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A Unified Characterization of Reproducing Systems Generated by a Finite Family

By *Demetrio Labate*

ABSTRACT. This article presents a general result from the study of shift-invariant spaces that characterizes tight frame and dual frame generators for shift-invariant subspaces of $L^2(\mathbb{R}^n)$. A number of applications of this general result are then obtained, among which are the characterization of tight frames and dual frames for Gabor and wavelet systems.

1. Introduction

The aim of this article is to provide an unified approach to the characterization of a large class of systems satisfying a reproducing formula of the form

$$f = \sum_{i \in \mathcal{I}} \langle f, \psi_i \rangle \, \psi_i \ , \tag{1.1}$$

or, more generally,

$$f = \sum_{i \in \mathcal{I}} \langle f, \phi_i \rangle \, \psi_i \ , \tag{1.2}$$

where $f, \psi_i, \phi_i, i \in \mathcal{I}$, belong to $L^2(\mathbb{R}^n)$. The kind of reproducing systems that we will consider are generated by the action of translations, dilations, and modulations on a finite family of functions. To keep the notation to a minimum and focus on the main ideas that we will present in this article, let us restrict our attention, for the moment, to one-dimensional systems generated by a single function. The Gabor system, for example, is generated by the action of the translations T_{ck}, $k \in \mathbb{Z}$, $c > 0$, and modulations M_{bm}, $m \in \mathbb{Z}$, $b > 0$, on a function $\psi \in L^2(\mathbb{R})$, where $T_{ck}\, \psi(x) = \psi(x - ck)$ and $M_{bm}\, \psi(x) = e^{2\pi i bmx}\, \psi(x)$. The system thus obtained is $\{\psi_i\}_{i \in \mathcal{I}} = \{M_{bm}\, T_{ck}\, \psi\}_{m,k \in \mathbb{Z}}$, where the indexing set, in this case, is $\mathcal{I} = \{(m, k) : m, k \in \mathbb{Z}\}$. Wavelets are obtained in a similar way, with the dilations D_{2^j}, where $D_{2^j}\, \psi(x) = 2^{j/2}\, \psi(2^j x)$, $j \in \mathbb{Z}$, replacing the modulations. The system thus obtained has the form $\{\psi_i\}_{i \in \mathcal{I}} = \{D_{2^j}\, T_{ck}\, \psi\}_{j,k \in \mathbb{Z}}$, where $\mathcal{I} = \{(j, k) : j, k \in \mathbb{Z}\}$, and is often referred to as an affine system. In the case of the Gabor system, the order of the modulation and translation operators can be reversed; however, this

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