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A Unified Characterization of Reproducing Systems Generated by a Finite Family, II

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ABSTRACT. *By a "reproducing" method for $\mathcal{H} = L^2(\mathbb{R}^n)$ we mean the use of two countable families $\{e_\alpha : \alpha \in \mathcal{A}\}$, $\{f_\alpha : \alpha \in \mathcal{A}\}$, in $\mathcal{H}$, so that the first "analyzes" a function $h \in \mathcal{H}$ by forming the inner products $\{< h, e_\alpha >: \alpha \in \mathcal{A}\}$, and the second "reconstructs" h from this information: $h = \sum_{\alpha\in\mathcal{A}} < h, e_\alpha > f_\alpha$.*

*A variety of such systems have been used successfully in both pure and applied mathematics. They have the following feature in common: they are generated by a single or a finite collection of functions by applying to the generators two countable families of operators that consist of two of the following three actions: dilations, modulations, and translations. The **Gabor systems**, for example, involve a countable collection of modulations and translations; the **affine systems** (that produce a variety of wavelets) involve translations and dilations.*

A considerable amount of research has been conducted in order to characterize those generators of such systems. In this article we establish a result that "unifies" all of these characterizations by means of a relatively simple system of equalities. Such unification has been presented in a work by one of the authors. One of the novelties here is the use of a different approach that provides us with a considerably more general class of such reproducing systems; for example, in the affine case, we need not to restrict the dilation matrices to ones that preserve the integer lattice and are expanding on $\mathbb{R}^n$. Another novelty is a detailed analysis, in the case of affine and quasi-affine systems, of the characterizing equations for different kinds of dilation matrices.

1. Introduction

The terms **reproducing systems** or **reproducing formulae** are applied to any of several methods that "analyze" a vector v (or function) and, then, "reconstructs" v in terms of this analysis. In order to fix our ideas, let us consider a specific way in which this procedure is carried out that will help us explain the principal features of this article.

A countable family $\{e_\alpha : \alpha \in \mathcal{A}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a **frame** if

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