# Reliability Analysis of Mechanical Components Using the Stochastic Finite Element Code PERMAS-RA

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## 1 INTRODUCTION

In many engineering applications and models data enter the analysis which must be considered as random or stochastic. This also applies to static analysis, since loads (i.c. wind, snow and earthquake to name a few), material properties (i.e. modulus of elasticity and material stiffnes) as well as geometric data (production tolerances like thickness and coordinates of nodes) are of a non-deterministic nature. These parameters must be described by suitable probability density functions with appropriate parameters for example, their mean and variance. Furthermore, the parameters themselves can turn out to have a stochastic character. PERMAS-RA allows a very good handling of these interconnected problems for a suitable model that is to be calculated.

## 2 THEORY

The basic problem of computing the failure probability of a structural component can be formulated as follows. Let $Z_s = (Z_{S1}, Z_{S2},\dots)$ be the random vector of load effects, $Z_r = (Z_{r1}, Z_{r2},\dots)$ the random vector of resistance quantities and $V_i = \{z = (z_r,z_s) \mid g(z_r, z_s) \le 0\}$ the failure domain with $g(z_r,z_s)$ the so-called limit state function. By convention $g(z_r,z_s) < 0$ defines failure and $g(z_r,z_s) > 0$ survival. The relation $g(z_r,z_s) = 0$ defines the limit state. The state variables Z, in general, are functions $Z = \mathcal{Z}(X)$ of the basic variable vector X collecting the uncertain variables such as loads, geometrical and material properties. The vector of basic variables $X = (X_1, X_2,\dots)$ has joint distribution function $F_X(x) = \mathbb{P}(\bigcap^n_{i=1} \{X_i \le x_i\})$. The failure probability then must be computed from

$$P_f = \mathbb{P}(F) = \mathbb{P}(g(Z) \le 0) = \int_V f_X(x)dx \qquad (1)$$

with $f_X(x)$ the joint probability density of X. A suitable approximation of this integral with sufficient accuracy for engineering practice can be evaluated by the so-called FORM method (First-Oder Reliability Method).

The FORM method transforms the integral (1) into a space of independent, standard normal variables U by a probability preserving transformation $X = \mathbb{T}(U)$ and approximates the limit state function by a linear expansion such that a well-known analytical result for a linear combination of normal variables can be used (Hohenbichler/Rackwitz 1981).

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$$P_f = \mathbb{P}(g(\mathbb{Z}(X)) \leq 0) = \mathbb{P}(g(\mathbb{Z}(\mathbb{T}(U))) \leq 0) = \int_{V} \Phi_U(u)du \approx \mathbb{P}(\alpha^T U \leq -\beta) = \phi(-\beta) \quad (2)$$

where

$$\beta = \| u^x \| = \min \{\|u\|\} \text{ for } \{u: g(\mathbb{Z}(\mathbb{T}(u))) \leq 0\} \quad (3)$$

and $u^x = -\beta\,\alpha$ the expansion point (ß-point). ß is so-called reliability index and $\Phi(.)$ the unit Gaussian distribution function. The choice of the ß-point as the expansion point is justified by arguments of asymptotic analysis. $u^x$ must be located by a suitable search algorithm. If the distribution of X is at least piecewise continuous and the limit state function is at least piecewise differentiable a gradient based search algorithm can be used which usually locates the ß-point in a few iteration steps. It is important to note that for many problems both the 'mechanical' transformation $\mathbb{Z}$ (X) as performed by the FEM code as well as the probability transformation $\mathbb{T}$ (U) are analytic so that the required gradient (in the u-space) can be given as (Abdo/Rackwitz 1990)

$$\nabla_u g^T = \nabla_x g^T J_{x,u} = \nabla_z g^T J_{z,x} J_{x,u} \quad (4)$$

where

$$\nabla_y g^T = [\frac{\delta g}{\delta y_1}, \frac{\delta g}{\delta y_2}, \ldots \frac{\delta g}{\delta y_n}]$$ = the gradient of the state function with respect to Y and

$$J_{v,w} = \left[ \frac{\delta v_i}{\delta w_j} \right]$$ = the Jacobian of the transformation $v = \mathbb{T}(w)$.

If the Jacobians $J_{z,x}$ and $J_{x,u}$ are analytic only the gradient $\nabla_z g$ needs to be evaluated numerically in a space with usually much smaller dimension than the x-space.

If neccessary FORM results can be improved by efficient importance-sampling schemes at the expense of some additional numerical effort (Hohenbichler/Rackwitz 1989). Also, the asymptotically exact Second-Order Reliability Method (SORM) can be employed (Breitung 1984).

## 3 CALCULATION STEPS

Ascertaining the reliability of a given structure is done with the help of a comparison with a given failure-criteria. This failure-criteria must be supplied in the form of a limit state function g(z,d), where the vector d contains all the deterministic and z the stochastic variables. Some elements of z can contain a basic variable directly while the rest of the elements in the z-vector depend on other basic variables through a mechanical transformation (FEM-Analysis). All basic variables (material properties, geometric data, loads) are collected in the vector of the basic variable x. The limit state function must be supplied by the user in a subroutine which makes PERMAS-RA an open system with regard to failure-criteria.

Calculation of the so-called reliability index ß (ß-point) is an optimization problem that can be solved iteratively by the program COMREL. Fig. 1

shows the dataflow of PERMAS-RA.

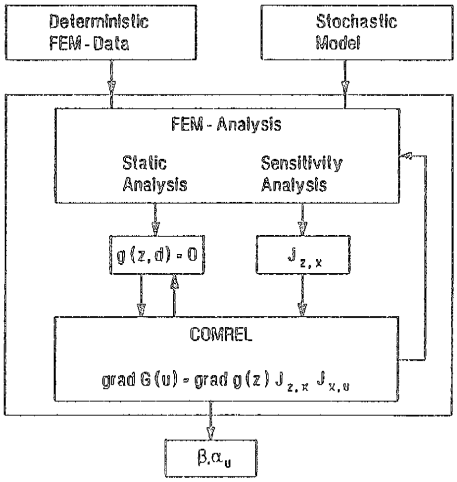


Fig. 1: Dataflow of PERMAS-RA

An iteration can be partitioned into the following problems: The program COMREL hands the current basic variables x to PERMAS. To start the iteration, suitable starting values (i.e. the mean value of the basic variables) are used. A control-program relates these values to the vector z of the limit state function or to the data-books of the FEM-algorithm. The FEM-analysis that follows is divided into two steps. During the static analysis $[r(x) = K^{-1}(R_o+R_R(x))]$ the system displacement vector r and the derived values for stress, strain and energy density that have an influence on the z-vector are calculated. The next step is a sensitivity analysis $[r(x)_x = K^{-1}(R_R(x)_x - K(x)_x\ r(x))]$. While the derivatives of the load vector R of the system with respect to the basic variables can easily be found, the derivatives of the system matrix K demand a large amount of computation. Dealing with this step on the element level can greatly reduce the amount of calculations involved. Thus, PERMAS can handle a great number of basic variables.

Finding the Jacobi-matrix $[J_{z,x}]$ is the aim of this sensitivity analysis. This matrix is then passed to the program COMREL which in turn calculates the reliability index ß through the correspondence with the limit state function g(z,d). To this end a transformation of the basic variable to the unit Gaussian distribution is performed $(X = \mathbb{T}(U))$. If the difference $\epsilon = \Delta\beta$ of two successive iteration steps is below a specified bound defined by the user, the iteration process is aborted. If the variation from the last iteration is not yet small enough, COMREL takes the current value for x as input to the next iteration. Another reason for abandoning the iteration would be to exceed a previously defined maximum number of iteration loops.

On exit of the calculation the PERMAS-RA user is supplied not only with a reliability estimate, but also with other information that allow an estimate of how sensitive the reliability index ß is with regard to the basic variables, selected parameters (i.e. deviation) or other deterministic factors. Given this information it becomes possible to limit the amount of basic variables to the essential ones in subsequent investigations. At present only first-order results for the reliability estimates are realized.

## 4 DATA INPUT

In addition to the usual input of the finite element model the definition of the stochastic model is needed. Three different kinds of information have to be provided: Definition of the basic variables, of the design parameters and of the limit state function including its parameters. Basic variables may be used as parameters of other basic variables, e.g. if the mean value of a property is again a stochastic variable, as design parameters which influence the stiffness of the model, as a load factor influencing the loading of the structure or as direct parameter of the limit state function. The type of the probability distribution function may be chosen out of 13 commonly used types. Upto several thousands of basic variables are possible. Correlation coefficients between normal or log-normal distributed variables may be specified. A design parameter is a vector of values that will influence the stiffness of the FE model. Three design parameter classes are possible: Sizing parameter such as plate or membrane thickness or cross sections, material properties and nodal point coordinates. The element set has to be defined which is influenced by the design variable.

The limit state function is a subroutine given by the user and describes the mode of failure of the structure. It is supplied with two types of parameters, namely deterministic and stochastic parameters resulting either from the FE calculation or from the input as a basic variable. Stresses, strains and displacements are possible as parameters. The search for their extreme values may be extended to the whole net or only parts of it.

## 5 EXAMPLES

All examples are calculated on a STARDENT GS 2000 with 4 processors and a vector processor.

### 5.1 Ten bar truss

The truss structure shown in Fig. 2 is loaded by two vertical forces at nodes 2 and 4. The cross section areas of the ten elements (log normal), the Youngs modulus (normal) and the loadings (normal) are the uncertain variables with a coefficient of variation of 10 %. The y displacement at node 2 is the stochastic constraint (normal) used in the limit state function. The reliability index $\beta$ = 0.961 (probability of failure $p_f$ = 0.1683) will be reached with $\epsilon$ = 0.001 after 4 iterations. The sensitivity parameters show that the Youngs modulus and the cross sections of element 1 and 3 are dominant. The computation time is 15.6 sec.

### 5.2 Pipe Junction

The pipe junction of Fig. 3 loaded with internal pressure has about 4800 DOF. Basic variables are the Youngs modulus (log normal, v = 5 %) and the load (normal, v = 10 %). The yield limit of the material is a deterministic value. It is compared with the max. v. Mises stress within the net. Two iterations are needed to get the $\beta$ value = 1.433 ($p_f$ = $0.759 \cdot 10^{-1}$ ($\epsilon$ = 0.001)) after 478 sec. With a stochastic yield limit (log normal, v = 10 %) one gets $\beta$ = 0.927.

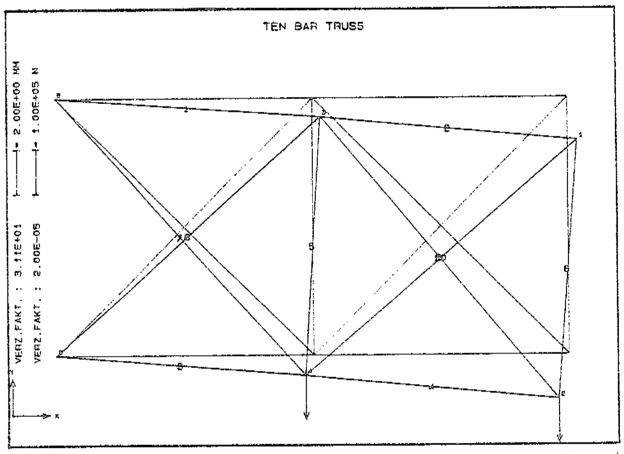


Fig. 2: Deformation of a ten bar truss

5.3 Tube

The axisymmetric model of the tube with 5 groups of elements is shown in Fig. 4. The model has 2500 unknowns and is loaded with a vertical load. Basic variables are the Youngs moduli (log normal), the loading (log normal) and the yield limit of the material (log normal). Failure occurs if the mean value of the v. Mises equivalent stresses in the left half of section A-A reaches the yield limit of 10 $N/mm^2$. This is comparable with a plastification to the middle. After 3 iterations and 253 sec the ß value ($\epsilon$ = 0.01) of 0.997 ($p_f$ = 0.1594) was reached. If the yield limit is deterministic the ß value is equal to 1.448.

5.4 Beam

A three dimensional beam with 7 different element groups is fixed at the left side area, suspended at the lower right edge and loaded on the top (Fig. 5). Basic variables are the Youngs moduli (log normal, v = 27 %) and the loading (normal, v = 10 %). The max. displacement is compared with a preset deterministic value. The ß value 1.551 ($p_f = 0.605 \cdot 10^{-1}$) is reached after 4 iterations (CPU = 341 sec). If only the dominant element groups 1, 2 and 6 are considered ß has the value of 1.588. The time is reduced in this case to 294 sec.

REFERENCES

Abdo Sarras, T., Rackwitz, R., Reliability of Uncertain Structural Systems. Proc. Finite Elements in Engineering Applications, Straßburg, 26./27.4.1990, INTES GmbH. 1990, pp. 161-176

Breitung, K., Asymptotic Approximations for Multinormal Integrals, Journ. of the Eng. Mech. Div., Vol. 110, No. 3, 1984, pp. 357-366

Hohenbichler, M., Rackwitz, R., Non-Normal Dependent-Vectors in Structural Safety. Journ. of the Eng. Mech. Div., ASCE, Vol. 107, No. 6, 1981, pp. 1227-1249

Hohenbichler, M., Rackwitz, R., Improvement of Second-Order Reliability Estimates by Importance Sampling, Journ. of Eng. Mechanics, ASCE, 1986, submitted.

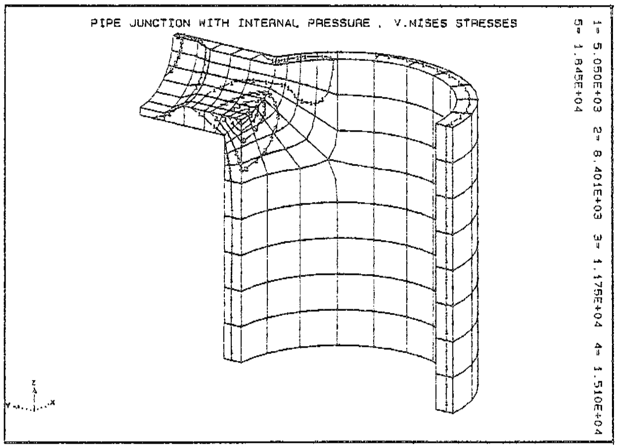


Fig. 3: Stress distribution in a pipe junction

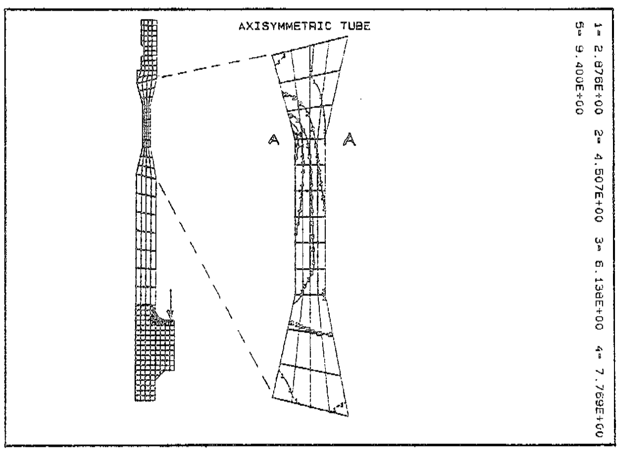


Fig. 4: Stress distribution in an axisymmetric tube

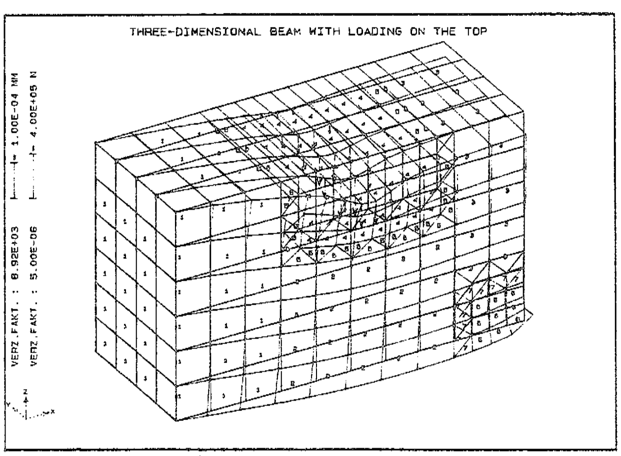


Fig. 5: Deformation of a three dimensional beam