$$U_\beta=\pm 2^{-4}=\pm\frac{1}{16}.$$

Using the $\sigma$-invariance, we then have the following:

$$\begin{aligned} U_{\alpha_3} &= U_{\alpha_1} e^{2\pi \mathrm{i}\left(-\frac{1}{2}+\frac{1}{2}\alpha_{(0)}\right)} = -U_{\alpha_1} e^{\pi \mathrm{i}\alpha_{(0)}}, \\ U_{\alpha} &= \varepsilon(\alpha_1,\alpha_3) B_{\alpha_1,\alpha_3} U_{\alpha_1} U_{\alpha_3} = -4U_{\alpha_1}^2 e^{\pi \mathrm{i}\alpha_{(0)}}, \\ U_{-\alpha_3} &= \varepsilon(\alpha_3,-\alpha_3) B_{\alpha_3,-\alpha_3}^{-1} U_{\alpha_3}^{-1} = \frac{1}{4} U_{\alpha_1}^{-1} e^{\pi \mathrm{i}\alpha_{(0)}}, \\ U_{\beta} &= \varepsilon(\alpha_1,-\alpha_3) B_{\alpha_1,-\alpha_3} U_{\alpha_1} U_{-\alpha_3} = \frac{1}{16} e^{\pi \mathrm{i}\alpha_{(0)}}. \end{aligned}$$

Consider the vector space $P = \mathbb{C}[q, q^{-1}, p, p^{-1}]$. Since each nontrivial action is determined by $U_{\alpha_1}$ and $U_{\alpha_2}$, we can let $U_{\alpha_1}$ act as multiplication by $q$ and $U_{\alpha_2}$ act as multiplication by $p(-1)^{q\frac{\partial}{\partial q}}$. Note that these actions ensure that $p$ and $q$ are commuting variables since the operators $U_{\alpha_1}$ and $U_{\alpha_2}$ anticommute. In order to determine the action of $e^{\pi \mathrm{i}\alpha_{(0)}}$, consider the commutation relation with $U_\gamma$:

$$e^{\pi \mathrm{i}\alpha_{(0)}} U_\gamma e^{-\pi \mathrm{i}\alpha_{(0)}} = e^{\pi \mathrm{i}(\alpha|\gamma)} U_\gamma.$$

For $\gamma = \alpha_1$, we have $e^{\pi \mathrm{i}\alpha_{(0)}} q e^{-\pi \mathrm{i}\alpha_{(0)}} = q$. Thus $e^{\pi \mathrm{i}\alpha_{(0)}} q^n = q^n e^{\pi \mathrm{i}\alpha_{(0)}}(1)$. Since $e^{2\pi \mathrm{i}\alpha_{(0)}} = 1$, we must have $e^{\pi \mathrm{i}\alpha_{(0)}}(1) = \pm 1$ so that $e^{\pi \mathrm{i}\alpha_{(0)}} q^n = \pm q^n$. Similarly for $\gamma = \alpha_2$, we have $e^{\pi \mathrm{i}\alpha_{(0)}} p^n = \pm p^n$. Thus on the space $P$ we have the following:

$$\begin{aligned} U_{\alpha_1} &= q, \\ U_{\alpha_2} &= p(-1)^{q\frac{\partial}{\partial q}}, \\ U_{\alpha_3} &= -U_{\alpha_1} e^{\pi \mathrm{i}\alpha_{(0)}} = \mp q, \\ U_{\alpha} &= = -4U_{\alpha_1}^2 e^{\pi \mathrm{i}\alpha_{(0)}} = \mp 4q^2, \\ U_{\beta} &= = \frac{1}{16} e^{-\pi \mathrm{i}\alpha_{(0)}} = \pm \frac{1}{16}. \end{aligned}$$

The automorphism $\sigma$ acts on each of these modules. To determine the action of $\sigma$, we calculate $\sigma(p^m q^n) = \sigma(U_{\alpha_2}^m U_{\alpha_1}^n \cdot 1)$ on the module $P_2$ corresponding to the positive action of $U_\beta$:

$$\begin{aligned} \sigma(U_{\alpha_2}^m U_{\alpha_1}^n \cdot 1) &= U_{\alpha_2}^m U_{\alpha_3}^n \cdot 1 \\ &= (-1)^n p^m q^n. \end{aligned}$$

Thus $P_2$ decomposes into two eigenspaces of $\sigma$ with eigenvalues $\pm 1$. The $+1$-eigenspace $P_2^+$ is generated by the elements $q^m p^n$, where $m$ is even, and the $-1$-eigenspace $P_2^-$ is generated by the elements $q^m p^n$, where $m$ is odd. Similarly, $\sigma(U_{\alpha_2}^m U_{\alpha_1}^n \cdot 1) = p^m q^n$ on the module $P_1$ corresponding to the negative action of $U_\beta$. Therefore $\sigma$ acts as the identity on $P_1$.

The entire $\sigma$-twisted $V_Q$-module is then $M(\sigma) \otimes P_i$ for $i = 1, 2$. Since $M(\sigma)$ itself decomposes into $\pm 1$-eigenspaces of $\sigma$, there are a total of 4 distinct orbifold modules of twisted type:

$$M(\sigma)^{\pm} \otimes P_1,$$

$$\left(M(\sigma)^{\pm} \otimes P_2^{+}\right) \bigoplus \left(M(\sigma)^{\mp} \otimes P_2^{-}\right).$$

In addition, we have the following correspondence:

$$\begin{aligned} M(\sigma)^{\pm} \otimes P_1 &\simeq \left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\right), \\ \left(M(\sigma)^{\pm} \otimes P_2^{+}\right) \bigoplus \left(M(\sigma)^{\mp} \otimes P_2^{-}\right) &\simeq \left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\pm}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\mp}\right). \end{aligned}$$

## 5.4 The Root Lattice $D_n$, $n \geq 4$

Consider the $D_n$ simple roots $\{\alpha_1, \ldots, \alpha_n\}$, where $n \geq 4$. The nondegenerate symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ is given by

$$((\alpha_i|\alpha_j))_{i,j} = \begin{pmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \vdots \\ & \ddots & \ddots & \ddots & & 0 \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}.$$

The associated even lattice is $Q = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$. Consider the Dynkin diagram automorphism $\sigma : \alpha_{n-1} \longleftrightarrow \alpha_n$ and $\alpha_i \longleftrightarrow \alpha_i$ for $i = 1, \ldots, n-2$. Set $\alpha = \alpha_{n-1} + \alpha_n$ and $\beta = \alpha_{n-1} - \alpha_n$. Then $\alpha$ and $\beta$ are eigenvectors for $\sigma$ with eigenvalues 1 and $-1$, respectively. Also, $(\alpha|\alpha) = 4 = (\beta|\beta)$ and $(\alpha|\beta) = 0$. Since $(\alpha_{n-1}|\alpha_n) = 0$ and $(\alpha_i|\alpha_i) = 2$ for $i = 1, \ldots, n-2$, we have that $Q \subset \bar{Q}$.

Therefore

$$\begin{aligned} Q &= \bar{Q}, && (5.22)\\ L_+ &= \bigoplus_{i=1}^{n-2} \mathbb{Z}\alpha_i + \mathbb{Z}\alpha, && (5.23)\\ L_- &= \mathbb{Z}\beta, && (5.24)\\ Q/L &= \{L, \alpha_{n-1} + L\}. && (5.25) \end{aligned}$$

Hence by Proposition 4.2.2, we have that

$$V_Q^\sigma \simeq \left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^+\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^+\right) \tag{5.26}$$

since $(\alpha_{n-1})_+ = \frac{\alpha}{2}$ and $(\alpha_{n-1})_- = \frac{\beta}{2}$.

In order to compute the $V_L^\sigma$-modules, we first determine $L_+^*/L_+$ using the Gram matrix for $L_+$. Ordering the basis of $L_+$ as $\{\alpha, \alpha_1, \ldots, \alpha_{n-2}\}$, the Gram matrix for $L_+$ is given by

$$G = \begin{pmatrix} 4 & & & & & -2 \\ & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ -2 & & & & -1 & 2 \end{pmatrix}.$$

The inverse is given by

$$G^{-1} = \begin{pmatrix} \frac{n-1}{4} & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \cdots \\ \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 2 & 2 & 2 & 2 & \cdots \\ \frac{3}{2} & 1 & 2 & 3 & 3 & 3 & \cdots \\ 2 & 1 & 2 & 3 & 4 & 4 & \cdots \\ \frac{5}{2} & 1 & 2 & 3 & 4 & 5 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

Note that $\det G = 4$ for all $n$.

**Lemma 5.4.1** *The group $L_+^*/L_+$ is given by*

$$L_+^*/L_+ = \left\{L_+, \frac{n-1}{4}\alpha + \theta + L_+, \frac{1}{2}\alpha + L_+, \frac{n+1}{4}\alpha + \theta + L_+\right\}, \tag{5.27}$$

*where $\theta = \frac{1}{2}\sum_{i=0}^{\lfloor\frac{n-3}{2}\rfloor} \alpha_{2i+1}$.*

**Proof** The element corresponding to the first column of $G^{-1}$ modulo $L_+$ is $\frac{n-1}{4}\alpha + \theta + L_+$. The nonzero element corresponding to all other columns is $\frac{1}{2}\alpha + L_+$. Thus the group $L_+^*/L_+$ is generated by these two elements. When $n$ is even we have

$$L_+^*/L_+ = \left\{L_+, \frac{1}{4}\alpha + \theta + L_+, \frac{1}{2}\alpha + L_+, \frac{3}{4}\alpha + \theta + L_+\right\}, \tag{5.28}$$

and $L_+^*/L_+$ is cyclic. When $n$ is odd we have

$$L_+^*/L_+ = \left\{L_+, \theta + L_+, \frac{1}{2}\alpha + L_+, \frac{1}{2}\alpha + \theta + L_+\right\}, \tag{5.29}$$

and $L_+^*/L_+$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The statement follows.

Since $(\mathbb{Z}\beta)^* = \mathbb{Z}\frac{\beta}{4}$ we have that

$$(\mathbb{Z}\beta)^*/\mathbb{Z}\beta \quad = \quad \left\{\mathbb{Z}\beta, \frac{\beta}{4} + \mathbb{Z}\beta, \frac{\beta}{2} + \mathbb{Z}\beta, \frac{3\beta}{4} + \mathbb{Z}\beta\right\}.$$

Since $\sigma\left(\frac{\beta}{4}\right) = \frac{3\beta}{4} \mod \mathbb{Z}\beta$, the automorphism $\sigma$ identifies two of the corresponding modules so that the set of distinct irreducible $V_{\mathbb{Z}\beta}^+$-modules is parameterized by the set

$$\left\{\mathbb{Z}\beta, \frac{\beta}{4} + \mathbb{Z}\beta, \frac{\beta}{2} + \mathbb{Z}\beta\right\},$$

with corresponding modules of untwisted type

$$V_{\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\beta}{4}+\mathbb{Z}\beta},$$

and corresponding modules of twisted type

$$V_{\mathbb{Z}\beta}^{T_i,\pm}, \quad i = 1, 2.$$

From the proof of Corollary 4.4.3 and (5.25), the irreducible $V_L^\sigma$-modules are obtained from elements $\lambda \in L_+^*$ and $\mu \in (\mathbb{Z}\beta)^*$ such that $(\lambda + \mu|\alpha_1) \in \mathbb{Z}$. Since $(\theta|\alpha_{n-1}) = \begin{cases} -\frac{1}{2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$ and $(\frac{\beta}{4}|\alpha_{n-1}) = \frac{1}{2}$, the irreducible $V_L^\sigma$-modules are one of the following:

$$V_{L_+} \otimes V_{\mathbb{Z}\beta}^{\pm}, \quad V_{L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta},$$

$$V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{\pm}, \quad V_{\frac{n-1}{4}\alpha+\theta+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}, \quad V_{\frac{n+1}{4}\alpha+\theta+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta},$$

$$V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_i,\pm}, \quad V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_i,\pm}, \quad i = 1, 2.$$

We show below that $\eta$ can be taken as trivial. Thus by Theorem 4.4.2, each irreducible $V_Q^\sigma$-module of untwisted type is isomorphic to one of the following:

$$\left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{\pm}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}\right), \quad \left(V_{L_+} \otimes V_{\frac{\beta}{2}+\mathbb{Z}\beta}^{\pm}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{\pm}\right),$$

$$\left(V_{\frac{n-1}{4}\alpha+\theta+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\right) \bigoplus \left(V_{\frac{n+1}{4}\alpha+\theta+L_+} \otimes V_{\frac{\beta}{4}+\mathbb{Z}\beta}\right).$$

The irreducible $V_Q^\sigma$-modules of twisted type are obtained using the fusion rules for $V_{\mathbb{Z}\beta}^+$ with $M_1 = V_{\frac{\beta}{2}+\mathbb{Z}\beta}^+$. Recall each irreducible character $\chi : 2L_-^* \cap L_- \longrightarrow \{\pm 1\}$ for which $\chi(e_{-\alpha}) = \chi(e_\alpha)$ can be written as

$$\chi(e_\alpha) = (-1)^{(\alpha|\mu)}$$

for some $\mu \in (2L_-^* \cap L_-)^*$. Thus

$$\begin{aligned} \chi_\mu^{(\beta/2)}(e_\beta) &= (-1)^{(\beta|\frac{\beta}{2})}(-1)^{(\beta|\mu)} && (5.30) \\ &= (-1)^{(\beta|\mu)} && (5.31) \\ &= \chi_\mu(e_\beta) && (5.32) \end{aligned}$$

so that the module $V_{\frac{\beta}{2}+\mathbb{Z}\beta}^+$ sends $V_{\mathbb{Z}\beta}^{T_i}$ to $V_{\mathbb{Z}\beta}^{T_i}$. Since $c_\chi\left(\frac{\beta}{2}\right) = \chi(e_\beta)$, we have that

$$c_i\left(\frac{\beta}{2}\right) = (-1)^{i-1} \qquad (i = 1, 2) \tag{5.33}$$

corresponding to the twisted modules $V_{\mathbb{Z}\beta}^{T_i}$. So the eigenspaces for $V_{\mathbb{Z}\beta}^{T_1}$ remain the same in each summand of the orbifold module but will switch for $V_{\mathbb{Z}\beta}^{T_2}$. Hence by Theorem 4.4.2, each irreducible $V_Q^\sigma$-module of twisted type is isomorphic to one of the following:

$$\left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\right), \quad \left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\pm}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\mp}\right).$$

We now construct the orbifold modules of twisted type using Section 3.3.2. The 2-cocycle $\varepsilon$ satisfies the following:

$$\begin{aligned}
\varepsilon(\alpha_i,\alpha_i) &= (-1)^{2\binom{2+1}{2}} = -1, \\
\varepsilon(\alpha_{i+1},\alpha_i)\varepsilon(\alpha_i,\alpha_{i+1}) &= (-1)^{-1} = -1, \quad i = 1,\dots,n-2, \\
\varepsilon(\alpha_n,\alpha_{n-2})\varepsilon(\alpha_{n-2},\alpha_n) &= (-1)^{-1} = -1, \\
\varepsilon(\alpha_i,\alpha_j)\varepsilon(\alpha_j,\alpha_i) &= 1 \quad \text{otherwise.}
\end{aligned}$$

Set $\varepsilon$ to be the following on the basis:

$$\begin{array}{lcl}
\varepsilon(\alpha_i,\alpha_{i+1}) = 1, & & \varepsilon(\alpha_{i+1},\alpha_i) = -1, \quad i = 1,\dots,n-3, \\
\varepsilon(\alpha_{n-2},\alpha_{n-1}) = 1 & , & \varepsilon(\alpha_{n-1},\alpha_{n-2}) = -1, \\
\varepsilon(\alpha_n,\alpha_{n-2}) = -1, & & \varepsilon(\alpha_{n-2},\alpha_n) = 1.
\end{array}$$

Then using bimultiplicativity, we have $\varepsilon(\alpha,\alpha) = \varepsilon(\beta,\beta) = \varepsilon(\alpha,\beta) = \varepsilon(\beta,\alpha) = 1$. With these notions, we have that $\varepsilon(\sigma\gamma_1,\sigma\gamma_2) = \varepsilon(\gamma_1,\gamma_2)$ for all $\gamma_1,\gamma_2 \in Q$ so that we can take $\eta$ to be trivial, that is, $\eta(\gamma) = 1$ for all $\gamma \in Q$.

The dual lattice to $D_n$ is spanned by the elements $\lambda_1,\dots,\lambda_{n-1},\lambda_n$, where

$$\begin{aligned}
\lambda_i &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{n-2}) + \frac{1}{2}i(\alpha_{n-1}+\alpha_n), \quad i < n-1, \\
\lambda_{n-1} &= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n\right), \\
\lambda_n &= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}n\alpha_n\right).
\end{aligned}$$

Since $\lambda_i + Q = \dfrac{\alpha}{2} + Q$ for $i$ odd such that $i < n-1$, we have that

$$Q^*/Q = \left\{Q, \frac{\alpha}{2} + Q, \lambda_{n-1} + Q, \lambda_n + Q\right\}.$$

The only $\sigma$-invariant elements in $Q^*/Q$ are $Q$ and $\frac{\alpha}{2} + Q$ and $\sigma$ identifies the other two corresponding modules since $\sigma(\lambda_{n-1}) = \lambda_n$. Thus there are 5 distinct orbifold modules of untwisted type:

$$V_Q^{\pm}, \quad V^{\pm}_{\frac{\alpha}{2}+Q}, \quad V_{\lambda_{n-1}+Q}.$$

For the orbifold modules of twisted type, the quantities that will be needed are the following:

$$\begin{aligned}
B_{\beta,-\beta} &= 2^8, \\
B_{\alpha_{n-1},\alpha_n} &= 4, \\
B_{\alpha_{n-1},-\alpha_n} &= \frac{1}{4}, \\
B_{\alpha_n,-\alpha_n} &= 4, \\
b_\beta &= \frac{1}{2}(0-4) = -2, \\
b_\alpha &= \frac{1}{2}(4-4) = 0, \\
b_{\alpha_{n-1}} &= \frac{1}{2}\,(1-2) = -\frac{1}{2}, \\
C_{\gamma,\theta} &= (-1)^{(\gamma|\theta)}.
\end{aligned}$$

The elements in $G_\sigma$ must satisfy $U_{\sigma\gamma} = U_\gamma e^{2\pi \mathrm{i}(b_\gamma + \gamma_0)}$. In particular, $U_{-\beta} = U_\beta e^{-4\pi \mathrm{i}} = U_\beta$ so that $U_\beta^2 = U_\beta U_{-\beta} = \varepsilon(\beta,-\beta)B_{\beta,-\beta}^{-1} = 2^{-8}$. Thus there are two possible actions of $U_\beta$:

$$U_\beta = \pm 2^{-4} = \pm\frac{1}{16}.$$

Using the $\sigma$-invariance, we then have the following:

$$\begin{aligned}
U_{\alpha_n} &= U_{\alpha_{n-1}} e^{2\pi \mathrm{i}\left(-\frac{1}{2}+\frac{1}{2}\alpha_{(0)}\right)} = -U_{\alpha_{n-1}} e^{\pi \mathrm{i}\alpha_{(0)}}, \\
U_\alpha &= \varepsilon(\alpha_{n-1},\alpha_n) B_{\alpha_{n-1},\alpha_n} U_{\alpha_{n-1}} U_{\alpha_n} = -4U_{\alpha_{n-1}}^2 e^{\pi \mathrm{i}\alpha_{(0)}}, \\
U_{-\alpha_n} &= \varepsilon(\alpha_n,-\alpha_n) B_{\alpha_n,-\alpha_n}^{-1} U_{\alpha_n}^{-1} = \frac{1}{4} U_{\alpha_{n-1}}^{-1} e^{\pi \mathrm{i}\alpha_{(0)}}, \\
U_\beta &= \varepsilon(\alpha_{n-1},-\alpha_n) B_{\alpha_{n-1},-\alpha_n} U_{\alpha_{n-1}} U_{-\alpha_n} = \frac{1}{16} e^{\pi \mathrm{i}\alpha_{(0)}}.
\end{aligned}$$

Consider the vector space $P(n) = \mathbb{C}[p_1^{\pm 1}, \ldots, p_{n-1}^{\pm 1}]$. Since each nontrivial action is determined by $U_{\alpha_i}$ for $i \leq n-1$, let $U_{\alpha_i}$ act as multiplication by $p_i(-1)^{p_{i+1}\frac{\partial}{\partial p_{i+1}}}$ for $i < n-1$ and $U_{\alpha_{n-1}}$ act as multiplication by $p_{n-1}$. Note that these actions ensure that $p_1, \ldots, p_{n-1}$ are commuting variables since the operators $U_{\alpha_i}$ and $U_{\alpha_{i+1}}$ anticommute. In order to determine the action of

$e^{\pi \mathrm{i} \alpha_{(0)}}$, consider the commutation relation with $U_\gamma$:

$$e^{\pi \mathrm{i} \alpha_{(0)}} U_\gamma e^{-\pi \mathrm{i} \alpha_{(0)}} = e^{\pi \mathrm{i} (\alpha|\gamma)} U_\gamma .$$

For $\gamma = \alpha_i$ with $i \leq n-1$, we have $e^{\pi \mathrm{i} \alpha_{(0)}} p_i e^{-\pi \mathrm{i} \alpha_{(0)}} = p_i$ Thus $e^{\pi \mathrm{i} \alpha_{(0)}} p_i^n = p_i^n e^{\pi \mathrm{i} \alpha_{(0)}}(1)$. Since $e^{2\pi \mathrm{i} \alpha_{(0)}} = 1$, we must have $e^{\pi \mathrm{i} \alpha_{(0)}}(1) = \pm 1$ so that $e^{\pi \mathrm{i} \alpha_{(0)}} p_i^n = \pm p_i^n$. Thus on the space $P(n)$ we have the following:

$$\begin{aligned}
U_{\alpha_{n-1}} &= p_{n-1}, \\
U_{\alpha_i} &= p_i (-1)^{p_{i+1} \frac{\partial}{\partial p_{i+1}}}, \quad i < n-1, \\
U_{\alpha_n} &= -U_{\alpha_{n-1}} e^{\pi \mathrm{i} \alpha_{(0)}} = \mp p_{n-1}, \\
U_\alpha &= = -4 U_{\alpha_{n-1}}^2 e^{\pi \mathrm{i} \alpha_{(0)}} = \mp 4 p_{n-1}^2, \\
U_\beta &= = \frac{1}{16} e^{\pi \mathrm{i} \alpha_{(0)}} = \pm \frac{1}{16}.
\end{aligned}$$

The automorphism $\sigma$ acts on each of these modules. To determine the action of $\sigma$, we calculate $\sigma(q^m p_1^{k_1} \cdots p_{n-2}^{k_{n-2}}) = \sigma(U_{\alpha_{n-1}}^m U_{\alpha_1}^{k_1} \cdots U_{\alpha_{n-2}}^{k_{n-2}} \cdot 1)$ on the module $P_2(n)$ corresponding to the positive action of $U_\beta$:

$$\begin{aligned}
\sigma(U_{\alpha_{n-1}}^m U_{\alpha_1}^{k_1} \cdots U_{\alpha_{n-2}}^{k_{n-2}} \cdot 1) &= U_{\alpha_n}^m U_{\alpha_1}^{k_1} \cdots U_{\alpha_{n-2}}^{k_{n-2}} \cdot 1 \\
&= (-1)^m U_{\alpha_{n-1}}^m U_{\alpha_1}^{k_1} \cdots U_{\alpha_{n-2}}^{k_{n-2}} \cdot 1 \\
&= (-1)^m q^m p_1^{k_1} \cdots p_{n-2}^{k_{n-2}}.
\end{aligned}$$

Thus $P_2(n)$ decomposes into two eigenspaces of $\sigma$ with eigenvalues $\pm 1$. The $+1$-eigenspace $P_2(n)^+$ is generated by the elements $q^m p_1^{k_1} \cdots p_{n-2}^{k_{n-2}}$, where $m$ is even, and the $-1$-eigenspace $P_2(n)^-$ is generated by the elements $q^m p_1^{k_1} \cdots p_{n-2}^{k_{n-2}}$, where $m$ is odd. Similarly, $\sigma(U_{\alpha_{n-1}}^m U_{\alpha_1}^{k_1} \cdots U_{\alpha_{n-2}}^{k_{n-2}} \cdot 1) = q^m p_1^{k_1} \cdots p_{n-2}^{k_{n-2}}$ on the module $P_1(n)$ corresponding to the negative action of $U_\beta$. Therefore $\sigma$ acts as the identity on $P_1(n)$.

The entire $\sigma$-twisted $V_Q$-module is then $M(\sigma) \otimes P_i(n)$ for $i = 1, 2$. Since $M(\sigma)$ itself decomposes into $\pm 1$-eigenspaces of $\sigma$, there are a total of 4 distinct orbifold modules of twisted type:

$$M(\sigma)^\pm \otimes P_1(n),$$

$$\left(M(\sigma)^\pm \otimes P_2(n)^+\right) \bigoplus \left(M(\sigma)^\mp \otimes P_2(n)^-\right).$$

In addition, we have the following correspondence:

$$\begin{aligned} M(\sigma)^{\pm} \otimes P_1(n) &\simeq \left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_1,\pm}\right), \\ (M(\sigma)^{\pm} \otimes P_2(n)^{+}) \bigoplus (M(\sigma)^{\mp} \otimes P_2(n)^{-}) &\simeq \left(V_{L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\pm}\right) \bigoplus \left(V_{\frac{\alpha}{2}+L_+} \otimes V_{\mathbb{Z}\beta}^{T_2,\mp}\right). \end{aligned}$$

## 5.5 The Root Lattice $A_n$, $n$ odd

Consider the $A_n$ simple roots $\{\alpha_1, \ldots, \alpha_n\}$, where $n$ is odd. The nondegenerate symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ is given by

$$((\alpha_i|\alpha_j))_{i,j} = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The associated even lattice is $Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$. Throughout this section, set $l = \frac{n-1}{2}$. Consider the Dynkin diagram automorphism $\sigma : \alpha_i \longleftrightarrow \alpha_{n-i+1}$. This is also an automorphism of $Q$ with a fixed point $\alpha_{l+1}$. For $i < l + 1$, set

$$\alpha^i = \alpha_i + \alpha_{n-i+1}, \tag{5.34}$$

$$\beta^i = \alpha_i - \alpha_{n-i+1}. \tag{5.35}$$

Then $\alpha^i$ and $\beta^i$ are eigenvectors for $\sigma$ with eigenvalues 1 and $-1$, respectively. Products between these elements are as follows:

$$\begin{aligned} &(\alpha^i|\alpha^i) = 4 = (\beta^i|\beta^i), \\ &(\alpha^i|\alpha^{i+1}) = -2 = (\beta^i|\beta^{i+1}), \qquad i = 1, \ldots, l-1, \\ &(\alpha^i|\alpha^j) = 0 = (\beta^i|\beta^j), \qquad \text{otherwise}, \\ &(\alpha^i|\beta^j) = 0. \end{aligned}$$

Since $(\alpha_i|\alpha_{n-i+1}) = 0$ for $i \leq l$ and $(\alpha_{l+1}|\alpha_{l+1}) = 2$, we have that $Q \subset \bar{Q}$. Therefore

$$Q = \bar{Q}, \tag{5.36}$$

$$L_+ = \sum_{i=1}^{l} \mathbb{Z}\alpha^i + \mathbb{Z}\alpha_{l+1}, \tag{5.37}$$

$$L_- = \sum_{i=1}^{l} \mathbb{Z}\beta^i. \tag{5.38}$$

The cosets $Q/L$ are in correspondence with $\{0,1\}$-valued $l$-tuples via

$$(a_1, \ldots, a_l) \longleftrightarrow \sum_{i=1}^{l} a_i\alpha_i + L \tag{5.39}$$

so that $|Q/L| = 2^l$. Hence by Proposition 4.2.2, we have that

$$V_Q^\sigma \simeq \bigoplus_{(b_1,\ldots,b_l)} \left( V_{\frac{1}{2}\sum b_i\alpha^i + L_+} \otimes V^+_{\frac{1}{2}\sum b_i\beta^i + L_-} \right), \tag{5.40}$$

where $b_i \in \{0,1\}$ and there are $2^l$ summands.

In order to compute the $V_L^\sigma$-modules, we first determine $L_+^*/L_+$ using the Gram matrix for $L_+$. Ordering the basis of $L_+$ as $\{\alpha_{l+1}, \alpha^1, \ldots, \alpha^l\}$, the Gram matrix for $L_+$ is given by

$$G = \begin{pmatrix} 2 & & & & & -2 \\ & 4 & -2 & & & \\ & -2 & 4 & \ddots & & \\ & & \ddots & \ddots & -2 & \\ & & & -2 & 4 & -2 \\ -2 & & & & -2 & 4 \end{pmatrix}.$$

The inverse is given by

$$G^{-1} = \begin{pmatrix} \frac{n+1}{4} & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \\ 1 & \frac{1}{2} & 1 & 1 & 1 & 1 & \cdots \\ \frac{3}{2} & \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \cdots \\ 2 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & 2 & \cdots \\ \frac{5}{2} & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

**Lemma 5.5.1** *The group $L_+^*/L_+$ is generated by the set*

$$\left\{ L_+, \frac{1}{2}\alpha_{l+1} + L_+, \frac{1}{2}\alpha^1 + L_+, \ldots, \frac{1}{2}\alpha^l + L_+ \right\} \tag{5.41}$$

*and* $|L_+^*/L_+| = \det G = 2^{l+1}$.

**Proof** It is clear that the elements in (5.41) are in $L_+^*/L_+$ and form a linearly independent set over $\mathbb{Z}$. Since the operation is addition modulo $L_+$, each nonintegral fraction in $G^{-1}$ can be replaced with $\frac{1}{2}$. Then every column of $G^{-1}$ can be written as a linear combination of the elements (5.41).

Each $V_{L_+}$-module can therefore be represented by a $\{0, 1\}$-valued $(l+1)$-tuple given by

$$(a, b_1, \ldots, b_l) \longrightarrow \frac{1}{2}\left(a\alpha_{l+1} + b_1\alpha^1 + \cdots + b_l\alpha^l\right) + L_+. \tag{5.42}$$

We determine $L_-^*/L_-$ using the Gram matrix for $L_-$. Ordering the basis of $L_-$ as $\{\beta^1, \ldots, \beta^l\}$, the Gram matrix for $L_-$ is given by

$$M = \begin{pmatrix} 4 & -2 & & \\ -2 & 4 & \ddots & \\ & \ddots & \ddots & -2 \\ & & -2 & 4 \end{pmatrix}.$$

Since $M$ is twice the Gram matrix for $A_l$, the dual basis for $L_-$ is given by the dual basis of $A_l$ divided by a factor of $\sqrt{2}$. Let $K$ be the root lattice for $A_l$ with basis $\{\nu_1, \ldots \nu_l\}$ such that $\sigma(\nu_i) = -\nu_i$. Then $L_- = \sqrt{2}K$ and $L_-^* = \frac{1}{\sqrt{2}}K^*$. We also have that $|L_-^*/L_-| = \det M = 2^l\,(l+1)$.

The dual lattice to $A_l$ is spanned by the elements $\lambda_1, \ldots, \lambda_l$, where

$$\lambda_i = \frac{1}{l+1}\left((l-i+1)\nu_1 + 2(l-i+1)\nu_2 + i(l-i+1)\nu_i + i(l-i)\nu_{i+1} + \cdots + i\nu_l\right),$$

and the fundamental group $K^*/K$ is cyclic generated by the element $\lambda_1 + K$. Let $\frac{1}{2}\mu_i$ be the $i$th fundamental dominant weight of $L_-$ so that

$$\frac{1}{2}\mu_i = \frac{1}{\sqrt{2}}\lambda_i$$

and $\sigma(\mu_i) = -\mu_i$. Then $L_-^*/L_-$ is related to $K^*$ and $K$ by

$$L_-^*/L_- = \frac{1}{\sqrt{2}}K^*/\sqrt{2}K \simeq K^*/2K,$$

where the isomorphism is given by multiplication by $\sqrt{2}$. Now the space $K^*/2K$ is generated by the elements $\{\lambda_1 + \sum b_j\nu_j + 2K | b_j = 0, 1\}$. Thus we have

$$\begin{aligned} L_-^*/L_- &= \frac{1}{\sqrt{2}}K^*/\sqrt{2}K \\ &= \left\{ \frac{1}{\sqrt{2}}a\lambda_1 + \frac{1}{\sqrt{2}}\sum_{j=1}^{l} b_j\nu_j + \sqrt{2}K \,\middle|\, a \in \{0, \ldots, l\}, b_j \in \{0,1\} \right\} \\ &= \left\{ \frac{1}{2}a\mu_1 + \frac{1}{2}\sum_{j=1}^{l} b_j\beta^j + L_- \,\middle|\, a \in \{0, \ldots, l\}, b_j \in \{0,1\} \right\}. \end{aligned}$$

The $\sigma$-invariant elements of $L_-^*/L_-$ are those for which $a = 0$. Thus there are $2 \cdot 2^l = 2^{l+1}$ distinct irreducible $V_{L_-}^+$-modules corresponding to the $l$-tuples $(b_1, \ldots, b_l)$, where $b_i \in \{0,1\}$ and $a = 0$. The remaining elements in $L_-^*/L_-$ are identified by $\sigma$ in pairs. Hence there are

$$\frac{2^l(l+1) - 2^l}{2} = 2^{l-1}l$$

distinct irreducible $V_{L_-}^+$-modules corresponding to the $l$-tuples $(b_1, \ldots, b_l)$, where $b_i \in \{0,1\}$ and $a \neq 0$. Hence the total number of distinct irreducible $V_{L_-}^+$-modules of untwisted type is $2 \cdot 2^l + 2^{l-1}l = 2^{l-1}(l+4)$.

**Lemma 5.5.2** *The distinct $V_{L_-}^+$-modules that decompose into eigenspaces correspond to the set*

$$\left(L_-^*/L_-\right)^\sigma = \left\{ \frac{1}{2}\sum_{j=1}^{l} b_j\beta^j + L_- \,\middle|\, b_j \in \{0,1\} \right\}.$$

*The other distinct $V_{L_-}^+$-modules correspond to the set*

$$\left\{ \frac{1}{2}\mu_i + \frac{1}{2}\sum_{j=1}^{l} b_j\beta^j + L_- \,\middle|\, b_j \in \{0,1\}, 1 \le i \le \frac{l+1}{2} \right\}.$$

*Furthermore, when $l$ is odd and $k = \frac{l+1}{2}$, a minimal spanning set corresponding to the distinct $V_{L_-}^+$-modules contains only elements of the form*

$$\frac{1}{2}\mu_k + \frac{1}{2}\sum_{j=1}^{l-1} b_j\beta^j.$$

**Proof** For $\mu = \frac{1}{2}\mu_i + \frac{1}{2}\sum b_j\beta^j \in L_-^*$, we have

$$\begin{aligned} \sigma(\mu) &= -\mu \\ &= \frac{1}{2}\mu_{l-i+1} + \frac{1}{2}\sum_{j=1}^{l} b_j\beta^j + \gamma_i \mod L_- \end{aligned}$$

where $\gamma_i = \frac{1}{2}\sum a_{ij}\beta^j$ for some $a_{ij} \in \{0,1\}$. In particular, when $l$ is odd and $k = \frac{l+1}{2}$,

$$\begin{aligned} \frac{1}{2}\mu_k &= \frac{1}{4k}\left(k\beta^1 + 2k\beta^2 + \cdots + k^2\beta^k + \cdots + 2k\beta^{l-1} + k\beta^l\right) \\ &= \frac{1}{4}\left(\beta^1 + 2\beta^2 + \cdots + k\beta^k + \cdots + 2\beta^{l-1} + \beta^l\right), \end{aligned}$$

so that

$$-\frac{1}{2}\mu_k = \frac{1}{2}\mu_k + \frac{1}{2}\left(\beta^1 + \beta^3 + \cdots + \beta^l\right) \mod L_-.$$

Thus we can take $\gamma_k = \frac{1}{2}\left(\beta^1 + \beta^3 + \cdots + \beta^l\right)$ and just use representatives with $b_l = 0$.

From the proof of Corollary 4.4.3 and (5.39), the irreducible $V_L^\sigma$-modules are obtained from elements $\lambda \in L_+^*$ and $\mu \in L_-^*$ such that $(\lambda + \mu|\alpha_i) \in \mathbb{Z}$ for $i = 1, \ldots, l$.

**Lemma 5.5.3** *We have for $j < l+1$,*

$$\begin{aligned}\left(\frac{1}{2}\mu_i \middle| \alpha_j\right) &= \frac{1}{2}\delta_{ij},\\ \left(\frac{1}{2}\beta^i \middle| \alpha_j\right) &= \left(\frac{1}{2}\alpha^i \middle| \alpha_j\right) = \delta_{ij} - \frac{1}{2}\left(\delta_{i,j-1} + \delta_{i,j+1}\right).\end{aligned}$$

**Proof** The statement follows from the following calculations:

$$\begin{aligned}(\beta^i|\alpha_j) &= (\alpha^i|\alpha_j) = -\delta_{i,j-1} + 2\delta_{ij} - \delta_{i,j+1},\\ (\mu_i|\alpha_i) &= \frac{1}{l+1}\left(-(i-1)(l-i+1) + 2i(l-i+1) - i(l-i)\right) = 1,\\ j < i : (\mu_i|\alpha_j) &= \frac{l-i+1}{l+1}\left(-(j-1) + 2j - (j+1)\right) = 0,\\ j > i : (\mu_i|\alpha_j) &= \frac{i}{l+1}\left(-(l-(j-1)+1) + 2(l-j+1) - (l-(j+1)+1)\right) = 0.\end{aligned}$$

**Lemma 5.5.4** *There are $l+4$ $V_Q^\sigma$-modules of untwisted type, where $l = \frac{n-1}{2}$.*

**Proof** The only elements in $L_+^*/L_+$ that have integral products with each $\alpha_i$ are

$$\frac{1}{2}\left(\alpha^1 + \alpha^3 + \cdots + \alpha^l\right) + L_+, \quad \text{if } l \text{ is odd, and} \tag{5.43}$$

$$\frac{1}{2}\left(\alpha_{l+1} + \alpha^1 + \alpha^3 + \cdots + \alpha^{l-1}\right) + L_+, \quad \text{if } l \text{ is even.} \tag{5.44}$$

Due to (5.40), the orbifold module corresponding to any representative in this list is the same as the orbifold module corresponding to the representative for either $V_{L_+}$ or $V_{\frac{1}{2}\alpha_{l+1}+L_+}$.

For $l$ odd, the following is a list of all elements in $L_-^*/L_-$ which have integral products with each $\alpha_i$:

$$L_-, \qquad \gamma + L_- = \tfrac{1}{2}\left(\beta^1 + \beta^3 + \cdots + \beta^l\right) + L_-,$$

$$\tfrac{1}{2}\mu_{2k} + \tfrac{1}{2}\left(\beta^1 + \beta^3 + \cdots + \beta^{2k-1}\right) + L_-, \qquad k = 1, \ldots, q = \begin{cases}\frac{l+1}{4}, & 4|(l+1)\\ \frac{l-1}{4}, & 4|(l+3)\end{cases},$$

$$\tfrac{1}{2}\mu_{2k} + \tfrac{1}{2}\left(\beta^1 + \beta^3 + \cdots + \beta^{2k-1}\right) + \gamma + L_-, \qquad k = 1, \ldots, q' = \begin{cases}\frac{l+1}{4} - 1, & 4|(l+1)\\ \frac{l-1}{4}, & 4|(l+3)\end{cases}.$$

The following is a list of all $x + L_- \in L_-^*/L_-$ which have half-integral products with each $\alpha_i$

and satisfy $(x+\frac{1}{2}\alpha_{l+1}|\alpha_i)\in\mathbb{Z}$:

$$\frac{1}{2}\mu_l + L_-,$$
$$\frac{1}{2}\mu_{l-2k}+\frac{1}{2}\left(\beta^{l-1}+\beta^{l-3}+\cdots+\beta^{l-2k+1}\right)+L_-, \qquad k=1,\ldots,r=\begin{cases}\frac{l-3}{4}, & 4|(l+1)\\ \frac{l-1}{4}, & 4|(l+3)\end{cases},$$

$$\frac{1}{2}\mu_l+\gamma+L_-,$$
$$\frac{1}{2}\mu_{l-2k}+\frac{1}{2}\left(\beta^{l-1}+\beta^{l-3}+\cdots+\beta^{l-2k+1}\right)+\gamma+L_-, \qquad k=1,\ldots,r'=\begin{cases}\frac{l-3}{4}, & 4|(l+1)\\ \frac{l-1}{4}-1, & 4|(l+3)\end{cases}.$$

The only $\sigma$-invariant elements in these lists are $L_-$ and $\gamma+L_-$ so that the two orbifold modules with representatives $V_{L_+}\otimes V_{L_-}$ and $V_{L_+}\otimes V_{\gamma+L_-}$ will split into eigenspaces under $\sigma$. For elements in $L_-^*/L_-$ that are not $\sigma$-invariant, there are $\frac{l-1}{2}$ elements that have integral products with each $\alpha_i$, and $\frac{l+1}{2}$ elements in $L_-^*/L_-$ that have half-integral products with each $\alpha_i$. Thus there are a total of $2(2)+\frac{l-1}{2}+\frac{l+1}{2}=l+4$ orbifold modules of untwisted type.

For $l$ even, the following is a list of all elements in $L_-^*/L_-$ which have integral products with each $\alpha_i$:

$$L_-,$$

$$\frac{1}{2}\mu_{2k}+\frac{1}{2}\left(\beta^{1}+\beta^{3}+\cdots+\beta^{2k-1}\right)+L_-, \qquad k=1,\ldots,s=\begin{cases}1, & l=2\\ \frac{l}{4}, & 4|l\\ \frac{l-2}{4}, & 4|(l+2), l>2\end{cases},$$

$$\frac{1}{2}\mu_{2k+1}+\frac{1}{2}\left(\beta^{2k+2}+\beta^{2k+4}+\cdots+\beta^{l}\right)+L_-, \qquad k=0,\ldots,t=\begin{cases}\frac{l-4}{4}, & 4|l\\ \frac{l-2}{4}, & 4|(l+2)\end{cases}.$$

The only $\sigma$-invariant element in this list is $L_-$ so that the orbifold module with representative $V_{L_+}\otimes V_{L_-}$ will split into eigenspaces under $\sigma$. The $V_{L_-}^+$-modules paired with $V_{\frac{1}{2}\alpha_{l+1}+L_+}$ correspond to the elements in the above list added to the element $\frac{1}{2}\left(\beta^1+\beta^3+\cdots+\beta^{l-1}\right)$, which has half-integral products with all $\alpha_i$. For elements that are not $\sigma$-invariant, there are $\frac{l}{2}$ elements that have integral products with each $\alpha_i$. Thus there are a total of $2\left(2+\frac{l}{2}\right)=l+4$ orbifold modules of untwisted type.

**Theorem 5.5.5** *Let $l=\frac{n-1}{2}$. Then the list of $l+4$ $V_Q^\sigma$-modules of untwisted type is equivalent* *to the following list:*

$$\bigoplus_{(b_1,\ldots,b_l)}\left(V_{\frac{1}{2}\sum b_i\alpha^i+L_+}\otimes V^{\pm}_{\frac{1}{2}\sum b_i\beta^i+L_-}\right),\quad \bigoplus_{(b_1,\ldots,b_l)}\left(V_{\frac{1}{2}\alpha_{l+1}+\frac{1}{2}\sum b_i\alpha^i+L_+}\otimes V_{\frac{1}{2}\mu_l+\frac{1}{2}\sum b_i\beta^i+L_-}\right),$$
$$\bigoplus_{(b_1,\ldots,b_l)}\left(V_{\frac{1}{2}\sum b_i\alpha^i+L_+}\otimes V_{\mu_l+\frac{1}{2}\sum b_i\beta^i+L_-}\right),\quad \bigoplus_{(b_1,\ldots,b_l)}\left(V_{\frac{1}{2}\alpha_{l+1}+\frac{1}{2}\sum b_i\alpha^i+L_+}\otimes V_{\frac{3}{2}\mu_l+\frac{1}{2}\sum b_i\beta^i+L_-}\right),$$
$$\ldots,\quad \bigoplus_{(b_1,\ldots,b_l)}\left(V_{\frac{1}{2}\alpha_{l+1}+\frac{1}{2}\sum b_i\alpha^i+L_+}\otimes V^{\pm}_{\frac{l+1}{2}\mu_l+\frac{1}{2}\sum b_i\beta^i+L_-}\right),$$

*where* $b_i\in\{0,1\}$.

**Proof** It is sufficient to show the following identities among the cosets of $L_-^*/L_-$:

$$\frac{1}{2}\mu_{2k}+\frac{1}{2}\left(\beta^1+\beta^3+\cdots+\beta^{2k-1}\right)+L_-=k\mu_l+L_-,$$
$$\frac{1}{2}\mu_{l-2k}+\frac{1}{2}\left(\beta^{l-1}+\beta^{l-3}+\cdots+\beta^{l-2k+1}\right)+L_-=\frac{2k+1}{2}\mu_l+L_-,$$
$$\frac{1}{2}\mu_{2k+1}+\frac{1}{2}\left(\beta^{2k+2}+\beta^{2k+4}+\cdots+\beta^{l}\right)+L_-=\left(\frac{l}{2}-k\right)\mu_l+L_-.$$

For the first identity, we consider separately the coefficients of $\beta^i$ for $i<2k$ and $i\geq 2k$. For $i<2k$ we also consider separately $i$ even and $i$ odd. For $i<2k$ odd, the coefficient of $\beta^i$ is

$$i(l-2k+1)+l+1=(i+1)(l+1)-2ik.$$

Since $i+1$ is even, we may take the coefficient to be $-2ik$. For $i<2k$ even, the coefficient of $\beta^i$ is

$$i(l-2k+1)=i(l+1)-2ik.$$

Since $i$ is even, we may take the coefficient to be $-2ik$. For $i\geq 2k$, the coefficient of $\beta^i$ is

$$2k(l-i+1)=2k(l+1)-2ik$$

which can be replaced with $-2ik$. Thus we have

$$\begin{aligned}\frac{1}{2}\mu_{2k+1} + \frac{1}{2}\left(\beta^{2k+2} + \beta^{2k+4} + \cdots + \beta^{l}\right) + L_{-} &= \frac{2k}{2(l+1)}\sum_{i=1}^{l}(-i\beta^{i}) + L_{-} \\ &\mapsto \frac{k}{l+1}\sum_{i=1}^{l} i\beta^{i} + L_{-} \\ &= k\mu_{l} + L_{-}.\end{aligned}$$

For the second identity, we consider separately the coefficients of $\beta^i$ for $i < l-2k$ and $i \geq l-2k$. For $i \geq l-2k$ we also consider separately $i$ even and $i$ odd. For $i \geq l-2k$ odd, the coefficient of $\beta^i = \beta^{l-2k+2j+1}$ can be written as

$$(l-2k)(2k-2j) + l + 1 + j(2l+2) = (2k+1)(l-2k+2j+1) = (2k+1)i$$

and for $i \geq l-2k$ even, the coefficient of $\beta^i = \beta^{l-2k+2j}$ can be written as

$$(l-2k)(2k-2j+1) + j(2l+2) = (2k+1)(l-2k+2j) = (2k+1)i.$$

For $i < l-2k$, the coefficient of $\beta^i$ is $(2k+1)i$. Thus we have

$$\begin{aligned}\frac{1}{2}\mu_{l-2k} + \frac{1}{2}\left(\beta^{l-1} + \beta^{l-3} + \cdots + \beta^{l-2k+1}\right) + L_{-} &= \frac{2k+1}{2(l+1)}\sum_{i=1}^{l} i\beta^{i} + L_{-} \\ &= \frac{2k+1}{2}\mu_{l} + L_{-}.\end{aligned}$$

For the third identity, we consider separately the coefficients of $\beta^i$ for $i < 2k+1$ and $i \geq 2k+1$. For $i \geq 2k+1$ we also consider separately $i$ even and $i$ odd. For $i \geq 2k+1$ even, the coefficient of $\beta^i = \beta^{2k+1+2j}$ can be written as

$$(2k+1)(l-2k-2j-1) + l + 1 + j(2l+2) = (l-2k)(2k+2+2j) = (l-2k)i$$

and for $i \geq 2k+1$ odd, the coefficient of $\beta^i = \beta^{2k+1+2j+1}$ can be written as

$$(2k+1)(l-2k-2j-2) + (j+1)(2l+2) = (l-2k)(2k+2+2j+1) = (l-2k)i.$$

For $i < 2k+1$, the coefficient of $\beta^i$ is $(l-2k)i$. Thus we have

$$\begin{aligned}\frac{1}{2}\mu_{2k+1} + \frac{1}{2}\left(\beta^{2k+2} + \beta^{2k+4} + \cdots + \beta^l\right) + L_- &= \frac{l-2k}{2(l+1)}\sum_{i=1}^{l} i\beta^i + L_- \\ &= \left(\frac{l}{2} - k\right)\mu_l + L_-.\end{aligned}$$

We also have $\frac{l+1}{2}\mu_l + L_- = \gamma + L_-$, where $\gamma = \frac{1}{2}\left(\beta^1 + \beta^3 + \cdots + \beta^l\right)$. This completes the proof.

For convenience, the modules in Theorem 5.5.5 are referred to by using the summand corresponding to the $l$-tuple $(0, \ldots, 0)$.

Recall the irreducible twisted $V_{L_-}^+$-modules are given by $V_{L_-}^{T_\chi,\pm}$, where $T_\chi$ is an irreducible $\hat{L}_-/K$-module with central character $\chi$. The irreducible $V_Q^\sigma$-modules of twisted type are obtained using the fusion rules for $V_{L_-}^+$ with $M_1 = V^+_{\frac{1}{2}\sum b_i\beta^i + L_-}$, where $i = 1, \ldots, l$ and $b_i \in \{0,1\}$ (cf. Definition 3.3.16). Note that each irreducible character $\chi : 2L_-^* \cap L_- \longrightarrow \{\pm 1\}$ for which $\chi(e_{-\alpha}) = \chi(e_\alpha)$ can be written as

$$\chi(e_\alpha) = (-1)^{(\alpha|\mu)}$$

for some $\mu \in (2L_-^* \cap L_-)^*$. Since

$$\begin{aligned}\left(\sum_{i=1}^{l} b_i\beta^i \middle| \frac{1}{2}\sum_{j=1}^{l} b_j\beta^j\right) &= \sum_{i<j} b_ib_j(\beta^i|\beta^j) + \frac{1}{2}\sum_{i=1}^{l} b_i^2(\beta^i|\beta^i) \\ &= -2\sum_{i<j} b_ib_j + 2\sum_{i=1}^{l} b_i^2 \in 2\mathbb{Z},\end{aligned}$$

we have that

$$\chi_\mu^{(\frac{1}{2}\sum b_i\beta^i)}(e_{\sum b_i\beta^i}) = \chi_\mu(e_{\sum b_i\beta^i}) \tag{5.45}$$

and hence every module $V^+_{\frac{1}{2}b_i\beta^i + L_-}$ sends $V_{L_-}^{T_\chi}$ to $V_{L_-}^{T_\chi}$. Since $c_\chi\left(\frac{1}{2}\sum b_i\beta^i\right) = \chi(e_{\sum b_i\beta^i})$ and linear characters are homomorphisms, we have that

$$c_\chi\left(\frac{1}{2}\sum_{i=1}^{l} b_i\beta^i\right) = \prod_{i=1}^{l}\chi(e_{b_i\beta^i}), \tag{5.46}$$

where $b_i \in \{0,1\}$ and $\chi(e_0) = 1$. Thus each irreducible $V_Q^\sigma$-module of twisted type corresponding to a choice for $\chi$ is obtained in the following way. For each element $\gamma_- = \frac{1}{2}\sum b_i\beta^i$, the eigenspaces are determined by computing the product in (5.46). There are a total of $2 \cdot 2^l = 2^{l+1}$ irreducible $V_Q^\sigma$-modules of twisted type.

**Example 5.5.6** *As an example of composing orbifold modules of twisted type, consider the case $l = 3$ and the character $\chi$ defined on the basis of $L_-$ by $\chi(e_{\beta^1}) = -1, \chi(e_{\beta^2}) = -1$ and $\chi(e_{\beta^3}) = 1$. Then the corresponding orbifold modules of twisted type are*

$$\left(V_{L_+} \otimes V_{L_-}^{T_\chi,\pm}\right) \bigoplus \left(V_{\frac{\alpha^1}{2}+L_+} \otimes V_{L_-}^{T_\chi,\mp}\right) \bigoplus \left(V_{\frac{\alpha^2}{2}+L_+} \otimes V_{L_-}^{T_\chi,\mp}\right)$$
$$\bigoplus \left(V_{\frac{\alpha^3}{2}+L_+} \otimes V_{L_-}^{T_\chi,\pm}\right) \bigoplus \left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V_{L_-}^{T_\chi,\pm}\right) \bigoplus \left(V_{\frac{\alpha^2+\alpha^3}{2}+L_+} \otimes V_{L_-}^{T_\chi,\mp}\right)$$
$$\bigoplus \left(V_{\frac{\alpha^1+\alpha^3}{2}+L_+} \otimes V_{L_-}^{T_\chi,\mp}\right) \bigoplus \left(V_{\frac{\alpha^1+\alpha^2+\alpha^3}{2}+L_+} \otimes V_{L_-}^{T_\chi,\pm}\right).$$

We now construct the orbifold modules using Section 3.3.2. The 2-cocycle $\varepsilon$ satisfies the following:

$$\begin{aligned}
\varepsilon(\alpha_i, \alpha_i) &= (-1)^{2\left(\frac{2+1}{2}\right)} = -1, \\
\varepsilon(\alpha_i, \alpha_{i+1})\varepsilon(\alpha_{i+1}, \alpha_i) &= (-1)^{-1} = -1, \\
\varepsilon(\alpha_i, \alpha_j)\varepsilon(\alpha_j, \alpha_i) &= (-1)^0 = 1, \quad \text{otherwise.}
\end{aligned}$$

Using bimultiplicativity, we also have $\varepsilon(\alpha^i, \alpha^j) = \varepsilon(\beta^i, \beta^j) = \varepsilon(\alpha^i, \beta^j) = \varepsilon(\beta^i, \alpha^j) = 1$. Set $\varepsilon(\alpha_i, \alpha_{i+1}) = \varepsilon(\alpha_{n-i+1}, \alpha_{n-i}) = 1$ and $\varepsilon(\alpha_{i+1}, \alpha_i) = \varepsilon(\alpha_{n-i}, \alpha_{n-i+1}) = -1$. Then with these notions, we have that $\varepsilon(\sigma\gamma_1, \sigma\gamma_2) = \varepsilon(\gamma_1, \gamma_2)$ for all $\gamma_1, \gamma_2 \in Q$. Thus we can take $\eta$ to be trivial, that is, $\eta(\gamma) = 1$ for all $\gamma \in Q$.

The dual lattice to $A_n$ is spanned by the fundamental dominant weights

$$\lambda_i = \frac{1}{n+1}\left((n-i+1)\alpha_1 + \cdots + i(n-i+1)\alpha_i + i(n-i)\alpha_{i+1} + \cdots + i\alpha_n\right),$$

for $i = 1, \ldots, n$. The only $\sigma$-invariant elements in $Q^*/Q$ are $Q$ and $\lambda_{l+1} + Q$ and $\sigma$ identifies $l$ of the corresponding modules since $\sigma(\lambda_i) = \lambda_{n-i+1}$. Thus there are $l + 4$ distinct orbifold modules of untwisted type given by

$$V_Q^{\pm}, \quad V_{\lambda_1+Q}, \quad \ldots, \quad V_{\lambda_l+Q}, \quad V_{\lambda_{l+1}+Q}^{\pm}.$$

For the orbifold modules of twisted type, the quantities that will be needed are the following:

$$\begin{aligned}
B_{\beta^i,-\beta^i} &= 2^8, \\
B_{\alpha_i,\alpha_{n-i+1}} &= 4, \\
B_{\alpha_i,-\alpha_{n-i+1}} &= \frac{1}{4}, \\
B_{\alpha_{n-i+1},-\alpha_{n-i+1}} &= 4, \\
b_{\beta^i} &= \frac{1}{2}(0-4) = -2, \\
b_{\alpha^i} &= \frac{1}{2}(4-4) = 0, \\
b_{\alpha_{\frac{n+1}{2}}} &= \frac{1}{2}(2-2) = 0, \\
b_{\alpha_i} &= \frac{1}{2}\,(1-2) = -\frac{1}{2}, \\
C_{\gamma,\theta} &= (-1)^{(\gamma|\theta)}.
\end{aligned}$$

The elements in $G_\sigma$ must satisfy $U_{\sigma\gamma} = U_\gamma e^{2\pi \mathrm{i}(b_\gamma+\gamma_0)}$. In particular, $U_{-\beta^i} = U_{\beta^i} e^{-4\pi\mathrm{i}} = U_{\beta^i}$ so that $U_{\beta^i}^2 = U_{\beta^i}U_{-\beta^i} = \varepsilon(\beta^i,-\beta^i)B^{-1}_{\beta^i,-\beta^i} = 2^{-8}$. Thus there are two possible actions of $U_{\beta^i}$:

$$U_{\beta^i} = \pm 2^{-4} = \pm\frac{1}{16}.$$

Using the $\sigma$-invariance, we then have the following:

$$\begin{aligned}
U_{\alpha_{n-i+1}} &= U_{\alpha_i} e^{2\pi\mathrm{i}\left(-\frac{1}{2}+\frac{1}{2}\alpha^i\right)} = -U_{\alpha_i} e^{\pi\mathrm{i}\alpha^i}, \\
U_{\alpha^i} &= \varepsilon(\alpha_i,\alpha_{n-i+1})B_{\alpha_i,\alpha_{n-i+1}}U_{\alpha_i}U_{\alpha_{n-i+1}} = -4U_{\alpha_i}^2 e^{\pi\mathrm{i}\alpha^i}, \\
U_{-\alpha_{n-i+1}} &= \varepsilon(\alpha_{n-i+1},-\alpha_{n-i+1})B^{-1}_{\alpha_{n-i+1},-\alpha_{n-i+1}}U^{-1}_{\alpha_{n-i+1}} = \frac{1}{4}U^{-1}_{\alpha_i}e^{\pi\mathrm{i}\alpha^i}, \\
U_{\beta^i} &= \varepsilon(\alpha_i,-\alpha_{n-i+1})B_{\alpha_i,-\alpha_{n-i+1}}U_{\alpha_i}U_{-\alpha_{n-i+1}} = \frac{1}{16}e^{\pi\mathrm{i}\alpha^i}.
\end{aligned}$$

Consider the vector space $P(n) = \mathbb{C}[q_1, q_1^{-1}, \ldots, q_{l+1}, q_{l+1}^{-1}]$, where $l = \frac{n-1}{2}$. Since each nontrivial action is determined by $U_{\alpha_i}$, for $i \le l{+}1$, let $U_{\alpha_i}$ act as multiplication by $q_i(-1)^{q_{i+1}\frac{\partial}{\partial q_{i+1}}}$ for $i < l{+}1$ and $U_{\alpha_{l+1}}$ act as multiplication by $q_{l+1}$. Note that these actions ensure that $q_1, \ldots, q_{l+1}$ are commuting variables since the operators $U_{\alpha_i}$ and $U_{\alpha_{i+1}}$ anticommute. In order to determine the action of $e^{\pi\mathrm{i}\alpha_{(0)}}$, consider the commutation relation with $U_\gamma$:

$$e^{\pi\mathrm{i}\alpha_{(0)}}U_\gamma e^{\pi\mathrm{i}\alpha_{(0)}} = e^{\pi\mathrm{i}(\alpha|\gamma)}U_\gamma.$$

For $\gamma = \alpha_i$, we have $e^{\pi i \alpha^i_{(0)}} q_i e^{-\pi i \alpha^i_{(0)}} = q_i$. Thus $e^{\pi i \alpha^i_{(0)}} q_i^n = q_i^n e^{\pi i \alpha^i_{(0)}}(1)$. Since $e^{2\pi i \alpha^i_{(0)}} = 1$, we must have $e^{\pi i \alpha^i_{(0)}}(1) = \pm 1$ so that $e^{\pi i \alpha^i_{(0)}} q_i^n = \pm q_i^n$. Similarly $e^{\pi i \alpha^i_{(0)}} q_{i+1}^n = \mp q_{i+1}^n$.

Thus on the space $P(n)$ we have the following for $i \leq l$:

$$\begin{aligned} U_{\alpha_{l+1}} &= q_{l+1} \\ U_{\alpha_i} &= q_i (-1)^{q_{i+1} \frac{\partial}{\partial q_{i+1}}}, \\ U_{\alpha_{n-i+1}} &= -U_{\alpha_i} e^{\pi i \alpha^i_{(0)}} = \mp q_i, \\ U_{\alpha^i} &= = -4U_{\alpha_i}^2 e^{\pi i \alpha^i_{(0)}} = \mp 4 q_i^2, \\ U_{\beta^i} &= = \frac{1}{16} e^{-\pi i \alpha^i_{(0)}} = \pm \frac{1}{16}. \end{aligned}$$

The automorphism $\sigma$ acts on each of these modules. To determine the action of $\sigma$, we calculate $\sigma\left(p^n \prod_{i=1}^l q_i^{m_i}\right) = \sigma\left(U_{\alpha_{l+1}}^n \prod_{i=1}^l U_{\alpha_i}^{m_i} \cdot 1\right)$ on the module $P_{\chi_-}(n)$ corresponding to the character $\chi_-(e_{\beta^i}) = -1$ for all $i$. Since

$$\sigma\left(U_{\alpha_{l+1}}^n \prod_{i=1}^l U_{\alpha_i}^{m_i} \cdot 1\right) = p^n \prod_{i=1}^l q_i^{m_i},$$

the automorphism $\sigma$ acts as the identity on $P_{\chi_-}(n)$. For other characters $\chi$, $\sigma$ is determined by

$$\sigma\left(U_{\alpha_i} \cdot 1\right) = U_{\alpha_{n-i+1}} \cdot 1 = \begin{cases} q_i, & \chi(e_{\beta^i}) = -1 \\ -q_i, & \chi(e_{\beta^i}) = 1 \end{cases}.$$

Thus $P_\chi(n)$ decomposes into two eigenspaces of $\sigma$ with eigenvalues $\pm 1$. The $+1$-eigenspace $P_\chi(n)^+$ is generated by products $p^n \prod_{i=1}^l q_i^{m_i}$, where $\sum m_j$ is even for each $j$ with $\chi(e_{\beta^j}) = 1$. The $-1$-eigenspace $P_\chi(n)^-$ is generated by products $p^n \prod_{i=1}^l q_i^{m_i}$, where $\sum m_j$ is odd for each $j$ with $\chi(e_{\beta^j}) = 1$.

The entire $\sigma$-twisted $V_Q$-module is then $M(\sigma) \otimes P_\chi(n)$. Since $M(\sigma)$ itself decomposes into $\pm 1$-eigenspaces of $\sigma$, there are a total of $2 \cdot 2^l = 2^{l+1}$ orbifold modules of twisted type:

$$M(\sigma)^\pm \otimes P_{\chi_-}(n),$$

$$\left(M(\sigma)^\pm \otimes P_\chi(n)^+\right) \bigoplus \left(M(\sigma)^\mp \otimes P_\chi(n)^-\right), \quad \chi \neq \chi_-.$$

We now present a correspondence between the two constructions. It is clear that each construction produces the same number of orbifold modules of untwisted and twisted type and also that the orbifold modules of twisted type both correspond to the same set of characters. The following lemma will be used to identify the orbifold modules of untwisted type.

**Lemma 5.5.7** *The orbifold module $V_{\lambda_1+Q}$ can be identified with the orbifold module with representative $V_{\frac{1}{2}\alpha_{l+1}+L_+} \otimes V_{\frac{1}{2}\mu_l+L_-}$.*

**Proof** We first show that $\sigma\left(\frac{1}{2}\mu_1\right) = \frac{1}{2}\mu_l - \frac{1}{2}\sum\beta^i$:

$$\begin{aligned}
\tfrac{1}{2}\mu_1 &= \frac{1}{2(l+1)}\sum_{i<l+1}(l-i+1)\beta^i \\
&= \frac{1}{2}\sum_{i<l+1}\beta^i - \frac{1}{2(l+1)}\sum_{i<l+1} i\beta^i \\
&\mapsto -\frac{1}{2}\sum_{i<l+1}\beta^i + \frac{1}{2(l+1)}\sum_{i<l+1} i\beta^i \\
&= -\frac{1}{2}\sum_{i<l+1}\beta^i + \frac{1}{2}\mu_l.
\end{aligned}$$

Now $\lambda_1$ can be written as follows:

$$\begin{aligned}
\lambda_1 &= \frac{1}{n+1}\sum_i (n-i+1)\alpha_i \\
&= \frac{1}{2(n+1)}\sum_{i<l+1}\left((n-i+1)\left(\alpha^i+\beta^i\right) + (l+1-i)\left(\alpha^{l-i+1}-\beta^{l-i+1}\right)\right) + \frac{n-l}{n+1}\alpha_{l+1} \\
&= \frac{1}{2(n+1)}\sum_{i<l+1}\left((n+1)\alpha^i + (n-2i+1)\beta^i\right) + \frac{1}{2}\alpha_{l+1} \\
&= \frac{1}{2}\sum_{i<l+1}\alpha^i + \frac{1}{n+1}\sum_{i<l+1}(l+1-i)\beta^i + \frac{1}{2}\alpha_{l+1} \\
&= \frac{1}{2}\sum_{i<l+1}\alpha^i + \frac{1}{2}\mu_1 + \frac{1}{2}\alpha_{l+1}.
\end{aligned}$$

Due to (5.52), the orbifold module with representative corresponding to $\lambda_1$ will have another summand with representative corresponding to each $\alpha^i$ replaced with $\beta^i$. Thus, since

$$\sigma : \frac{1}{2}\mu_1 + \frac{1}{2}\sum_{i<l+1}\beta^i + \frac{1}{2}\alpha_{l+1} \mapsto \frac{1}{2}\mu_l + \frac{1}{2}\alpha_{l+1},$$

we have that $V_{\lambda_1+Q}$ is isomorphic to the orbifold module with representative $V_{\frac{1}{2}\alpha_{l+1}+L_+} \otimes V_{\frac{1}{2}\mu_l+L_-}$.

Recall the space $Q^*/Q$ for $A_n$ is cyclic generated by $\lambda_1 + Q$. This can easily be seen by writting each root as $\alpha_i = v_i - v_{i+1}$, where $v_1, \ldots, v_{n+1}$ is an orthonormal basis for an $(n+1)$-

dimensional vector space (cf. the end of Section 2.2). Then the fundamental dominant weights are given as $\lambda_i = v_1 + \cdots + v_i$. Using induction and that $\lambda_{j+1} = \lambda_j + (v_{j+1} - v_1) + v_1$, we also have

$$\begin{aligned}\lambda_{j+1} + Q &= \lambda_j + \lambda_1 + Q \\ &= (j+1)\lambda_1 + Q.\end{aligned}$$

From these identities, other orbifold modules can be identified using Lemma 5.5.7 and that $\sigma : V_{\lambda_i+Q} \to V_{\lambda_{n-i+1}+Q}$. From the proof of Lemma 5.5.7 and that $Q^*/Q$ is cyclic, it is clear that the module $V_{\lambda_j+Q}$ can be identified with the orbifold module with representative $V_{\frac{j}{2}\alpha_{l+1}+L_+} \otimes V_{\frac{j}{2}\mu_l+L_-}$, where $j = 1, \ldots, l$, and that $V^{\pm}_{\lambda_{l+1}+Q}$ can be identified with the orbifold module with representative $V_{\frac{l+1}{2}\alpha_{l+1}+L_+} \otimes V^{\pm}_{\frac{l+1}{2}\mu_l+L_-}$. This completes the correspondence of orbifold modules of untwisted type.

To illustrate the correspondence of the orbifold modules of twisted type, consider the module of twisted type presented in Example 5.5.6. Then we have the following correspondence between eigenspaces:

$$\begin{aligned}M(\sigma)^{\pm} \otimes P_{\chi}(n)^{+} &\simeq \left(V_{L_+} \oplus V_{\frac{\alpha^3}{2}+L_+} \oplus V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \oplus V_{\frac{\alpha^1+\alpha^2+\alpha^3}{2}+L_+}\right) \otimes V_{L_-}^{T_\chi,\pm}, \\ M(\sigma)^{\mp} \otimes P_{\chi}(n)^{-} &\simeq \left(V_{\frac{\alpha^1}{2}+L_+} \oplus V_{\frac{\alpha^2}{2}+L_+} \oplus V_{\frac{\alpha^1+\alpha^3}{2}+L_+} \oplus V_{\frac{\alpha^2+\alpha^3}{2}+L_+}\right) \otimes V_{L_-}^{T_\chi,\mp}.\end{aligned}$$

## 5.6 The Root Lattice $A_n$, $n$ even

Consider the $A_n$ simple roots $\{\alpha_1, \ldots, \alpha_n\}$, where $n$ is even. The nondegenerate symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ is given by

$$((\alpha_i|\alpha_j))_{i,j} = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The associated even lattice is $Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$. Throughout this section, set $l = \frac{n}{2}$. Consider the Dynkin diagram automorphism $\sigma : \alpha_i \longleftrightarrow \alpha_{n-i+1}$, which is also an automorphism of $Q$. For

$i < l + 1$, set

$$\alpha^i = \alpha_i + \alpha_{n-i+1}, \tag{5.47}$$
$$\beta^i = \alpha_i - \alpha_{n-i+1}. \tag{5.48}$$

Then $\alpha^i$ and $\beta^i$ are eigenvectors for $\sigma$ with eigenvalues 1 and $-1$, respectively. Products between these elements are as follows:

$$\begin{array}{ll} (\alpha^i|\alpha^i) = 4 = (\beta^i|\beta^i), & i < l, \\ (\alpha^l|\alpha^l) = 2, \quad (\beta^l|\beta^l) = 6, & \\ (\alpha^i|\alpha^{i+1}) = -2 = (\beta^i|\beta^{i+1}), & i = 1, \ldots, l-1, \\ (\alpha^i|\alpha^j) = 0 = (\beta^i|\beta^j), & \text{otherwise}, \\ (\alpha^i|\beta^j) = 0. & \end{array}$$

In order to determine $\bar{Q}$, we find conditions for which $\gamma = \sum_{i=1}^n m_i \alpha_i \in Q$ satisfies $(\gamma|\sigma\gamma) \in 2\mathbb{Z}$. Since

$$\begin{aligned} (\gamma|\sigma\gamma) &= \sum_{i=1}^n m_i m_{n-j+1} (\alpha_i|\alpha_j) \\ &= 2\sum_{i=1}^{l-1} (m_i m_{n-i} + m_{n-i+1} m_{i+1})(\alpha_i|\alpha_{i+1}) + (m_l + m_{l+1})(\alpha_l|\alpha_{l+1}) \\ &= m_l + m_{l+1} \mod 2\mathbb{Z}, \end{aligned}$$

we have that

$$\begin{aligned} \bar{Q} &= \left\{ \sum_{i=1}^n m_i \alpha_i \,\middle|\, m_l = m_{l+1} \mod 2\mathbb{Z} \right\} \\ &= \sum_{i=1}^{l-1} \mathbb{Z}\alpha_i + \mathbb{Z}\alpha^l + \mathbb{Z}\beta^l + \sum_{i=l+2}^n \mathbb{Z}\alpha_i. \end{aligned}$$

Therefore

$$L_+ = \sum_{i=1}^l \mathbb{Z}\alpha^i, \tag{5.49}$$
$$L_- = \sum_{i=1}^l \mathbb{Z}\beta^i. \tag{5.50}$$

The cosets $\bar{Q}/L$ are in correspondence with $\{0,1\}$-valued $(l-1)$-tuples via

$$(a_1, \ldots, \alpha_{l-1}) \longleftrightarrow \sum_{i=1}^{l-1} a_i \alpha_i + L \tag{5.51}$$

so that $|\bar{Q}/L| = 2^{l-1}$. Hence by Proposition 4.2.2, we have that

$$V_Q^\sigma \simeq \bigoplus_{(b_1,\ldots,b_{l-1})} \left( V_{\frac{1}{2}\sum b_i \alpha^i + L_+} \otimes V^+_{\frac{1}{2}\sum b_i \beta^i + L_-} \right), \tag{5.52}$$

where $b_i \in \{0,1\}$ and there are $2^{l-1}$ summands.

In order to compute the $V_L^\sigma$-modules, we first determine $L_+^*/L_+$ using the Gram matrix for $L_+$. Ordering the basis of $L_+$ as $\{\alpha^1, \ldots, \alpha^l\}$, the Gram matrix for $L_+$ is given by

$$G = \begin{pmatrix} 4 & -2 & & & \\ -2 & 4 & \ddots & & \\ & \ddots & \ddots & -2 & \\ & & -2 & 4 & -2 \\ & & & -2 & 2 \end{pmatrix}.$$

The inverse is given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \\ \frac{1}{2} & 1 & 1 & 1 & 1 & \cdots \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \cdots \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & 2 & \cdots \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

**Lemma 5.6.1** *The group $L_+^*/L_+$ is generated by the set*

$$\left\{ L_+, \frac{1}{2}\alpha^1 + L_+, \ldots, \frac{1}{2}\alpha^l + L_+ \right\} \tag{5.53}$$

*and $|L_+^*/L_+| = \det G = 2^l$.*

**Proof** It is clear that the elements in (5.53) are in $L_+^*/L_+$ and form a linearly independent set. Since the operation is addition modulo $L_+$, each nonintegral fraction in $G^{-1}$ can be replaced with $\frac{1}{2}$. Then the columns of $G^{-1}$ can be obtained as linear combinations of the elements in (5.53).

Each $V_{L_+}$-module can therefore be represented by a $\{0,1\}$-valued $l$-tuple given by

$$(b_1, \ldots, b_l) \longrightarrow \frac{1}{2}\left(b_1\alpha^1 + \cdots + b_l\alpha^l\right) + L_+.$$

We determine $L_-^*/L_-$ using the Gram matrix for $L_-$. Ordering the basis of $L_-$ as $\{\beta^1, \ldots, \beta^l\}$, the Gram matrix for $L_-$ is given by

$$M = \begin{pmatrix} 4 & -2 & & & \\ -2 & 4 & \ddots & & \\ & \ddots & \ddots & -2 & \\ & & -2 & 4 & -2 \\ & & & -2 & 6 \end{pmatrix}.$$

We now describe the dual basis of $L_-$. Since each entry of $M$ is even, it is sufficient to find a dual basis for $M' = \frac{1}{2}M$. The inverse of $M'$ is given by

$$\frac{1}{2l+1}\begin{pmatrix} 2l-1 & 2l-3 & 2l-5 & 2l-7 & \cdots & 1 \\ 2l-3 & 2(2l-3) & 2(2l-5) & 2(2l-7) & \cdots & 2 \\ 2l-5 & 2(2l-5) & 3(2l-5) & 3(2l-7) & \cdots & 3 \\ 2l-7 & 2(2l-7) & 3(2l-7) & 4(2l-7) & \cdots & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \ldots & l \end{pmatrix}.$$

We also have that $\det M' = 2l+1$ so that $\det M = 2^l(2l+1)$.

**Proposition 5.6.2** *A dual basis for $L_-$ is given by the elements*

$$\frac{1}{2}\mu_i = \frac{1}{2(2l+1)}\left((2l-2i+1)\beta^1 + \cdots + i(2l-2i+1)\beta^i + i(2l-2i-1)\beta^{i+1} + \cdots + i\beta^l\right),$$

*where $i = 1, \ldots, l$.*

**Proof** We show that $\left(\beta_i \,\middle|\, \frac{1}{2}\mu_j\right) = \delta_{ij}$ in cases:

$$\begin{aligned}
i \neq l : \left(\beta^i \,\middle|\, \frac{1}{2}\mu_i\right) &= \frac{-2(i-1)(2l-2i+1)+4i(2l-2i+1)-2i(2l-2i-1)}{2(2l+1)} = 1,\\
\left(\beta^l \,\middle|\, \frac{1}{2}\mu_l\right) &= \frac{1}{2(2l+1)}\left(-2(l-1)+6l\right) = 1,\\
i < j : \left(\beta^i \,\middle|\, \frac{1}{2}\mu_j\right) &= \frac{2l-2i+1}{2(2l+1)}\left(-2(i-1)+4i-2(i+1)\right) = 0,\\
i > j : \left(\beta^i \,\middle|\, \frac{1}{2}\mu_j\right) &= \frac{j\left(-2(2l-2(i-1)+1)+4(2l-2i+1)-2(2l-2(i+1)+1)\right)}{2(2l+1)} = 0.
\end{aligned}$$

Since

$$\begin{aligned}
\mu_i &= \frac{1}{2l+1}\left((2l-2i+1)\beta^1+\cdots+i(2l-2i+1)\beta^i+i(2l-2i-1)\beta^{i+1}+\cdots+i\beta^l\right)\\
&= \frac{1}{2l+1}\left(-2i\beta^1-\cdots-2i^2\beta^i+i(2l-2(i+1)+1)\beta^{i+1}+\cdots+i\beta^l\right) \mod L_-\\
&= \frac{i}{2l+1}\sum_{i=1}^{l}(2l-2i+1)\beta^i \mod L_-\\
&= i\mu_1 \mod L_-,
\end{aligned}$$

the fundamental group $L_-^*/L_-$ is generated by the elements $\frac{1}{2}\mu_1 + L_-$ and $\frac{1}{2}\beta^i + L_-$, where $i = 1, \ldots, l$. Thus we have that

$$L_-^*/L_- = \left\{ \frac{1}{2}a\mu_i + \frac{1}{2}\sum_{j=1}^{l} b_j\beta^j + L_- \,\middle|\, a \in \{0,\ldots,2l\}, b_j \in \{0,1\} \right\}.$$

The $\sigma$-invariant elements of $L_-^*/L_-$ are those for which $a = 0$. Thus there are $2 \cdot 2^l = 2^{l+1}$ distinct irreducible $V_{L_-}^+$-modules of untwisted type corresponding to the $l$-tuples $(b_1, \ldots, b_l)$, where $b_i \in \{0,1\}$ and $a = 0$. The remaining elements in $L_-^*/L_-$ are identified by $\sigma$ in pairs. Hence there are

$$\frac{2^l(2l+1)-2^l}{2} = 2^l l$$

distinct irreducible $V_{L_-}^+$-modules corresponding to the $l$-tuples $(b_1, \ldots, b_l)$, where $b_i \in \{0,1\}$ and $a \neq 0$. Hence the total number of distinct irreducible $V_{L_-}^+$-modules of untwisted type is $2 \cdot 2^l + 2^l l = 2^l(l+2)$.

**Lemma 5.6.3** *The distinct $V_{L_-}^+$-modules that decompose into eigenspaces corresponds to the set*

$$\left(L_-^*/L_-\right)^\sigma = \left\{ \frac{1}{2}\sum b_j\beta^j + L_- \,\middle|\, b_j \in \{0,1\}, j = 1,\ldots,l \right\}.$$

*The other distinct $V_{L_-}^+$-modules correspond to the set*

$$\left\{ \frac{1}{2}\mu_i + \frac{1}{2}\sum b_j\beta^j + L_- \,\middle|\, b_j \in \{0,1\}, k \le i \le l \right\},$$

*where* $k = \begin{cases} \frac{l+1}{2}, & l \ \ odd \\ \frac{l}{2}+1, & l \ \ even \end{cases}.$

**Proof** We show the following identity:

$$\sigma\left(\frac{1}{2}\mu_i\right) = \frac{1}{2}\mu_{l-i+1} + \frac{1}{2}\mu_l + \gamma_i \mod L_-,$$

where $\gamma_i = \frac{1}{2}\sum_{j=1}^l a_{ij}\beta^j$ for some $a_{ij} \in \{0,1\}$. The automorphism $\sigma$ acts on each $\mu_i$ by

$$\sigma(\mu_i) = \frac{1}{2l+1}\big((-2l+2i-1)\beta^1 + \cdots + i(-2l+2i-1)\beta^i \\ + i(-2l+2i+1)\beta^{i+1} + \cdots + i(-1)\beta^l\big)$$

The coefficient of $\beta^j$ for $j = 1,\ldots,i$ can be written as

$$j(-2l+2i-1) + j(2l+1) = 2ij$$

and the coefficient of $\beta^{i+j}$ for $j = 1,\ldots,l-i$ can be written as

$$i(-2l+2i+2j-1) + i(2l+1) = 2i(i+j).$$

Thus we have that

$$\begin{aligned}
\sigma(\mu_i) &= \frac{1}{2l+1}\left(2i\beta^1 + \cdots + 2i^2\beta^i + 2i(i+1)\beta^{i+1} + \cdots + 2il\beta^l\right) \mod L_- \\
&= \frac{2i}{2l+1}\left(\beta^1 + \cdots + i\beta^i + (i+1)\beta^{i+1} + \cdots + l\beta^l\right) \mod L_- \\
&= 2i\mu_l \mod L_-.
\end{aligned}$$

Now

$$\mu_{l-i+1} = \frac{1}{2l+1}\Big((2i-1)\beta^1 + \cdots + (l-i+1)(2i-1)\beta^{l-i+1} \\ +(l-i+1)(2i-3)\beta^{l-i+2} + \cdots + (l-i+1)\beta^l\Big).$$

The coefficient of $\beta^{l-i+1+j}$ for $j = 1, \ldots, i-1$ can be written as

$$(l-i+1)(2i-1-2j) + j(2l+1) = (2i-1)(l-i+1+j).$$

Therefore

$$\begin{aligned} \mu_{l-i+1} &= \frac{2i-1}{2l+1}\Big(\beta^1 + \cdots + i\beta^i + (i+1)\beta^{i+1} + \cdots + l\beta^l\Big) \mod L_- \\ &= (2i-1)\mu_l \mod L_- \end{aligned}$$

and hence

$$\begin{aligned} \sigma(\mu_i) &= 2i\mu_l \mod L_- \\ &= (2i-1)\mu_l + \mu_l \mod L_- \\ &= \mu_{l-i+1} + \mu_l \mod L_-. \end{aligned}$$

Thus we have that

$$\sigma\left(\frac{1}{2}\mu_i\right) = \frac{1}{2}\mu_{l-i+1} + \frac{1}{2}\mu_l + \gamma_i \mod L_-.$$

The lemma now follows.

From the proof of Corollary 4.4.3 and (5.51), the irreducible $V_L^\sigma$-modules are obtained from elements $\lambda \in L_+^*$ and $\mu \in L_-^*$ such that $(\lambda + \mu|\alpha_i) \in \mathbb{Z}$ for $i = 1, \ldots, l$.

**Lemma 5.6.4** *We have for $i < l+1$ and $j < l$,*

$$\begin{aligned}
\left(\frac{1}{2}\mu_i\middle|\alpha_j\right) &= \frac{1}{2}\delta_{ij},\\
\left(\frac{1}{2}\beta^i\middle|\alpha_j\right) &= \left(\frac{1}{2}\alpha^i\middle|\alpha_j\right) = \delta_{ij} - \frac{1}{2}\left(\delta_{i,j-1} + \delta_{i,j+1}\right),\\
\left(\frac{1}{2}\beta^l\middle|\alpha_j\right) &= -\frac{1}{2}\delta_{j,l-1} + \frac{3}{2}\delta_{jl},\\
\left(\frac{1}{2}\alpha^l\middle|\alpha_j\right) &= -\frac{1}{2}\delta_{j,l-1} + \frac{1}{2}\delta_{jl}.
\end{aligned}$$

**Proof** The statement is clear from the following calculations:

$$\begin{aligned}
(\beta^i|\alpha_j) &= (\alpha^i|\alpha_j) = -\delta_{i,j-1} + 2\delta_{ij} - \delta_{i,j+1},\\
(\beta^l\alpha_j) &= -\delta_{j,l-1} + 3\delta_{jl},\\
(\alpha^l|\alpha_j) &= -\delta_{j,l-1} + \delta_{jl},\\
(\mu_i|\alpha_i) &= \frac{1}{2l+1}\left(-(i-1)(2l-2i+1) + 2i(2l-2i+1) - i(2l-2i-1)\right) = 1,\\
j < i : (\mu_i|\alpha_j) &= \frac{2l-2i+1}{2l+1}\left(-(j-1) + 2j - (j+1)\right) = 0,\\
j > i : (\mu_i|\alpha_j) &= \frac{i}{2l+1}(-(2l-2i+1-2(j-1-i))\\
&\quad +2(2l-2i+1-2(j-i)) - (2l-2i+1-2(j+1-i))) = 0.
\end{aligned}$$

**Lemma 5.6.5** *There are $l+4$ $V_Q^\sigma$-modules of untwisted type, where $l = \frac{n}{2}$.*

**Proof** By Proposition 5.6.4, the only element in $L_+^*/L_+$ that has integral products with each $\alpha_i$ is the trivial coset $L_+$. Therefore all orbifold modules can be represented using only $V_{L_+}$. Set

$$s = \begin{cases} l, & l \text{ odd} \\ l-1, & l \text{ even} \end{cases} \quad \text{and} \quad t = \begin{cases} l-1, & l \text{ odd} \\ l, & l \text{ even} \end{cases}. \tag{5.54}$$

Also set

$$\gamma = \frac{1}{2}\left(\beta^1 + \beta^3 + \cdots + \beta^s\right). \tag{5.55}$$

Then we have the following list of elements which have integral products with each $\alpha_i$ for

$i = 1, \ldots, l-1$:

$$\tfrac{1}{2}\mu_{2k+1} + \tfrac{1}{2}\left(\beta^{2k+2} + \beta^{2k+4} + \cdots + \beta^{t}\right) + \gamma, \qquad k = 0, \ldots, q = \begin{cases} \frac{l-1}{2}, & l \text{ odd} \\ \frac{l-2}{2}, & l \text{ even} \end{cases},$$

$$\tfrac{1}{2}\mu_{2k} + \tfrac{1}{2}\left(\beta^{1} + \beta^{3} + \cdots + \beta^{2k-1}\right), \qquad k = 1, \ldots, r = \begin{cases} \frac{l-1}{2}, & l \text{ odd} \\ \frac{l}{2}, & l \text{ even} \end{cases},$$

along with the cosets $\gamma + L_-$ and $L_-$. The only $\sigma$-invariant elements in this list are $\gamma + L_-$ and $L_-$ so that the two orbifold modules with representatives $V_{L_+} \otimes V_{L_-}$ and $V_{L_+} \otimes V_{\gamma+L_-}$ will each split into eigenspaces with eigenvalues $\pm 1$. There are $l$ elements that have integral products with each $\alpha_i$ that are not $\sigma$-invariant. Thus there are a total of $l + 2(2) = l + 4$ orbifold modules of untwisted type.

**Theorem 5.6.6** *The list of $l+4$ $V_Q^\sigma$-modules of untwisted type is equivalent to the following list:*

$$\bigoplus_{(b_1,\ldots,b_{l-1})} \left(V_{\frac{1}{2}\sum b_i\alpha^i + L_+} \otimes V^{\pm}_{\frac{1}{2}\sum b_i\beta^i + L_-}\right)$$

$$\bigoplus_{(b_1,\ldots,b_{l-1})} \left(V_{\frac{1}{2}\sum b_i\alpha^i + L_+} \otimes V_{k\mu_l + \frac{1}{2}\sum b_i\beta^i + L_-}\right), \quad k = 1, \ldots, l,$$

$$\bigoplus_{(b_1,\ldots,b_{l-1})} \left(V_{\frac{1}{2}\sum b_i\alpha^i + L_+} \otimes V^{\pm}_{\gamma + \frac{1}{2}\sum b_i\beta^i + L_-}\right)$$

*where $\gamma$ is given in* (5.55) *and $b_i \in \{0, 1\}$.*

**Proof** It is sufficient to show the following identities among the cosets of $L_-^*/L_-$:

$$\frac{1}{2}\mu_{2k} + \frac{1}{2}\left(\beta^{1} + \beta^{3} + \cdots + \beta^{2k-1}\right) + L_- = 2k\mu_l + L_-,$$

$$\frac{1}{2}\mu_{2k+1} + \frac{1}{2}\left(\beta^{2k+2} + \beta^{2k+4} + \cdots + \beta^{t}\right) + \gamma + L_- = (2k+1)\mu_l + L_-.$$

For the first identity, we consider separately the coefficients of $\beta^i$ for $i < 2k$ and $i \geq 2k$. For $i < 2k$ we also consider separately $i$ even and $i$ odd. For $i < 2k$ odd, the coefficient of $\beta^i$ can be written as

$$i(2l-4k+1) = i(2l+1) - 4ik = (i-1)(2l+1) - 4ik + (2l+1).$$

Since $i-1$ is even, we may take the coefficient to be $-4ik+(2l+1)$. For $i < 2k$ even, the coefficient of $\beta^i$ can be written as

$$i(2l-4k+1) = i(2l+1) - 4ik.$$

Since $i$ is even, we may take the coefficient to be $-4ik$. For $i \geq 2k$, the coefficient of $\beta^i$ can be written as

$$2k(2l-2i+1) = 2k(2l+1) - 4ik.$$

Hence we may take the coefficient to be $-4ik$. Thus we have

$$\begin{aligned}
\frac{1}{2}\mu_{2k} + \frac{1}{2}\left(\beta^1+\beta^3+\ldots+\beta^{2k-1}\right) + L_- &= \frac{-4k}{2(2l+1)}\sum_{i=1}^{l}(i\beta^i) + L_- \\
&\mapsto \frac{2k}{2l+1}\sum_{i=1}^{l} i\beta^i + L_- \\
&= 2k\mu_l + L_-.
\end{aligned}$$

For the second identity, we consider separately the coefficients of $\beta^i$ for $i < 2k+1$ and $i \geq 2k+1$. For $i < 2k+1$ we also consider separately $i$ even and $i$ odd. For $i < 2k+1$ odd, the coefficient of $\beta^i$ can be written as

$$i(2l-2(2k+1)+1) = (i-1)(2l+1) - 2(2k+1)i + (2l+1).$$

Since $i-1$ is even, we may take the coefficient to be $-2(2k+1)i+(2l+1)$. For $i < 2k+1$ even, the coefficient of $\beta^i$ can be written as

$$i(2l-2(2k+1)+1) = i(2l+1) - 2(2k+1)i.$$

Since $i$ is even, we may take the coefficient to be $-2(2k+1)i$. For $i \geq 2k+1$, the coefficient of $\beta^i$ can be written as

$$(2k+1)(2l-2i+1) = 2k(2l+1) - 2(2k+1)i + (2l+1).$$

Hence we may take the coefficient to be $-2(2k+1)i+(2l+1)$. Thus we have

$$\begin{aligned}
\frac{1}{2}\mu_{2k+1}+L_- &= \frac{-2(2k+1)}{2(2l+1)}\sum_{i=1}^{l} i\beta^i + \frac{1}{2}\left(\beta^1+\beta^3+\ldots+\beta^{2k-1}\right)+\frac{1}{2}\sum_{i\geq 2k+1}\beta^i + L_- \\
&\mapsto \frac{2k+1}{2l+1}\sum_{i=1}^{l} i\beta^i + \frac{1}{2}\left(\beta^1+\beta^3+\ldots+\beta^{2k-1}\right)+\frac{1}{2}\sum_{i\geq 2k+1}\beta^i + L_- \\
&= (2k+1)\mu_l + \frac{1}{2}\left(\beta^{2k+2}+\beta^{2k+4}+\ldots+\beta^t\right)+\gamma+L_-.
\end{aligned}$$

This completes the proof.

For convenience, the modules in Theorem 5.6.6 are referred to by using the summand corresponding to the $(l-1)$-tuple $(0,\ldots,0)$.

The irreducible twisted $V_{L_-}$-modules are $V_{L_-}^{T_\chi}$, where $T_\chi$ is an irreducible $\hat{L}_-/K$-module with central character $\chi$. The irreducible $V_Q^\sigma$-modules of twisted type are obtained using the fusion rules for $V_{L_-}^+$ with $M_1 = V^+_{\frac{1}{2}\sum b_i\beta^i+L_-}$, where $i=1,\ldots,l$ and $b_i\in\{0,1\}$ (cf. Definition 3.3.16). Recall each irreducible character $\chi : 2L_-^*\cap L_- \longrightarrow \{\pm 1\}$ for which $\chi(e_{-\alpha})=\chi(e_\alpha)$ can be written as

$$\chi(e_\alpha) = (-1)^{(\alpha|\mu)}$$

for some $\mu\in(2L_-^*\cap L_-)^*$. Since

$$\begin{aligned}
\left(\sum_{i=1}^{l} b_i\beta^i \middle| \frac{1}{2}\sum_{j=1}^{l} b_j\beta^j\right) &= \sum_{i<j} b_ib_j(\beta^i|\beta^j)+\frac{1}{2}\sum_{i=1}^{l} b_i^2(\beta^i|\beta^i) \\
&= -2\sum_{i<j} b_ib_j + 2\sum_{i=1}^{l} b_i^2 + 3b_l^2,
\end{aligned}$$

we have that

$$\chi_\mu^{(\frac{1}{2}\sum b_i\beta^i)}(e_{\sum b_i\beta^i}) = \begin{cases} \chi_\mu(e_{\sum b_i\beta^i}), & b_l = 0 \\ -\chi_\mu(e_{\sum b_i\beta^i}), & b_l = 1 \end{cases}. \tag{5.56}$$

Hence every module $V^+_{\frac{1}{2}b_i\beta^i+L_-}$ sends $V_{L_-}^{T_\chi}$ to $V_{L_-}^{T_\chi}$, for $i<l$, and $V^+_{\frac{1}{2}b_l\beta^l+L_-}$ sends $V_{L_-}^{T_\chi}$ to

$V_{L_-}^{T_{-\chi}}$. Since $c_\chi\left(\frac{1}{2}\sum b_i\beta^i\right) = \chi(e_{\sum b_i\beta^i})$ and linear characters are homomorphisms, we have that

$$c_\chi\left(\frac{1}{2}\sum_{i=1}^{l} b_i\beta^i\right) = (-1)^{b_l}\prod_{i=1}^{l}\chi(e_{b_i\beta^i}), \tag{5.57}$$

where $b_i \in \{0,1\}$ and $\chi(e_0) = 1$. Thus each irreducible $V_Q^\sigma$-module of twisted type corresponding to a choice for $\chi$ is obtained in the following way. For each element $\gamma_- = \frac{1}{2}\sum b_i\beta^i$, the eigenspaces are determined by computing the products in (5.57). There are a total of $2\cdot 2^l = 2^{l+1}$ irreducible $V_Q^\sigma$-modules of twisted type.

**Example 5.6.7** *As an example of composing orbifold modules of twisted type, consider the case $l = 2$ and the character $\chi$ defined on the basis of $L_-$ by $\chi(e_{\beta^1}) = -1$, and $\chi(e_{\beta^2}) = 1$. Then the corresponding orbifold modules of twisted type are*

$$\left(V_{L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\bigoplus\left(V_{\frac{\alpha^1}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right)\bigoplus\left(V_{\frac{\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_{-\chi},\pm}\right)\bigoplus\left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_{-\chi},\mp}\right).$$

We now construct the orbifold modules using Section 3.3.2. The 2-cocycle $\varepsilon$ satisfies the following:

$$\begin{aligned}
\varepsilon(\alpha^l,\alpha^l) &= (-1)^{2\binom{2+1}{2}} = -1,\\
\varepsilon(\beta^l,\beta^l) &= (-1)^{6\binom{6+1}{2}} = -1,\\
\varepsilon(\alpha^l,\beta^l)\varepsilon(\beta^l,\alpha^l) &= 1,\\
\varepsilon(\alpha_i,\alpha_i) &= (-1)^{2\binom{2+1}{2}} = -1,\\
\varepsilon(\alpha_i,\alpha_{i+1})\varepsilon(\alpha_{i+1},\alpha_i) &= (-1)^{-1} = -1,\\
\varepsilon(\alpha_i,\alpha_j)\varepsilon(\alpha_j,\alpha_i) &= 1 \quad \text{otherwise,}
\end{aligned}$$

where $i,j < l$. For $i < l-1$, set

$$\begin{aligned}
\varepsilon(\alpha_i,\alpha_{i+1}) &= \varepsilon(\alpha_{n-i+1},\alpha_{n-i}) = 1,\\
\varepsilon(\alpha_{i+1},\alpha_i) &= \varepsilon(\alpha_{n-i},\alpha_{n-i+1}) = -1,\\
\varepsilon(\alpha^l,\beta^l) &= \varepsilon(\beta^l,\alpha^l) = 1.
\end{aligned}$$

Then $\varepsilon$ is $\sigma$-invariant on $\bar{Q}$ so that we may take $\eta$ to be trivial, that is, $\eta(\gamma) = 1$ for all $\gamma\in\bar{Q}$.

**Proposition 5.6.8** *The fundamental weights of $\bar{Q}$ are given by*

$$\begin{aligned}\lambda_i &= \frac{1}{2(2l+1)}(2(2l-i+1)\alpha_1 + \cdots + 2i(2l-i+1)\alpha_i + 2i(2l-i)\alpha_{i+1} \\ &\qquad +2i(2l-i-1)\alpha_{i+2} + \cdots + 2i(l+2)\alpha_{l-1} + i(2l+1)\alpha^l + i\beta^l \\ &\qquad +2i(l-1)\alpha_{l+2} + \cdots + 2i\alpha_n)\end{aligned}$$

*for $i = 1, \ldots, l-1$,*

$$\begin{aligned}\lambda_l &= \frac{1}{2}(\alpha_1 + \ldots + l\alpha^l + (l-1)\alpha_{l+2} + \ldots + \alpha_n), \\ \lambda_{l+1} &= \frac{1}{2(2l+1)}(\alpha_1 + \ldots + (l-1)\alpha_{l-1} + l\beta^l - (l-1)\alpha_{l+2} - \ldots - \alpha_n), \\ \lambda_i &= \frac{1}{2(2l+1)}(2(2l-i+1)\alpha_1 + \cdots + 2(2l-i+1)(l-1)\alpha_{l-1} + (2l+1)(2l-i+1)\alpha^l \\ &\qquad -(2l-i+1)\beta^l + 2(2l-i+1)(l+2)\alpha_{l+2} + \cdots + 2i(2l-i+1)\alpha_i \\ &\qquad +2i(2l-i)\alpha_{i+1} + \cdots + 2i\alpha_n),\end{aligned}$$

*for $i = l+2, \ldots, n$.*

**Proof** We show that each $\lambda_i$ is a fundamental weight of $\bar{Q}$ in cases. For $i < l$ we have

$$\begin{aligned}(\lambda_1|\alpha_1) &= \frac{8l-(4l-2)}{2(2l+1)} = 1, \\ (\lambda_i|\alpha_i) &= \frac{-2(i-1)(2l-i+1) + 4i(2l-i+1) - 2i(2l-i)}{2(2l+1)} = 1, \quad i = 2, \ldots, l-1, \\ (\lambda_i|\alpha_j) &= \frac{2l-i+1}{2(2l+1)}(-2(j-1) + 4j - 2(j+1)) = 0, \quad j < i, \\ (\lambda_i|\alpha_j) &= \frac{2i}{2(2l+1)}(-2(2l-j+2) + 4(2l-j+1) - 2(2l-j)) = 0, \quad i < j < l-1, \\ (\lambda_i|\alpha_{l-1}) &= \frac{-2i(l+3) + 4i(l+2) - i(2l+1) - i}{2(2l+1)} = 0, \quad i \neq l-1, \\ (\lambda_i|\alpha^l) &= \frac{-2(l+2) + 2i(2l+1) - 2i(l-1)}{2(2l+1)} = 0, \\ (\lambda_i|\beta^l) &= \frac{-2(l+2) + 6i + 2i(l-1)}{2(2l+1)} = 0, \\ (\lambda_i|\alpha_j) &= \frac{2i(-(j-4) + 2(j-3) - (j-2))}{2(2l+1)} = 0, \quad l < j < n, \\ (\lambda_i|\alpha_n) &= \frac{-4i+4i}{2(2l+1)} = 0.\end{aligned}$$

For $\lambda_l$ and $\lambda_{l+1}$ we have

$$\begin{aligned}
(\lambda_l|\alpha^l) &= \frac{-(l-1)+2l-(l-1)}{2} = 1, \\
(\lambda_l|\beta^l) &= \frac{-(l-1)+(l-1)}{2} = 0, \\
(\lambda_l|\alpha_j) &= \frac{-(j-1)+2j-(j+1)}{2} = 0, \quad j \leq l, \\
(\lambda_l|\alpha_j) &= \frac{-(j-4)+2(j-3)-(j-2)}{2} = 0, \quad j > l+1,
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{l+1}|\beta^l) &= \frac{-(l-1)+6l-(l-1)}{2(2l+1)} = 1, \\
(\lambda_{l+1}|\alpha^l) &= \frac{-(l-1)+(l-1)}{2(2l+1)} = 0, \\
(\lambda_{l+1}|\alpha_j) &= \frac{-(j-1)+2j-(j+1)}{2(2l+1)} = 0, \quad j < l, \\
(\lambda_{l+1}|\alpha_j) &= \frac{-(j-4)+2(j-3)-(j-2)}{2(2l+1)} = 0, \quad l+1 < j < n, \\
(\lambda_{l+1}|\alpha_n) &= \frac{2-2}{2(2l+1)} = 0.
\end{aligned}$$

For $i > l+1$ we have

$$\begin{aligned}
(\lambda_{l+2}|\alpha_{l+2}) &= \frac{-(l-1)+4(l-1)(l+2)-2(l-2)(l+2)-(2l+1)(l-1)}{2(2l+1)} = 1,\\
(\lambda_i|\alpha_i) &= \frac{(-2(i-1)+4i)(2l-i+1)-2i(2l-i)}{2(2l+1)} = 1, \quad i = l+3,\ldots,n,\\
(\lambda_i|\alpha_j) &= \frac{2l-i+1}{2(2l+1)}(-2(j-1)+4j-2(j+1)) = 0, \quad j < l-1,\\
(\lambda_i|\alpha_{l-1}) &= \frac{2l-i+1}{2(2l+1)}(-2(l-2)+4(l-1)-(2l+1)+1) = 0, \quad i \neq l-1,\\
(\lambda_i|\alpha^l) &= \frac{2(2l-i+1)}{2(2l+1)}(-(l-1)+(2l+1)-(l+2)) = 0,\\
(\lambda_i|\beta^l) &= \frac{2(2l-i+1)}{2(2l+1)}(-(j-1)+2j-(j+1)) = 0,\\
(\lambda_i|\alpha_j) &= \frac{2(2l-i+1)}{2(2l+1)}(-(l-1)+(2l+1)-(l+2)), \quad l < j < i,\\
(\lambda_i|\alpha_j) &= \frac{2i}{2(2l+1)}(-(2l-j+2)+2(2l-j+1)-(2l-j)), \quad i < j < n,\\
(\lambda_i|\alpha_n) &= \frac{-4i+4i}{2(2l+1)} = 0.
\end{aligned}$$

This completes the proof.

The fundamental group $\bar{Q}^*/\bar{Q}$ is generated by $\lambda_1+\bar{Q}$ and $\lambda_l+\bar{Q}$, where $l\lambda_1 = \lambda_l+\lambda_{l+1} \mod \bar{Q}$, $(l+1)\lambda_1 = \lambda_l - \lambda_{l+1} \mod \bar{Q}$ and $(2l+1)\lambda_1 = 0 \mod \bar{Q}$. In addition, $\lambda_i = i\lambda_1 \mod \bar{Q}$ for $i = 1,\ldots,l-1$. The only $\sigma$-invariant elements of $\bar{Q}^*/\bar{Q}$ are $\bar{Q}$ and $\lambda_l+\bar{Q}$ and $\sigma$ identifies $l-1$ of the corresponding modules since $\sigma(\lambda_i) = \lambda_{n-i+1}$ for $i = 1,\ldots,l-1$. In addition, $\sigma(\lambda_{l+1}) = -\lambda_{l+1}$. Thus there are $l+4$ distinct orbifold modules of untwisted type given by

$$V_{\bar{Q}}^{\pm}, \quad V_{\lambda_1+\bar{Q}}, \quad \ldots, V_{\lambda_{l-1}+\bar{Q}}, \quad V_{\lambda_l+\lambda_{l+1}+\bar{Q}}, \quad V_{\lambda_l+\bar{Q}}^{\pm}.$$

For the orbifold modules of twisted type, the quantities that will be needed are:

$$\begin{aligned}
B_{\beta^i,-\beta^i} &= 2^8, \quad i<l,\\
B_{\beta^l,-\beta^l} &= 2^{12},\\
B_{\alpha_i,\alpha_{n-i+1}} &= 4, \quad i<l,\\
B_{\alpha_l,\alpha_{l+1}} &= 8,\\
B_{\alpha_{n-i+1},-\alpha_{n-i+1}} &= 4,\\
b_{\beta^i} &= \frac{1}{2}(0-4) = -2, \quad i<l,\\
b_{\beta^l} &= \frac{1}{2}(0-6) = -3,\\
b_{\alpha^i} &= \frac{1}{2}(4-4) = 0,\\
b_{\alpha_i} &= \frac{1}{2}\left(1-2\right) = -\frac{1}{2},\\
C_{\gamma,\theta} &= (-1)^{(\gamma|\theta)}.
\end{aligned}$$

The elements in $G_\sigma$ must satisfy $U_{\sigma\gamma} = U_\gamma e^{2\pi\mathrm{i}(b_\gamma+\gamma_0)}$. In particular, $U_{-\beta^i} = U_{\beta^i}e^{-4\pi\mathrm{i}} = U_{\beta^i}$ for $i<l$ and $U_{-\beta^l} = U_{\beta^l}e^{-6\pi\mathrm{i}} = U_{\beta^l}$. Thus $U_{\beta^i}^2 = U_{\beta^i}U_{-\beta^i} = \varepsilon(\beta^i,-\beta^i)B_{\beta^i,-\beta^i}^{-1} = 2^{-8}$ for $i<l$ and $U_{\beta^l}^2 = -2^{-12}$ so that there are two possible actions of each $U_{\beta^i}$:

$$\begin{aligned}
U_{\beta^i} &= \pm 2^{-4} \quad \text{for} \quad i<l,\\
U_{\beta^l} &= \pm 2^{-6}\mathrm{i}.
\end{aligned}$$

Using the $\sigma$-invariance, we then have the following for $i<l$:

$$\begin{aligned}
U_{\alpha_{n-i+1}} &= U_{\alpha_i}e^{2\pi\mathrm{i}\left(-\frac{1}{2}+\frac{1}{2}\alpha^i_{(0)}\right)} = -U_{\alpha_i}e^{\pi\mathrm{i}\alpha^i_{(0)}},\\
U_{\alpha^i} &= \varepsilon(\alpha_i,\alpha_{n-i+1})B_{\alpha_i,\alpha_{n-i+1}}U_{\alpha_i}U_{\alpha_{n-i+1}} = -4U_{\alpha_i}^2 e^{\pi\mathrm{i}\alpha^i_{(0)}},\\
U_{\beta^i} &= \varepsilon(\alpha_i,-\alpha_{n-i+1})B_{\alpha_i,-\alpha_{n-i+1}}U_{\alpha_i}U_{-\alpha_{n-i+1}} = \frac{1}{16}e^{-\pi\mathrm{i}\alpha^i_{(0)}}.
\end{aligned}$$

Consider the vector space $P(n) = \mathbb{C}[q_1,q_1^{-1},\ldots,q_l,q_l^{-1}]$, where $l=\frac{n-1}{2}$. Since each nontrivial action is determined by $U_{\alpha_i}$ and $U_{\alpha^l}$, let $U_{\alpha_i}$ act as multiplication by $q_i(-1)^{\frac{\partial}{\partial q_{i+1}}}$ for $i<l$ and $U_{\alpha^l}$ act as multiplication by $q_l$. Note that these actions ensure that $q_1,\ldots,q_l$ are commuting variables since the operators $U_{\alpha_i}$, $U_{\alpha_{i+1}}$ for $i<l$ and $U_{\alpha_{l-1}}$, $U_{\alpha^l}$ anticommute. In order to

determine the action of $e^{\pi i \alpha_{(0)}}$, consider the commutation relation with $U_\gamma$:

$$e^{\pi i \alpha_{(0)}} U_\gamma e^{-\pi i \alpha_{(0)}} = e^{\pi i (\alpha|\gamma)} U_\gamma.$$

For $\gamma = \alpha_i$, we have $e^{\pi i \alpha^i_{(0)}} q_i e^{-\pi i \alpha^i_{(0)}} = q_i$. Thus $e^{\pi i \alpha^i_{(0)}} q_i^n = q_i^n e^{\pi i \alpha^i_{(0)}}(1)$. Since $e^{2\pi i \alpha^i_{(0)}} = 1$, we must have $e^{\pi i \alpha^i_{(0)}}(1) = \pm 1$ so that $e^{\pi i \alpha^i_{(0)}} q_i^n = \pm q_i^n$. Similarly $e^{\pi i \alpha^i_{(0)}} q_{i+1}^n = \mp q_{i+1}^n$.

Thus on the space $P(n)$ we have the following for $i < l$:

$$\begin{aligned} U_{\alpha_i} &= q_i(-1)^{\frac{\partial}{\partial q_{i+1}}}, \\ U_{\alpha^l} &= q_l, \\ e^{\pi i \alpha^l_{(0)}} &= \pm 1, \\ U_{\alpha_{n-i+1}} &= -U_{\alpha_i} e^{\pi i \alpha^i_{(0)}} = \mp q_i, \\ U_{\alpha^i} &= = -4U_{\alpha_i}^2 e^{\pi i \alpha^i_{(0)}} = \mp 4 q_i^2, \\ U_{\beta^i} &= = \frac{1}{16} e^{-\pi i \alpha^i_{(0)}} = \pm \frac{1}{16}, \\ U_{\beta^l} &= \pm \frac{1}{64} i. \end{aligned}$$

The automorphism $\sigma$ acts on these modules. To determine the action of $\sigma$, we calculate $\sigma\left(q_l^n \prod_{i=1}^{l-1} q_i^{m_i}\right) = \sigma\left(U_{\alpha^l}^n \prod_{i=1}^{l-1} U_{\alpha_i}^{m_i} \cdot 1\right)$ on the module $P_{\chi_-}(n)$ corresponding to the character $\chi_-(e_{\beta^i}) = -1$ for all $i < l$. Since

$$\sigma\left(U_{\alpha^l}^n \prod_{i=1}^{l-1} U_{\alpha_i}^{m_i} \cdot 1\right) = q_l^n \prod_{i=1}^{l-1} q_i^{m_i},$$

the automorphism $\sigma$ acts as the identiy on $P_{\chi_-}(n)$. For other characters $\chi$, $\sigma$ is determined by

$$\sigma\left(U_{\alpha_i} \cdot 1\right) = U_{\alpha_{n-i+1}} \cdot 1 = \begin{cases} q_i, & \chi(e_{\beta^i}) = -1 \\ -q_i, & \chi(e_{\beta^i}) = 1 \end{cases},$$

where $i < l$. Thus $P_\chi(n)$ decomposes into two eigenspaces of $\sigma$ with eigenvalues $\pm 1$. The $+1$-eigenspace $P_\chi(n)^+$ is generated by products $q_l^n \prod_{i=1}^{l-1} q_i^{m_i}$, where $\sum m_j$ is even for each $j$ with $\chi(e_{\beta^j}) = 1$. The $-1$-eigenspace $P_\chi(n)^-$ is generated by products $q_l^n \prod_{i=1}^{l-1} q_i^{m_i}$, where $\sum m_j$ is odd for each $j$ with $\chi(e_{\beta^j}) = 1$.

The entire $\sigma$-twisted $V_Q$-module is then $M(\sigma) \otimes P_\chi(n)$. Since $M(\sigma)$ itself decomposes into

$\pm 1$-eigenspaces of $\sigma$, there are a total of $2 \cdot 2^l = 2^{l+1}$ orbifold modules of twisted type:

$$M(\sigma)^{\pm} \otimes P_{\chi_-}(n),$$

$$(M(\sigma)^{\pm} \otimes P_{\chi}(n)^+) \bigoplus (M(\sigma)^{\mp} \otimes P_{\chi}(n)^-), \quad \chi \neq \chi_-.$$

We now present a correspondence between the two constructions. It is clear that each construction produces the same number of orbifold modules of untwisted and twisted type and also that the orbifold modules of twisted type both correspond to the same set of characters. The following lemma will be used to identify the orbifold modules of untwisted type.

**Lemma 5.6.9** *The orbifold module $V_{\lambda_1+\bar{Q}}$ can be identified with the orbifold module with representative $V_{L_+} \otimes V_{\mu_l+L_-}$.*

**Proof** We write $\lambda_1$ as follows:

$$\begin{aligned}
\lambda_1 &= \frac{1}{2(2l+1)}(4l\alpha_1 + 2(2l-1)\alpha_2 + \cdots + 2(l+2)\alpha_{l-1} + (2l+1)\alpha^l \\
&\quad +\beta^l + 2(l-1)\alpha_{l+2} + \cdots + 2\alpha_n) \\
&= \frac{1}{2(2l+1)}(2l(\alpha^1+\beta^1) + (2l-1)(\alpha^2+\beta^2) + \cdots + (l+2)(\alpha^{l-1}+\beta^{l-1}) + (2l+1)\alpha^l \\
&\quad +\beta^l + (l-1)(\alpha^{l-1}-\beta^{l-1}) + \cdots + (\alpha^1-\beta^1)) \\
&= \frac{1}{2}\sum_{i=1}^{l}\alpha^i + \frac{1}{2(2l+1)}((2l-1)\beta^1 + (2l-3)\beta^2 + \cdots + 3\beta^{l-1} + \beta^l) \\
&\mapsto \frac{1}{2}\sum_{i=1}^{l}(\alpha^i+\beta^i) + \frac{1}{2l+1}\sum_{j=1}^{l} j\beta^j \\
&= \frac{1}{2}\sum_{i=1}^{l}(\alpha^i+\beta^i) + \mu_l
\end{aligned}$$

Due to (5.52), the orbifold module corresponding to the element $\frac{1}{2}\sum_{i=1}^{l}(\alpha^i+\beta^i)+\mu_l$ will have another summand with representative corresponding to $\mu_l$. Hence $V_{\lambda_1+\bar{Q}}$ is isomorphic to the orbifold module with representative $V_{L_+} \otimes V_{\mu_l+L_-}$.

From these identities, other module representatives can be identified using Lemma 5.6.9 and that $\sigma : V_{\lambda_i+\bar{Q}} \to V_{\lambda_{n-i+1}+\bar{Q}}$. From Lemma 5.6.9, we have that the module $V_{\lambda_1+\bar{Q}}$ can be identified with the orbifold module with representative $V_{L_+} \otimes V_{\mu_l+L_-}$. Therefore, using that $\lambda_i = i\lambda_1 \mod \bar{Q}$ for $i = 1, \ldots, l-1$, the module $V_{\lambda_j+\bar{Q}}$ can be identified with the module with representative $V_{L_+} \otimes V_{j\mu_l+L_-}$, where $j = 1, \ldots, l-1$. In addition, we have that $V_{\lambda_l+\lambda_{l+1}+\bar{Q}}$

and $V^{\pm}_{\lambda_l+\bar{Q}}$ can be identified with the orbifold module with representative $V_{L_+} \otimes V_{l\mu_l+L_-}$ and $V_{L_+} \otimes V^{\pm}_{\gamma+L_-}$, respectively. This completes the correspondence of orbifold modules of untwisted type.

As an example of the correspondence of orbifold modules of twisted type, consider the module of twisted type presented in Example 4.3. Then we have the following correspondence between eigenspaces:

$$\begin{aligned} M(\sigma)^{\pm} \otimes P_{\chi}(n)^{+} &\simeq \left(V_{L_+} \otimes V_{L_-}^{T_{\chi},\pm}\right) \bigoplus \left(V_{\frac{\alpha^2}{2}+L_+} \otimes V_{L_-}^{T_{-\chi},\pm}\right), \\ M(\sigma)^{\mp} \otimes P_{\chi}(n)^{-} &\simeq \left(V_{\frac{\alpha^1}{2}+L_+} \otimes V_{L_-}^{T_{\chi},\mp}\right) \bigoplus \left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V_{L_-}^{T_{-\chi},\mp}\right). \end{aligned}$$

## 5.7 The Root Lattice $E_6$

Consider the $E_6$ simple roots $\{\alpha_1, \ldots, \alpha_6\}$. The nondegenerate symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ is given by

$$((\alpha_i|\alpha_j))_{i,j} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The associated even lattice is $Q = \sum_{i=1}^{6} \mathbb{Z}\alpha_i$. Consider the Dynkin diagram automorphism $\sigma : \alpha_1 \longleftrightarrow \alpha_6,\ \alpha_3 \longleftrightarrow \alpha_5$ with fixed points $\alpha_2$ and $\alpha_4$. Then $\sigma$ is also an automorphism of $Q$. Set

$$\alpha^1 = \alpha_1 + \alpha_6, \tag{5.58}$$

$$\beta^1 = \alpha_1 - \alpha_6, \tag{5.59}$$

$$\alpha^2 = \alpha_3 + \alpha_5, \tag{5.60}$$

$$\beta^2 = \alpha_3 - \alpha_5. \tag{5.61}$$

Then $\alpha^i$ and $\beta^i$ are eigenvectors for $\sigma$ with eigenvalues 1 and $-1$, respectively. Products between these elements are as follows:

$$(\alpha^i|\alpha^i) = 4 = (\beta^i|\beta^i),$$
$$(\alpha^1|\alpha^2) = -2 = (\beta^1|\beta^2).$$

Since $(\alpha_1|\alpha_6) = 0 = (\alpha_3|\alpha_5)$ and $(\alpha_2|\alpha_2) = 2 = (\alpha_4|\alpha_4)$, we have that $Q \subset \bar{Q}$. Therefore

$$Q = \bar{Q}, \tag{5.62}$$
$$L_+ = \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha^1 + \mathbb{Z}\alpha^2, \tag{5.63}$$
$$L_- = \mathbb{Z}\beta^1 + \mathbb{Z}\beta^2, \tag{5.64}$$
$$Q/L = \{L, \alpha_1 + L, \alpha_3 + L, \alpha_1 + \alpha_3 + L\}. \tag{5.65}$$

Hence by Proposition 4.2.2, we have that

$$\begin{aligned} V_Q^\sigma \simeq \quad & \left(V_{L_+} \otimes V_{L_-}^+\right) \oplus \left(V_{\frac{\alpha^1}{2}+L_+} \otimes V^+_{\frac{\beta^1}{2}+L_-}\right) \\ & \oplus \left(V_{\frac{\alpha^2}{2}+L_+} \otimes V^+_{\frac{\beta^2}{2}+L_-}\right) \oplus \left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V^+_{\frac{\beta^1+\beta^2}{2}+L_-}\right), \end{aligned} \tag{5.66}$$

since $(\alpha_1)_+ = \frac{\alpha^1}{2}$, $(\alpha_1)_- = \frac{\beta^1}{2}$, and $(\alpha_3)_+ = \frac{\alpha^2}{2}$, $(\alpha_3)_- = \frac{\beta^2}{2}$.

In order to compute the $V_L^\sigma$-modules, we first determine $L_+^*/L_+$ using the Gram matrix for $L_+$. Ordering the basis of $L_+$ as $\{\alpha_2, \alpha_4, \alpha^1, \alpha^2\}$, the Gram matrix for $L_+$ is given by

$$G = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & -2 \\ 0 & 0 & 4 & -2 \\ 0 & -2 & -2 & 4 \end{pmatrix}.$$

The inverse is given by

$$G^{-1} = \begin{pmatrix} 2 & 3 & 1 & 2 \\ 3 & 6 & 2 & 4 \\ 1 & 2 & 1 & \frac{3}{2} \\ 2 & 4 & \frac{3}{2} & 3 \end{pmatrix}.$$

Note that $\det G = 4$. Thus a $\mathbb{Z}$-basis for $L_+^*$ is given by $\{\frac{\alpha^1}{2}, \frac{\alpha^2}{2}\}$ so that

$$L_+^*/L_+ = \left\{L_+, \frac{\alpha^1}{2} + L_+, \frac{\alpha^2}{2} + L_+, \frac{\alpha^1+\alpha^2}{2} + L_+\right\}.$$

We now determine $L_-^*/L_-$ using the Gram matrix for $L_-$. Ordering the $\mathbb{Z}$-basis of $L_-$ as $\{\beta^1, \beta^2\}$, the Gram matrix for $L_-$ is given by

$$M = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$

with inverse

$$M^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Therefore $L_-^*/L_-$ is generated by the elements $\frac{1}{2}\mu_1 = \frac{1}{6}(2\beta^1 + \beta^2)$ and $\frac{1}{2}\mu_2 = \frac{1}{6}(\beta^1 + 2\beta^2)$ and $|L_-^*/L_-| = \det M = 12$. The 12 elements in $L_-^*/L_-$ can be described by the following list:

$$a_1\frac{\beta^1}{2} + b_1\frac{\beta^2}{2} + L_-, \tag{5.67}$$

$$\frac{1}{2}\mu_1 + a_2\frac{\beta^1}{2} + b_2\frac{\beta^2}{2} + L_-, \tag{5.68}$$

$$\frac{1}{2}\mu_2 + a_3\frac{\beta^1}{2} + b_3\frac{\beta^2}{2} + L_-, \tag{5.69}$$

where $a_i, b_i \in \{0, 1\}$ and $i = 1, 2, 3$.

Since the $\sigma$-invariant elements of $L_-^*/L_-$ correspond to all elements in (5.67), there are 4 distinct irreducible $V_{L_-}^+$-modules that can be written as a sum of eigenspaces for $\sigma$ with eigenvalues $\pm 1$. The remaining elements in $L_-^*/L_-$ are identified by $\sigma$ in pairs since $\sigma\left(\frac{1}{2}\mu_1\right) = \frac{1}{2}\mu_2 + \frac{1}{2}\beta^1 + \frac{1}{2}\beta^2$ and $\sigma\left(\frac{1}{2}\mu_2\right) = \frac{1}{2}\mu_1 + \frac{1}{2}\beta^1 + \frac{1}{2}\beta^2$. Thus there are $2(4) + 4 = 12$ distinct irreducible $V_{L_-}^+$-modules. We then obtain the following lemma.

**Lemma 5.7.1** *The distinct $V_{L_-}^+$-modules of untwisted type that decompose into eigenspaces corresponds to the set*

$$\left(L_-^*/L_-\right)^\sigma = \left\{ \frac{1}{2}b_1\beta^1 + \frac{1}{2}b_2\beta^2 + L_- \,\middle|\, b_j \in \{0, 1\} \right\}.$$

*The other distinct irreducible $V_{L_-}^+$-modules of untwisted type correspond to the set*

$$\left\{ \frac{1}{2}\mu_1 + \frac{1}{2}b_1\beta^1 + \frac{1}{2}b_2\beta^2 + L_- \,\middle|\, b_j \in \{0, 1\} \right\}.$$

From the proof of Corollary 4.4.3 and (5.65), the irreducible $V_Q^\sigma$-modules are obtained from elements $\lambda \in L_+^*$ and $\mu \in L_-^*$ such that $(\lambda + \mu|\alpha_i) \in \mathbb{Z}$ for $i = 1, 3$. However, due to (5.66), an orbifold module corresponding to a representative containing $V_{\frac{1}{2}a_1\alpha^1+\frac{1}{2}a_2\alpha^2+L_+}$ will have another summand containing $V_{L_+}$. Thus we only need to find those $\mu \in L_-^*$ for which $(\mu|\alpha_i) \in \mathbb{Z}$ for $i = 1, 3$. Since

$$\left(\frac{1}{2}\mu_1 \middle| \alpha_i\right) = \frac{1}{2}\delta_{1i}, \quad i = 1, 3,$$

$$\left(\frac{1}{2}\beta^i \middle| \alpha_j\right) = \delta_{ij} - \frac{1}{2}\delta_{i,j-2} - \frac{1}{2}\delta_{i,j+1}, \quad i = 1, 2,\ j = 1, 3,$$

the only cosets in $L_-^*/L_-$ that have integral products with both $\alpha_1$ and $\alpha_3$ are $L_-$ and $\frac{1}{2}\mu_1 + \frac{1}{2}\beta^2 + L_- = \mu_2 + L_-$. Therefore, the irreducible $V_Q^\sigma$-modules of untwisted type are equivalent to the following list:

$$\left(V_{L_+} \otimes V_{L_-}^{\pm}\right) \oplus \left(V_{\frac{\alpha^1}{2}+L_+} \otimes V^{\pm}_{\frac{\beta^1}{2}+L_-}\right) \oplus \left(V_{\frac{\alpha^2}{2}+L_+} \otimes V^{\pm}_{\frac{\beta^2}{2}+L_-}\right) \oplus \left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V^{\pm}_{\frac{\beta^1+\beta^2}{2}+L_-}\right),$$

$$\begin{aligned}\left(V_{L_+} \otimes V_{\mu_2+L_-}\right) &\oplus \left(V_{\frac{\alpha^1}{2}+L_+} \otimes V_{\mu_2+\frac{\beta^1}{2}+L_-}\right) \\ &\oplus \left(V_{\frac{\alpha^2}{2}+L_+} \otimes V_{\mu_2+\frac{\beta^2}{2}+L_-}\right) \oplus \left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+} \otimes V_{\mu_2+\frac{\beta^1+\beta^2}{2}+L_-}\right).\end{aligned}$$

For convenience, these modules are referred to by using the summands $V_{L_+} \otimes V_{L_-}^{\pm}$ and $V_{L_+} \otimes V_{\mu_2+L_-}$, respectively.

The irreducible twisted $V_{L_-}$-modules are $V_{L_-}^{T_\chi}$, where $T_\chi$ is an irreducible $\hat{L}_-/K$-module with central character $\chi$. The irreducible $V_Q^\sigma$-modules of twisted type are obtained using the fusion rules for $V_{L_-}^+$ with $M_1 = V^+_{\frac{1}{2}\sum b_i\beta^i+L_-}$ where $i = 1, 2$ and $b_i \in \{0, 1\}$ (cf. Definition 3.3.16). Recall each irreducible character $\chi : 2L_-^* \cap L_- \longrightarrow \{\pm 1\}$ for which $\chi(e_{-\alpha}) = \chi(e_\alpha)$ can be written as

$$\chi(e_\alpha) = (-1)^{(\alpha|\mu)}$$

for some $\mu \in (2L_-^* \cap L_-)^*$. Since

$$\left(b_1\beta^1 + b_2\beta^2 \middle| \frac{1}{2}b_1\beta^1 + \frac{1}{2}b_2\beta^2\right) \quad = \quad 2(b_1^2 + b_2^2 - b_1b_2) \in 2\mathbb{Z},$$

we have that

$$\chi_\mu^{(\frac{1}{2}\sum b_i\beta^i)}(e_{\sum b_i\beta^i}) \quad = \quad \chi_\mu(e_{\sum b_i\beta^i}). \tag{5.70}$$

Hence every module $V^{+}_{\frac{1}{2}b_i\beta^i+L_-}$ sends $V_{L_-}^{T_\chi}$ to $V_{L_-}^{T_\chi}$. Since $c_\chi\left(\frac{1}{2}\sum b_i\beta^i\right) = \chi(e_{\sum b_i\beta^i})$ and linear characters are homomorphisms, we have that

$$c_\chi\left(\frac{1}{2}b_1\beta^1 + \frac{1}{2}b_2\beta^2\right) = \chi(e_{b_1\beta^1})\chi(e_{b_2\beta^2}), \tag{5.71}$$

where $b_i \in \{0,1\}$ and $\chi(e_0) = 1$. Thus each irreducible $V_Q^\sigma$-module of twisted type corresponding to a choice for $\chi$ is obtained in the following way. For each element $\gamma_- = \frac{1}{2}\sum b_i\beta^i$, the eigenspaces are determined by computing the products in (5.71). There are a total of 4 distinct irreducible orbifold modules of twisted type given by the following:

If $\chi(e_{\beta^1}) = 1 = \chi(e_{\beta^2})$, we have

$$\left(V_{L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^1}{2}+L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\pm}\right).$$

If $\chi(e_{\beta^1}) = 1$ and $\chi(e_{\beta^2}) = -1$, we have

$$\left(V_{L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^1}{2}+L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right)\oplus\left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right).$$

If $\chi(e_{\beta^1}) = -1$ and $\chi(e_{\beta^2}) = 1$, we have

$$\left(V_{L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^1}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right)\oplus\left(V_{\frac{\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right).$$

If $\chi(e_{\beta^1}) = -1 = \chi(e_{\beta^2})$, we have

$$\left(V_{L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^1}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right)\oplus\left(V_{\frac{\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right)\oplus\left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\pm}\right).$$

We now construct the orbifold modules using Section 3.3.2. The 2-cocycle $\varepsilon$ satisfies the following:

$$\begin{aligned}
\varepsilon(\alpha_i, \alpha_i) &= (-1)^{2\left(\frac{2+1}{2}\right)} = -1, \\
\varepsilon(\alpha_1, \alpha_3)\varepsilon(\alpha_3, \alpha_1) &= -1, \\
\varepsilon(\alpha_3, \alpha_4)\varepsilon(\alpha_4, \alpha_3) &= -1, \\
\varepsilon(\alpha_4, \alpha_5)\varepsilon(\alpha_5, \alpha_4) &= -1, \\
\varepsilon(\alpha_5, \alpha_6)\varepsilon(\alpha_6, \alpha_5) &= -1, \\
\varepsilon(\alpha_2, \alpha_4)\varepsilon(\alpha_4, \alpha_2) &= -1, \\
\varepsilon(\alpha_i, \alpha_j)\varepsilon(\alpha_j, \alpha_i) &= 1, \quad \text{otherwise.}
\end{aligned}$$

Set $\varepsilon$ to be the following on the basis:

$$\begin{array}{ll}
\varepsilon(\alpha_1, \alpha_3) = 1, & \varepsilon(\alpha_3, \alpha_1) = -1, \\
\varepsilon(\alpha_3, \alpha_4) = 1, & \varepsilon(\alpha_4, \alpha_3) = -1, \\
\varepsilon(\alpha_4, \alpha_5) = -1, & \varepsilon(\alpha_5, \alpha_4) = 1, \\
\varepsilon(\alpha_5, \alpha_6) = -1, & \varepsilon(\alpha_6, \alpha_5) = 1, \\
\varepsilon(\alpha_2, \alpha_4) = 1, & \varepsilon(\alpha_4, \alpha_2) = -1.
\end{array}$$

With these notions, we have that $\varepsilon(\sigma\gamma_1, \sigma\gamma_2) = \varepsilon(\gamma_1, \gamma_2)$ for all $\gamma_1, \gamma_2 \in Q$. Thus we can take $\eta$ to be trivial, that is, $\eta(\gamma) = 1$ for all $\gamma \in Q$.

The dual lattice to $E_6$ is spanned by the elements

$$\begin{aligned}
\lambda_1 &= \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) \\
&= \frac{1}{3}(\alpha_1 + 2\alpha_3 + \alpha_5 + 2\alpha_6) \mod Q, \\
\lambda_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\
&= 0 \mod Q, \\
\lambda_3 &= \frac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6) \\
&= \frac{1}{3}(2\alpha_1 + \alpha_3 + 2\alpha_5 + \alpha_6) \mod Q, \\
\lambda_4 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 \\
&= 0 \mod Q, \\
\lambda_5 &= \frac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6) \\
&= \frac{1}{3}(\alpha_1 + 2\alpha_3 + \alpha_5 + 2\alpha_6) \mod Q = \lambda_1 \mod Q, \\
\lambda_6 &= \frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6) \\
&= \frac{1}{3}(2\alpha_1 + \alpha_3 + 2\alpha_5 + \alpha_6) \mod Q = \lambda_3 \mod Q.
\end{aligned}$$

The only $\sigma$-invariant element in $Q^*/Q$ is the trivial coset $Q$ and $\sigma$ identifies the other two corresponding modules since $\sigma(\lambda_1) = \lambda_3$. Thus there are 3 distinct orbifold modules of untwisted type given by

$$V_Q^{\pm}, \quad V_{\lambda_1+Q}.$$

For the orbifold modules of twisted type, the quantities that will be needed are the following:

$$\begin{aligned}
B_{\beta^i,-\beta^i} &= 2^8, \\
B_{\alpha_1,\alpha_6} &= 4 = B_{\alpha_3,\alpha_5}, \\
B_{\alpha_6,-\alpha_6} &= 4 = B_{\alpha_5,-\alpha_5}, \\
b_{\beta^i} &= \frac{1}{2}(0-4) = -2, \\
b_{\alpha_1} &= \frac{1}{2}(1-2) = -\frac{1}{2}, \\
b_{\alpha_3} &= \frac{1}{2}(1-2) = -\frac{1}{2}, \\
C_{\gamma,\theta} &= (-1)^{(\gamma|\theta)}.
\end{aligned}$$

The elements in $G_\sigma$ must satisfy $U_{\sigma\gamma} = U_\gamma e^{2\pi\mathrm{i}(b_\gamma+\gamma_0)}$. In particular, $U_{-\beta^i} = U_{\beta^i} e^{-4\pi\mathrm{i}} = U_{\beta^i}$ so that $U_{\beta^i}^2 = U_{\beta^i} U_{-\beta^i} = \varepsilon(\beta^i, -\beta^i) B_{\beta^i,-\beta^i}^{-1} = 2^{-8}$. Thus there are two possible actions of $U_{\beta^i}$:

$$U_{\beta^i} = \pm 2^{-4} = \pm\frac{1}{16}.$$

Using the $\sigma$-invariance, we then have the following:

$$\begin{aligned}
U_{\alpha_6} &= U_{\alpha_1} e^{2\pi\mathrm{i}\left(-\frac{1}{2}+\frac{1}{2}\alpha^1_{(0)}\right)} = -U_{\alpha_1} e^{\pi\mathrm{i}\alpha^1_{(0)}},\\
U_{\alpha_5} &= U_{\alpha_3} e^{2\pi\mathrm{i}\left(-\frac{1}{2}+\frac{1}{2}\alpha^2_{(0)}\right)} = -U_{\alpha_3} e^{\pi\mathrm{i}\alpha^2_{(0)}},\\
U_{\alpha^1} &= \varepsilon(\alpha_1, \alpha_6) B_{\alpha_1,\alpha_6} U_{\alpha_1} U_{\alpha_6} = -4U_{\alpha_1}^2 e^{\pi\mathrm{i}\alpha^1_{(0)}},\\
U_{\alpha^2} &= \varepsilon(\alpha_3, \alpha_5) B_{\alpha_3,\alpha_5} U_{\alpha_3} U_{\alpha_5} = -4U_{\alpha_3}^2 e^{\pi\mathrm{i}\alpha^2_{(0)}},\\
U_{-\alpha_6} &= \varepsilon(\alpha_6, -\alpha_6) B_{\alpha_6,-\alpha_6}^{-1} U_{\alpha_6}^{-1} = \frac{1}{4} U_{\alpha_1}^{-1} e^{\pi\mathrm{i}\alpha^1_{(0)}},\\
U_{-\alpha_5} &= \varepsilon(\alpha_5, -\alpha_5) B_{\alpha_5,-\alpha_5}^{-1} U_{\alpha_5}^{-1} = \frac{1}{4} U_{\alpha_3}^{-1} e^{\pi\mathrm{i}\alpha^2_{(0)}},\\
U_{\beta^1} &= \varepsilon(\alpha_1, -\alpha_6) B_{\alpha_1,-\alpha_6} U_{\alpha_1} U_{-\alpha_6} = \frac{1}{16} e^{\pi\mathrm{i}\alpha^1_{(0)}},\\
U_{\beta^2} &= \varepsilon(\alpha_3, -\alpha_5) B_{\alpha_3,-\alpha_5} U_{\alpha_3} U_{-\alpha_5} = \frac{1}{16} e^{\pi\mathrm{i}\alpha^2_{(0)}}.
\end{aligned}$$

Consider the vector space $P = \mathbb{C}[q_1, q_1^{-1}, q_2, q_2^{-1}, q_3, q_3^{-1}, q_4, q_4^{-1}]$. Since each nontrivial action is determined by $U_{\alpha_1}$, $U_{\alpha_3}$, $U_{\alpha_2}$ and $U_{\alpha_4}$, we can let these operators act as multiplication in the following way:

$$\begin{aligned}
U_{\alpha_1} &= q_1(-1)^{\frac{\partial}{\partial q_3}},\\
U_{\alpha_2} &= q_2(-1)^{\frac{\partial}{\partial q_4}},\\
U_{\alpha_3} &= q_3(-1)^{\frac{\partial}{\partial q_4}},\\
U_{\alpha_4} &= q_4.
\end{aligned}$$

Note that these actions ensure that $q_1, q_2, q_3, q_4$ are commuting variables since $(\alpha_1|\alpha_3), (\alpha_3|\alpha_4)$ and $(\alpha_2|\alpha_4)$ are all equal to $-1$.

In order to determine the action of $e^{\pi\mathrm{i}\alpha^i_{(0)}}$, consider the commutation relation with $U_\gamma$:

$$e^{\pi\mathrm{i}\alpha^i_{(0)}} U_\gamma e^{-\pi\mathrm{i}\alpha^i_{(0)}} = e^{\pi\mathrm{i}(\alpha^i|\gamma)} U_\gamma.$$

For $\gamma = \alpha_1$, we have $e^{\pi\mathrm{i}\alpha^1_{(0)}} q_1 e^{-\pi\mathrm{i}\alpha^1_{(0)}} = q_1$. Thus $e^{\pi\mathrm{i}\alpha^1_{(0)}} q_1^n = q_1^n e^{\pi\mathrm{i}\alpha^1_{(0)}}(1)$. Since $e^{2\pi\mathrm{i}\alpha^1_{(0)}} = 1$, we must have $e^{\pi i \alpha^1_{(0)}}(1) = \pm 1$ so that $e^{\pi i \alpha^1_{(0)}} q_1^n = \pm q_1^n$. Similarly we have

$$\begin{aligned}
e^{\pi i \alpha^1_{(0)}} q_3^n &= \mp q_3^n, \\
e^{\pi i \alpha^2_{(0)}} q_1^n &= \mp q_1^n, \\
e^{\pi i \alpha^2_{(0)}} q_3^n &= \pm q_3^n, \\
e^{\pi i \alpha^i_{(0)}} q_j^n &= \pm q_j^n, \quad i = 1, 2, \quad j = 2, 4.
\end{aligned}$$

Thus on the space $P$ we have the following:

$$\begin{aligned}
U_{\alpha_5} &= \mp q_3 (-1)^{\frac{\partial}{\partial q_4}}, \\
U_{\alpha_6} &= \mp q_1 (-1)^{\frac{\partial}{\partial q_3}}, \\
U_{\alpha^i} &= \mp 4 q_i^2, \quad i = 1, 2, \\
U_{\beta^i} &= \pm \frac{1}{16}, \quad i = 1, 2.
\end{aligned}$$

The signs for $U_{\alpha_6}$ and $U_{\alpha^1}$ are determined by the sign of $U_{\beta^1}$ and the signs for $U_{\alpha_5}$ and $U_{\alpha^2}$ are determined by the sign of $U_{\beta^2}$.

The automorphism $\sigma$ acts on each of these modules. To determine the action of $\sigma$, we calculate $\sigma(q_1^l q_2^m p_1^n p_2^k) = \sigma(U_{\alpha_1}^l U_{\alpha_3}^m U_{\alpha_2}^n U_{\alpha_4}^k \cdot 1)$ on the module $P_{\chi_-}(n)$ corresponding to the character $\chi_-(e_{\beta^1}) = -1 = \chi_-(e_{\beta^2})$. Since

$$\sigma\left(U_{\alpha_1}^l U_{\alpha_3}^m U_{\alpha_2}^n U_{\alpha_4}^k \cdot 1\right) = q_1^l q_2^m p_1^n p_2^k,$$

the automorphism $\sigma$ acts as the identity on $P_{\chi_-}(n)$. For other characters $\chi$, $\sigma$ is determined by

$$\sigma\left(U_{\alpha_1} \cdot 1\right) = U_{\alpha_6} \cdot 1 = \begin{cases} q_1, & \chi(e_{\beta^1}) = -1 \\ -q_1, & \chi(e_{\beta^1}) = 1 \end{cases},$$

$$\sigma\left(U_{\alpha_3} \cdot 1\right) = U_{\alpha_5} \cdot 1 = \begin{cases} q_2, & \chi(e_{\beta^2}) = -1 \\ -q_2, & \chi(e_{\beta^2}) = 1 \end{cases}.$$

Thus $P_\chi(n)$ decomposes into two eigenspaces of $\sigma$ with eigenvalues $\pm 1$. The $+1$-eigenspace $P_\chi(n)^+$ is generated by products $p_1^n p_2^k q_1^l q_2^m$, where $l$ is even if only $\chi(e_{\beta^1}) = 1$, $m$ is even if only $\chi(e_{\beta^2}) = 1$, and $l + m$ is even if $\chi(e_{\beta^1}) = 1 = \chi(e_{\beta^2})$. The $-1$-eigenspace $P_\chi(n)^-$ is generated by products $p_1^n p_2^k q_1^l q_2^m$, where $l$ is odd if only $\chi(e_{\beta^1}) = 1$, $m$ is odd if only $\chi(e_{\beta^2}) = 1$, and $l + m$ is odd if $\chi(e_{\beta^1}) = 1 = \chi(e_{\beta^2})$.

The entire $\sigma$-twisted $V_Q$-module is then $M(\sigma)\otimes P_\chi(n)$. Since $M(\sigma)$ itself decomposes into $\pm 1$-eigenspaces of $\sigma$, there are a total of $2\cdot 4 = 8$ orbifold modules of twisted type:

$$M(\sigma)^{\pm}\otimes P_{\chi_-}(n),$$

$$\left(M(\sigma)^{\pm}\otimes P_\chi(n)^{+}\right)\bigoplus\left(M(\sigma)^{\mp}\otimes P_\chi(n)^{-}\right),\quad \chi\neq\chi_-.$$

We now present a correspondence between the two constructions. It is clear that each construction produces the same number of orbifold modules of untwisted and twisted type and also that the orbifold modules of twisted type both correspond to the same set of characters. The orbifold modules $V_Q^{\pm}$ are identified with the orbifold modules corresponding to the representatives $V_{L_+}\otimes V_{L_-}^{\pm}$. The following lemma identifies the other orbifold module of untwisted type.

**Lemma 5.7.2** *The orbifold module $V_{\lambda_1+Q}$ can be identified with the orbifold module with representative $V_{L_+}\otimes V_{\mu_2+L_-}$.*

**Proof** We first write $\lambda_1$ in terms of elements from $L^*$.

$$\begin{aligned}\lambda_1 &= \frac{1}{3}(\alpha_1+2\alpha_3+\alpha_5+2\alpha_6)\\ &= \frac{1}{6}(3\alpha^1-\beta^1+3\alpha^2+\beta^2)\\ &= \frac{1}{2}\alpha^1+\frac{1}{2}\alpha^2+\frac{1}{6}(5\beta^1+\beta^2)\quad \text{mod } L_-\\ &= \frac{1}{2}\alpha^1+\frac{1}{2}\alpha^2+\frac{1}{2}\mu_1+\frac{1}{2}\beta^1\quad \text{mod } L_-.\end{aligned}$$

Due to (5.66), the orbifold module corresponding to $\lambda_1$ will have another summand with representative corresponding to $\frac{1}{2}\mu_1+\frac{1}{2}\beta^2=\mu_2$.

As an example of the correspondence of orbifold modules of twisted type, consider the case of the character $\chi(e_{\beta^1})=1$ and $\chi(e_{\beta^2})=-1$. We then have the following correspondence between eigenspaces:

$$\begin{aligned}M(\sigma)^{\pm}\otimes P_\chi(n)^{+} &\simeq \left(V_{L_+}\otimes V_{L_-}^{T_\chi,\pm}\right)\oplus\left(V_{\frac{\alpha^1}{2}+L_+}\otimes V_{L_-}^{T_\chi,\pm}\right),\\ M(\sigma)^{\mp}\otimes P_\chi(n)^{-} &\simeq \left(V_{\frac{\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right)\oplus\left(V_{\frac{\alpha^1+\alpha^2}{2}+L_+}\otimes V_{L_-}^{T_\chi,\mp}\right).\end{aligned}$$

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