A Model to Describe Anisotropy for Rate-Dependent Plasticity

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ABSTRACT

Two constitutive models for rate-dependent inelasticity are developed on the basis of the irreversible thermodynamics. A new representation of the internal state variables is postulated for characterizing a combined isotropic and anisotropic hardening. From specific forms of the free energy function and the dissipation energy function, simple evolution equations of the Bailey-Orowan type are derived. One model is a two-variables model, and the other is a modified three-variables model; the latter contains a non-proportional hardening variable. These models reduce to a single-variable model under monotonic proportional deformation.

1 INTRODUCTION

It is one of the most important issues to establish the material models which precisely describe an anisotropic-hardening state and its evolution with inelastic deformation. The anisotropy in strain-hardening of metals is usually attributed to an inhomogeneous distribution and a particular structure of dislocations (Poirier 1976). It, once developed, has a significant influence on the transient-state and steady-state responses in subsequent deformation. We can clearly observe the effect of the anisotropy under reversed stressings or multiaxial non-proportional stressings (Ohashi et al. 1986), while we cannot perceive it as far as a monotonic proportional deformation concerned.

For developing material models, on the other hand, the ease of material identification is of practical importance, especially from engineering points of view. Most phenomenological formulations adopt the effective stress concept in manipulating isotropic-hardening and anisotropic-hardening. For describing the anisotropic hardening, the effective stress is usually defined in terms of the difference between the applied stress s and the back stress ξ: $s^{*} = s - \xi$ (Malinin-Khadjinsky 1972). We can modify it by using the drag stress D to contain the isotropic hardening as well: $s^{**} = s^{*}/D$ (Miller 1976). It is noted here that the effective stress s^{*} uniquely corresponds to a single internal variable $\xi = s - s^{*}$. The inversion of $s^{**} = (s - \xi)/D$, however, is not unique, since two independent variables are integrated into a

single effective stress. Hence, we see that the multiplied form $\mathbf{s}^{**}$ makes it more difficult to split the role of the effective stress into the associated isotropic and anisotropic components; it implies some difficulty in material identification. In order to make the material identification easier, a simple expression of the effective stress will be necessary which even apparently has an unique relation to the associated internal variables representing the isotropic and anisotropic hardenings.

The present study aims at a formulation of simple inelastic constitutive equations of the phenomenological and unified type, with an emphasis on the problems pointed out above. For this purpose, following the approach based on the irreversible thermodynamics (Kratochvil and Dillon 1969; Lemaitre and Chaboche 1990), we describe a current internal state in terms of isotropic and anisotropic hardening variables, and develop their rate type evolution equations. The characterization of the combined isotropic and anisotropic hardening state is based on the previous study (Kawai 1991). The Helmholtz free energy function and the dissipation energy function are assumed to be of particular forms so that an additive relation among the thermodynamic forces associated with the internal state variables and a set of simple evolution equations of the Bailey-Orowan type are derived. The condition of non-negative dissipation rate is explicitly imposed on the inelastic constitutive model developed, since the condition is not automatically satisfied in the present model; it differs from the conventional modellings. Finally, a generalized version is discussed on the basis of the additive relation among the time rates of the state variables; it contains the third internal state variable which we can interpret as a non-proportional hardening variable.

2 THERMODYNAMIC APPROACH FOR CONSTITUTIVE MODELLING

In the irreversible thermodynamic framework of the internal-state-variable theories (Kratochvil and Dillon 1969; Lemaitre and Chaboche 1990), local state laws in thermodynamic equilibrium can be derived from the Helmholtz free energy ψ:

$$\psi = \hat{\psi}(\varepsilon^e, T, \xi) \tag{1}$$

which is defined in terms of the external and internal state variables postulated in advance; in eq.(1), ε^e, T, and ξ are the elastic strain, temperature, and the non-observable internal state variable, respectively.

In the case of no-coupling between elasticity and inelasticity, the free energy function can be decomposed into the reversible and irreversible parts:

$$\psi = \psi_e(\varepsilon^e, T) + \psi_p(\xi, T), \tag{2}$$

$$\psi_e = (1/2)(1/\rho^*)\varepsilon^e \cdot \mathbf{E}\varepsilon^e - (T - T_0)\boldsymbol{\beta} \cdot \varepsilon^e, \tag{3}$$

$$\psi_p = \hat{\psi}_p(\xi, T) \tag{4}$$

where $\mathbf{E}$, $\boldsymbol{\beta}$, and T_0 are a fourth order elastic tensor, a second order thermal expansion tensor, and a reference temperature, respectively.

The second law of thermodynamics imposes the non-negative rate of the internal entropy production which is the sum of the entropy production rates due to the internal dissipation δ and the heat flux $\mathbf{q} = \hat{\mathbf{q}}(\varepsilon^e, T, \nabla T, \xi)$, respectively. Assuming an additive decomposition of the infinitesimal total

strain rate $\mathring{\varepsilon}$ into the elastic part $\mathring{\varepsilon}^e$ and the inelastic part $\mathring{\varepsilon}^p$ (i.e. $\mathring{\varepsilon} = \mathring{\varepsilon}^e + \mathring{\varepsilon}^p$), we obtain the local state laws,

$$\sigma = \rho^*(\partial\psi/\partial\varepsilon^e), \quad x = \rho^*(\partial\psi/\partial\xi), \quad \eta = -\partial\psi/\partial T, \tag{5}$$

and the condition of non-negative dissipation rate D,

$$D = \sigma\circ\mathring{\varepsilon}^p - x\circ\mathring{\xi} - \nabla T\circ q/T \geq 0, \tag{6}$$

i.e. the Clausius-Duhem inequality. In the above relations, σ (Cauchy stress), x, and η (entropy) are the thermodynamic forces associated with ε^e, ξ, and T, respectively. The mass density is denoted here in terms of ρ^*.

Evolution equations of the postulated state variables ε^e and ξ are defined in terms of a dissipation energy function $\Omega^*(\sigma, x)$ as follows:

$$\mathring{\varepsilon}^p = \partial\Omega^*/\partial\sigma, \qquad \mathring{\xi} = -\partial\Omega^*/\partial x. \tag{7}$$

The function Ω^* is usually assumed to be a convex, non-negative, nested scalar function satisfying $\Omega^*(0, 0) = 0$.

In the present study we limit our discussion to a quasi-isothermal process. The internal state variable ξ is assumed to be a second order symmetric tensor. For the sake of simple notation, furthermore, we adopt the deviatoric vector representation (Il'yushin 1954); the norm of the vectors, i.e. $|\mathbb{A}| = (\mathbb{A}\circ\mathbb{A})^{1/2}$, agrees with the invariant of von Mises type; the operation ($\circ$) defines the inner product between two vectors.

3 INTERNAL STATE VARIABLES AND ASSOCIATED THERMODYNAMIC FORCES

In order to characterize the current hardening state, we postulate that the internal state variable ξ which contains anisotropic and isotropic parts is given by the following expression (Kawai 1991):

$$\xi = \alpha + \rho n \tag{8}$$

where α and ρ denote a second order symmetric deviatoric tensor and a scalar, respectively. The tensor n represents the normalized deviatoric stress tensor, i.e. $n = s/|s|$.

By using these internal state variables, we specify a quadratic form of the free energy function which neglects the coupling between α and ρ as follows:

$$\rho^*\psi_p(\xi; \alpha, \rho) = \rho^*\psi_{p1}(\alpha) + \rho^*\psi_{p2}(\rho) \tag{9}$$

where

$$\rho^*\psi_{p1}(\alpha) = (1/2)H\alpha\circ\alpha, \qquad \rho^*\psi_{p2}(\rho) = (1/2)\rho^2. \tag{10}$$

The thermodynamic forces p, a, and r which are associated with the internal state variables ξ, α, and ρ, respectively, are defined by

$$p = \rho^*(\partial\psi_p/\partial\xi) = H\xi, \tag{11}$$

$$a = \rho^*(\partial\psi_{p1}/\partial\alpha) = H\alpha, \tag{12}$$

$$r = \rho^*(\partial\psi_{p2}/\partial\rho) = H\rho \tag{13}$$

where we use eq.(8). It is obvious that the thermodynamic forces satisfy the additive expression:

$$p = a + rn, \tag{14}$$

as in eq.(8).

4 EVOLUTION EQUATIONS FOR INTERNAL STATE VARIABLES

Regarding the dissipation energy function, we assume the following form:

$$\Omega^*(\mathbf{s}, \mathbf{p}) = \Omega[U(\mathbf{s}, \mathbf{p})] + \Lambda[V(\mathbf{p})]$$
$$= K/(m+1)U^{m+1} + M/(\ell+1)V^{\ell+1} \qquad (15)$$

where $U = |\mathbf{s} - \mathbf{p}|$ and $V = |\mathbf{p}|$, respectively; and K, M, m, and ℓ are material constants. In the present modelling, we assume the von Mises type materials (Oytana et al. 1982) which satisfy the inelastic incompressibility and the simultaneous occurrence of hardening and softening.

In this case, the inelastic strain rate is given by

$$\mathring{\varepsilon}^p = \partial\Omega^*/\partial\mathbf{s} = K|\mathbf{s}^*|^m\mathbf{n}^* \qquad (16)$$

where $\mathbf{s}^* = \mathbf{s} - \mathbf{p}$ and $\mathbf{n}^* = \mathbf{s}^*/|\mathbf{s}^*|$, respectively.

The time rates of the state variables are defined by

$$\mathring{\xi} = -\partial\Omega^*/\partial\mathbf{p} = \mathbf{G} = \mathring{\varepsilon}^p - M|\mathbf{p}|^\ell\mathbf{p}/|\mathbf{p}|, \qquad (17)$$

$$\mathring{\alpha} = -\mu(\partial\Omega^*/\partial\mathbf{a}) = \mu\mathbf{G}, \qquad (18a)$$

$$\mathring{\rho} = -(1-\mu)(\partial\Omega^*/\partial r) = (1-\mu)\mathbf{G}\circ\mathbf{n} \qquad (18b)$$

where μ is a scalar; the value is assumed to be less than unity.

Substituting the above equations (18a) and (18b) for the inequality (6), we can derive the inequality for mechanical dissipation rate D_1 as follows:

$$D_1 = \sigma\circ\mathring{\varepsilon}^p - \mathbf{a}\circ\mathring{\alpha} - r\mathring{\rho} = (\mathbf{s} - \mathbb{X})\circ\mathring{\varepsilon}^p + \mathbb{X}\circ(\partial\Lambda/\partial\mathbf{p}) \geq 0 \qquad (19)$$

where

$$\mathbb{X} = \mu\mathbf{a} + (1-\mu)r\mathbf{n}. \qquad (20)$$

Consequently, we have developed the model, the first one, which consists of the evolution equations (16) and (18), together with the state equation (8) or (14), and with the condition of non-negative dissipation rate, eq.(19).

In most thermodynamic formulations, the dissipation energy function is introduced so as to automatically satisfy the non-negative dissipation rate; so that the dissipation inequality does not appear in the resulting mathematical expressions. As for the present modelling, however, the condition does not automatically satisfied at all points in the space of the thermodynamic forces, except in the particular cases of pure isotropic hardening ($\mu = 0$) and the pure kinematic hardening ($\mu = 1$). Hence, we explicitly retain the dissipation inequality (19) to prescribe the thermodynamic force states admissible to inelastic evolution.

5 A GENERALIZATION OF MODEL

Let us consider the difference between $\mathring{\xi}$ and $\mathring{\alpha} + \mathring{\rho}\mathbf{n}$. By using eqs.(17) and (18), we can write it as

$$\mathring{\xi} - (\mathring{\alpha} + \mathring{\rho}\mathbf{n}) = (1-\mu)[\mathbf{G} - (\mathbf{G}\circ\mathbf{n})\mathbf{n}]$$
$$= (1-\mu)[(\mathring{\varepsilon}^p - M|\mathbf{p}|^\ell\mathbf{p}/|\mathbf{p}|) - \{(\mathring{\varepsilon}^p - M|\mathbf{p}|^\ell\mathbf{p}/|\mathbf{p}|)\circ\mathbf{n}\}\mathbf{n}]$$
$$= (1-\mu)[\{\mathring{\varepsilon}^p - (\mathring{\varepsilon}^p\circ\mathbf{n})\mathbf{n}\} - M|\mathbf{p}|^{\ell-1}\{\mathbf{p} - (\mathbf{p}\circ\mathbf{n})\mathbf{n}\}]. \qquad (21)$$

From the above relation it is found that the right hand side vanishes for proportional deformation, while it does not for non-proportional deformation.

In connection with eq.(21), we discuss a generalization of the model developed above.

By substituting eq.(16) for the right hand side of eq.(21), we can see that the vector given by the first term in the bracket, $\mathring{\varepsilon}^p - (\mathring{\varepsilon}^p \circ n)n$, has the same direction as the second one, $p - (p \circ n)n$; the direction of the vector is perpendicular to n through p. Hence, in order to prescribe the direction of the right hand side, we need only one direction $\mathbb{N}$ defined by

$$\mathbb{N} = \zeta P \circ n / |P \circ n|, \qquad \zeta = n \circ n, \qquad P = pn - np, \tag{22}$$

so that we can write

$$\mathring{\xi} = \mathring{\alpha} + \mathring{\rho} n + \mathring{\kappa} \mathbb{N}. \tag{23}$$

Correspondingly we generalize the expressions of the internal state (8) and the free energy (9) to the following forms, respectively:

$$\xi = \alpha + \rho n + \kappa \mathbb{N}, \qquad 2\rho^* \psi_p / H = \alpha \circ \alpha + \rho^2 + \kappa^2. \tag{24}$$

Now we have reached the second model:

$$p = a + rm + k\mathbb{N}, \tag{25}$$

$$\mathring{a} = \mu H G, \quad \mathring{r} = (1 - \mu) H G \circ n, \quad \mathring{k} = (1 - \mu) H G \circ \mathbb{N}. \tag{26}$$

Notice that the additional internal variable κ appears only under non-proportional deformation; it gives rise to the thermodynamic force k which decreases the non-proportionality developed during the prior deformation.

The mechanical dissipation rate is expressible as

$$\begin{aligned} D_1 &= \sigma \circ \mathring{\varepsilon}^p - a \circ \mathring{\alpha} - r\mathring{\rho} - k\mathring{\kappa} \\ &= [\sigma - \{\mu a + (1 - \mu) r m + (1 - \mu) k \mathbb{N}\} \circ \partial \Omega / \partial s \\ &\qquad + [\mu a + (1 - \mu) r m + (1 - \mu) k \mathbb{N}] \circ \partial \Lambda / \partial p \\ &= [\sigma - \mathbb{X}(\mu)] \circ \partial \Omega / \partial s + \mathbb{X}(\mu) \circ \partial \Lambda / \partial p \geq 0 \end{aligned} \tag{27}$$

where

$$\mathbb{X}(\mu) = \mu a + (1 - \mu) r m + (1 - \mu) k \mathbb{N}. \tag{28}$$

On the second model again we explicitly impose the non-negative condition of the mechanical dissipation rate.

The effective stress s^*, in this case, is represented by

$$\begin{aligned} s^* &= s - p \\ &= s - (a + rm + k\mathbb{N}) \\ &= (|s| - r) n - (a + k\mathbb{N}). \end{aligned} \tag{29}$$

The description of the effective stress is characteristic of the present modellings; it has an additive expression; the direction of the inelastic flow is influenced by both the isotropic variable and the anisotropic variable.

6 CONCLUSION

Unified constitutive models were developed on the basis of the irreversible thermodynamic approach. The models, the first and the second ones, essentially differ from the material models proposed so far in the specific description of the internal state variables for the isotropic and anisotropic strain hardenings and that of their evolution equations.

The characteristic of the models is the expression of the internal state variable p. It is partitioned into the isotropic hardening r and the anisotropic hardening a in the first model, i.e. eq.(14), and furthermore into the non-proportional hardening k in the second model, i.e. eq.(25). The time derivatives of these variables are described by eqs.(18a) and (18b) in the first model, and by eqs.(26) in the second model. Notice that the difference between these models disappears under proportional deformation.

Emphasis to be placed on is the explicit constraint of the non-negative dissipation rate, eq.(19) or (27); the inelastic process can evolve only when the condition is satisfied.

As far as a monotonic proportional deformation concerned, the models reduce to a single-variable model; in this case the partition (14) or (25) is not necessary. Therefore, we can easily identify the material constants involved by using eqs.(16) and (17).

The second model, which is a modified version of the first, satisfies eq.(23) for arbitrary deformation. It includes the third state variable which appears only under non-proportional deformation; it represents some effect of non-proportionality; it will therefore be interesting to discuss the characteristic of the variable in more detail.

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