On the Development of a Beam Model for the Fracture of Plain and Reinforced Concrete

K. T. Sundara Raja IVENGAR, B. K. Raghu PRASAD
Indian Institute of Science, Bangalore, India
H. ANANTHAN
Mahanad College of Engineering, Hassan, India

1 INTRODUCTION

Several investigations have revealed that conventional linear Elastic Fracture Mechanics (LEFM) principles are not applicable to concrete. Fracture Toughness is found to be size dependent. It is also emphasized that, it is important to consider slow crack growth and formation of nonlinear fracture process zone while evaluating the fracture parameters. In view of this observation several analytical models have been proposed by different investigators for characterising the process of fracture in concrete. These can be broadly classified into two categories viz. (1) strain softening models (Hillerborg et al. 1976, Bazant and Oh, 1983) and (2) Modified LEFM model (Jeng and Shah,1985).

Recently a one-dimensional model called Softening Beam Model(SBM) has been developed at the Indian Institute of Science (Ananthan et al. 1990) for the fracture of Plain concrete beams and has been extended to explain the fracture behaviour of Reinforced concrete beams.

2 SBM FOR PLAIN CONCRETE

The essential characteristic of the model is given by the stress block distribution across the section under consideration at any stage of loading. Changes in stress distribution can be expected to follow the appropriate stress-strain law for the concrete under consideration. In the model developed, the following constitutive laws for concrete in tension and compression are adopted : - (i) Since concrete in tension undergoes softening, the stress-strain diagram is approximated by a bi-linear relationship as in Fig.1. (ii) Since the failure of Plain concrete beam is in tension, it is reasonable to expect that the beam will only be subjected to compressive stresses of very small magnitude. Hence a linear variation of stress with strain is assumed.

Fig.2(a) shows the longitudinal section of the beam and
Fig. 2(b) shows the cross-section at midspan. A notched beam has been considered while deriving the mathematical expressions. However the equations are valid for an unnotched beam also. A typical stress distribution diagram across the depth corresponding to some stage of loading at which softening of concrete in tension has occurred is shown in Fig. 3. Considering the equilibrium equations, the following equations are obtained (Ananthan et al. 1990)

\[
\alpha = \frac{-\left( \beta + E^* \beta - E^* \right) + \left( 2\beta E^* + 2\beta - E^* \right)^{0.5}}{1 + E^*} \]  

(1)

\[
M = \frac{3 - 2\alpha + E^* \left( \beta + 2 \right) \left( 2\beta + 2\alpha - 1 \right) - \left( \alpha + \beta \right)^2}{\beta} \left[ 1 + \frac{E^*(\beta + 2)}{\beta} \right] \]  

(2)

where \( M \) is the moment at the mid-section for any particular loading and \( M_n \) is a non-dimensionalizing quantity which is taken as \( \sigma_0 b D^2 / 6 \). \( E = E_s / E \) [Fig 1.], \( E_s \) = Strain softening modulus, \( E \) = initial tangent modulus of concrete, \( \sigma_t \) = tensile strength of concrete, \( b \) and \( D \) are the breadth and uncracked ligament depth of the beam. It can be verified from equation (2), that for \( E^* = \infty \), \( \alpha = \beta = 1/2, M/M_n = 1 \). Similarly for \( E^* = 0, \alpha = \beta = 0 \), \( M/M_n = 3 \). These two values are the extremes for brittle and perfectly plastic conditions. To determine the three parameters \( \alpha, \beta \) and \( E^* \), from the two equations (1) and (2), an iterative solution using some additional and valid physical conditions has been developed (Ananthan et al. 1990). It is shown that the fracture of plain concrete beams occurs at post peak value of \( M/M_n = 1 \) for all values of \( E^* \) other than 0. The variation of \( M_{\text{max}} / M_n \) with \( E^* \) is shown in Fig. 4. The length of fracture process zone at any stage of loading is given by the relation [Fig. 3].

\[
L_p = \left( 1 - \alpha - \beta \right) D \]  

(3)

The maximum value \( (L_p)_{\text{max}} \) is obtained at the onset of fracture instability and this is found to occur when the ratio of \( \alpha / \beta \) attains a maximum value. Denoting \( 1 - ((L_p)_{\text{max}} / D) \) as \( \Delta \), the following equation is obtained.

\[
E^* = \frac{\Delta}{1 - \Delta^2} \]  

(4)

For \( \Delta = 1 \), \( E^* \) tends to \( \infty \), indicating brittle fracture behaviour and \( (L_p)_{\text{max}} = 0 \) is implied. Similarly perfect plastic collapse is indicated when \( \Delta = 0 \) and \( (L_p)_{\text{max}} = D \). For \( E = 1 \), \( (L_p)_{\text{max}} = 0.293D \) This represents the transition in the fracture behaviour. From a plot of variation of \( (L_p)_{\text{max}} \) with \( \sigma_t (W - a)/E \), obtained from a number of experimental results and using a regression analysis the following empirical equation is obtained:-

\[
(L_p)_{\text{max}} = 5091.5 - \frac{\sigma_t}{E} (W - a) - 46485.14 \left[ \frac{\sigma_t}{E} \right]^2 (W - a) \]  

(5)

where \( (L_p)_{\text{max}} \) is in mm, \( \sigma_t \) & \( E \) are in N/mm², \( W \) the depth of
the beam and a the notch depth are in mm. For known values of \( c_t, E, W \) and \( a \), \((Lp)_{\text{max}}\) is obtained from (5) and \( E^*\) from equation (4). With this value of \( E^*\), \( (M)_{\text{max}}/M_0 \) can be determined from Fig.4 and hence the value of maximum load \( P_{\text{max}} \). This procedure is adopted to predict \( P_{\text{max}} \) in the cases of several plain concrete beams tested either in three-point or four-point bending by a number of investigators (Ananthan 1989). From a comparison of these values for about 101 test results, the mean, standard deviation and coefficient of variation of the ratio of \( P_{\text{max}} \) (predicted) to \( P_{\text{max}} \) (experimental) are 0.97, 0.133 and 13.74\% respectively. It is also shown that several nonlinear fracture parameters such as crack tip opening displacement, crack mouth opening displacement and fracture energy can be calculated using the model (Ananthan 1989).

3 APPLICATION TO REINFORCED CONCRETE BEAMS

It is known that to retain the philosophy of limit state design, it is essential to understand the complete progressive failure of a reinforced concrete beam. The model developed for the fracture of plain concrete beam is extended to reinforced concrete beam also with a view to predict the behaviour of a r.c beam from initial to failure load. This is done by considering the effect of reinforcement by replacing it by the force carried by it across the section under consideration. Three different stages of analysis are performed, namely crack initiation stage, crack propagation stage till the yielding of steel or crushing of concrete and crack propagation stage after the yielding of steel till the crushing of concrete.

In this analysis, the nonlinear behaviour of concrete in compression is taken into account. Steel is considered as a linear elastic and perfectly plastic material. Crack propagation is modelled by the spreading of process zone across the uncracked ligament. An iterative procedure is developed so that results pertaining to various stages are obtained in a single computer program. The results obtained at various stages of loading are found to be in good agreement with the experimental observations (Ananthan 1989).

Table 1. Comparison of balanced percentage of reinforcement

<table>
<thead>
<tr>
<th>( f_{\text{c,max}}/f_y ) Values in N/mm²</th>
<th>Balanced %age of reinforcement</th>
<th>Park and Paulay (1975)</th>
<th>Present model</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 / 250</td>
<td>4.138</td>
<td>3.75</td>
<td></td>
</tr>
<tr>
<td>30 / 250</td>
<td>6.207</td>
<td>5.75</td>
<td></td>
</tr>
<tr>
<td>40 / 250</td>
<td>8.275</td>
<td>7.80</td>
<td></td>
</tr>
<tr>
<td>20 / 415</td>
<td>2.100</td>
<td>1.75</td>
<td></td>
</tr>
<tr>
<td>30 / 415</td>
<td>3.150</td>
<td>2.75</td>
<td></td>
</tr>
<tr>
<td>40 / 415</td>
<td>4.200</td>
<td>3.75</td>
<td></td>
</tr>
</tbody>
</table>

\( f_{\text{c,max}} \) is the Compressive strength of concrete
\( f_y \) is the yield strength of steel
Table 1. shows the comparison of values for balanced
percentage of reinforcement obtained according to the conventional formula (Park and Pauley 1975) and the present approach. The salient feature of the proposed model is that at any stage of loading, knowledge regarding the length of process zone and the stress distribution across the depth of the beam can be obtained. With this knowledge, the Crack opening displacement (COD) can be easily evaluated. Since COD evaluated at any location corresponds to the width of the crack at that location, a rational analytical procedure to predict crack widths at different locations and various stages of loading is developed. Using this model the crack widths are predicted in the cases of beams tested by several investigators (Clark 1956, Hognessed 1962 and Bese et al.1966). Several statistical quantities are evaluated and presented in Table 2.

Table 2. Statistical quantities pertaining to the prediction of crack width

<table>
<thead>
<tr>
<th>Test results (Authors)</th>
<th>Number of observations</th>
<th>( \bar{w}<em>{pre} / w</em>{exp} )</th>
<th>Mean value</th>
<th>Standard deviation</th>
<th>Coefficient of variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clark</td>
<td>250</td>
<td>0.95</td>
<td>0.430</td>
<td>45.20%</td>
<td></td>
</tr>
<tr>
<td>Hognessed</td>
<td>117</td>
<td>1.40</td>
<td>0.412</td>
<td>29.42%</td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>147</td>
<td>1.10</td>
<td>0.180</td>
<td>16.26%</td>
<td></td>
</tr>
</tbody>
</table>

\( * \) \( w_{pre} \) is the predicted crack width

\( ** \) \( w_{exp} \) is the experimental crack width

4 CONCLUDING REMARKS

A beam model developed for the fracture of Plain concrete beams (notched or unnotched) has been described. The model takes into account the softening behaviour of concrete in tension. The influence of structural size in altering the fracture mode from brittle fracture to plastic collapse is explained through the stress distribution across the uncracked ligament obtained by varying the strain softening modulus. The model predictions for maximum loads and several nonlinear fracture parameters compare well with those from tests. An extension of the model to Reinforced concrete beams has been made, which predicts satisfactorily the ultimate strength, stresses in steel and also the crack width for a number of R C beams tested.

5 ACKNOWLEDGEMENT

The computations in the above investigation were completed using a P C/AT which was available in a research scheme sponsored by University Grants Commission, New Delhi, India. Therefore the authors wish to thank the University Grants Commission for the above and the other facilities provided.
Fig. 1 Stress-Strain diagram for Concrete in Tension

Fig. 2 Beam nomenclature (a) Longitudinal section of the beam (b) Cross-section at mid-span

Fig. 3 Stress Block for $\infty \times E^n > 0$
Fig. 4 $M_{\text{max}}/M_n$ for all values of $E^n$