ALimit Load Solution for an Edge Cracked Plate under Combined Independent Bi-Axial Membrane and Bending

Peter James\textsuperscript{a}, Dennis Hooton\textsuperscript{a}, and David Dean\textsuperscript{b}

\textsuperscript{a}Serco TAS, Birchwood Park, Risley, Warrington, WA3 6GA, UK, peter.james@sercoassurance.com
\textsuperscript{b}British Energy Generation Ltd., Barnett Way, Barnwood, Glouce., GL4 3RS, UK.

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1 ABSTRACT

Recent assessment work has shown that a means to estimate the reference stress for a residual stress distribution for a cylinder containing a circumferential crack would be greatly beneficial. This paper develops an improved limit load solution for combined bending and tensile loading both in plane (stresses acting perpendicular to the crack; akin to axial stress in a circumferentially cracked cylinder) and out of plane (stresses acting parallel to the crack; akin to the hoop stress in a circumferentially cracked cylinder).

The derived net stress limit load solution is for an edge cracked plate geometry and, as such, can provide a more appropriate solution when estimating the reference stress for use in both cracked plates and circumferentially cracked cylinders. This will in turn provide improved estimations of lifetime in creep and fracture assessments which include out of plane secondary stresses with the secondary reference stress based upon a limit load solution.

Current limit load solutions only consider circumstances where the out of plane stress is membrane; such as that contained within the Miller Compendium, with which comparisons are made. Also briefly considered is an adaption from an edge cracked plate to a semi-elliptical crack in a plate.

2 INTRODUCTION

Recently it has been suggested that an improved means to estimate the secondary, or combined, reference stress for a cylinder which contains out of plane secondary stresses is required. More specifically a solution is required which allows for biaxial loading and incorporates both membrane and bending stresses in both the in and out of plane directions. Currently, a Mises stress based solution for an edge cracked plate contained within Miller (1988, 227), shown in Equation (1) (with the terminology adapted to reflect that in this paper), is being applied to a cylinder under the assumption that the limit state for both should be similar. However, this “Miller Equation” only includes membrane (hoop) out of plane loading. It is also worth noting that other Limit Load solutions exist, such as Lei (2004a) which consider combined in plane loads.

\begin{equation}
\delta_{ref}^2 = \frac{3\delta_h^2}{4} + \frac{3Q^2}{4W^2b^2} + \frac{2M}{2Wb^2} + \left( \frac{4M^2}{4W^2b^4} + \left( \frac{\delta_h}{2} - \frac{N}{2Wb} \right)^2 \right)^{1/2} \tag{1}
\end{equation}

where \( \delta_{ref} \) is the reference stress, \( \delta_h \) is the hoop stress, \( Q \) is the shear stress, \( 2W \) is the plate width, \( M \) is the applied moment, \( b \) is the length of the uncracked ligament and \( N \) is the applied tensile load. This paper therefore develops a limit load solution for an edge cracked plate that includes the required biaxial loading conditions and compares this to the solution obtained for finite element models of both a plate and a cylinder. In doing so, the derived limit load solution has been validated and guidance for its applicability to a cylinder provided.

Given a value for the structural limit load, the reference stress can be derived from the limit load as shown in Equation (2) for a given applied load and yield stress. As such, the derivation of the limit load is a key input into procedures such as R5 (2003), R6 (2007) and BS7910 (2005). This is because the reference
stress provides a parameter “capable of providing accurate descriptions of energy dissipation, displacement rates, times for stress redistribution to the steady state, and the associated redistribution times” (Webster and Ainsworth (1994)), and is thus integral to both creep and fracture predictions.

\[
\dot{\sigma}_{ref} = \frac{P}{P_L(v, \alpha)} \dot{\gamma}
\]  

(2)

where, in Equation (2), \(P\) is the applied load, \(P_L\) is the limit load for the applied load with an elastic-perfectly-plastic material at the materials yield stress, \(\dot{\gamma}\), and crack size, \(\alpha\). In a real component, a secondary reference stress is hard to ascertain as the displacement controlled secondary stress will be reduced as the material approaches its yield stress. Therefore, it has been assumed that the difference between primary and secondary stresses is negligible prior to yielding. It is also noted that as secondary stresses are displacement controlled, gamma factors are generally applied to the in plane secondary loads to convert to a displacement controlled estimation from a load controlled limit load solution (see Equation (3)). As a result of this assumption, coupled with the use of gamma factors converting the stresses to displacement controlled solutions, the secondary reference stress can be obtained from the limit load solution derived for primary stresses.

\[
\dot{\sigma}^S_{ref} = f(\dot{\gamma}, \dot{\sigma}_m, \dot{\sigma}_b, N_1, N_2, m, M_1, M_2)
\]  

(3)

where, in Equation (3), \(\dot{\sigma}^S_{ref}\) is the secondary reference stress, \(\dot{\sigma}_m\) and \(\dot{\sigma}_b\) are the ratio of the displacement controlled and load controlled stress intensity factors for the tensile force and bending moment, \(N_1\) and \(N_2\) are the in plane and out of plane tensile force, \(M_1\) and \(M_2\) are the in plane and out of plane moments and \(f()\) is a general function describing the limit load for primary stresses. As a limit load established from primary loads can be used to provide an estimate of the secondary (and combined) reference stresses, this report only considers primary loads.

3 LIMIT LOAD SOLUTION FOR EDGE CRACKED PLATE

3.1 Estimation of Mises Stress

To estimate the limiting in plane stress, the Mises equation (see Equation (4)) was used and a similar method to Lei (2004b) adopted. However, the applicability to a cylinder is limited as a constant radial stress would have to be assumed. The derived solution for the in plane stress is shown below in Equation (5).

\[
\left(\dot{\sigma}_i^2 + \dot{\sigma}_j^2 + \dot{\sigma}_k^2\right) \left(\dot{\sigma}_i \dot{\sigma}_j + \dot{\sigma}_j \dot{\sigma}_k + \dot{\sigma}_k \dot{\sigma}_i\right) = \dot{\sigma}_y^2
\]  

(4)

\[
S_i^+ = \frac{1}{2} \left[ \left( S_2 + S_3 \right) \pm \sqrt{4 - 3 \left( \frac{S_2^2 + S_3^2 - S_2 S_3}{S_2^2 + S_3^2 + S_2 S_3} \right)} \right]
\]  

(5)

where \(\dot{\sigma}_i\) is the stress acting in the \(i^{th}\) direction and \(S_i = \dot{\sigma}_i / \dot{\sigma}_y\). This means that two values for \(S_i\) (\(S_i^+\) and \(S_i^-\)) can be obtained, based upon known values of \(S_2\) and \(S_3\). Therefore, the values of \(S_i^+\) and \(S_i^-\) can be used within the limit load solution to estimate the point at which the material has yielded.

The above equation is shown where all the principle stresses are assumed to be membrane. For the circumstances where the out of plane loading includes a bending stress the evaluation of a unique stress for use as \(S_2\) becomes more difficult. The easiest means to provide a unique value for the out of plane stress is to use an average through the thickness of the combined membrane and bending. However, this means that, for cases where the bending component has no net stress (i.e. residual), the bending effect is not considered.
However, if it is noted that the crack will only have a minor influence on the out of plane stress, the out of plane stresses can be considered independent of crack length. This means that if the out of plane stress is only considered for the section acting upon the uncracked ligament, as is considered for the in plane stress, the out of plane membrane and bending stress can be used to provide \( S_2 \) by the following equation.

\[
S_2 = \frac{1}{\delta_y L} \frac{\delta y a}{t} - \frac{\delta y a}{t}
\]

(6)

where \( \delta y a \) and \( \delta y a \) are the membrane and bending components of the stress acting in the second principle axis, \( a \) is the crack depth and \( t \) is the wall thickness.

### 3.2 Arbitrary Shaped Crack

Figure 1 provides an illustration of the plate model and axis with which this equation is based. For cases where a combined tensile and bending moment is applied, two simultaneous equations are formed. Equation (7) is for the tensile failure load, \( N_L \); where \( \delta y a \) and \( \delta y a \) are the net-section (tension and compression) stresses at failure, \( x \) is the distance along the plate width, \( a(x) \) is the crack depth as a function of \( x \) and \( \dot{a} \) is the position of the neutral axis. Under combined loading the proportion of the tensile load at failure to the uncracked body under purely tensile loads, \( N_0 \), is provided by \( n_L \) (see Equation (8)), which is evaluated in Equation (9).

\[
N_L = \frac{1}{2} \left( \delta y a - 2Wt - \int a(x) dx \right) + \left( \delta y a - \delta y a \right) \dot{a} \dot{a}
\]

(7)

\[
n_L = \frac{N_L}{N_0} = \frac{N_L}{2W \dot{a} y}
\]

(8)

\[
n_L = \frac{\dot{a}}{t} \left( \delta y a - S x \right) - \left( \delta y a + S x \right) \left( \int - \frac{1}{2Wt} \int a(x) dx \right)
\]

(9)

The same process is defined for the bending moment contribution to provide the normalised limit moment, \( m_L \), for an arbitrary shaped crack shown in (12). Where \( M_L \) is the applied moment at failure and \( M_0 \) is the applied moment at failure for the uncracked body under pure bending.

\[
M_L = \frac{\delta y a - \delta y a}{8} \left\{ 2Wt^2 - 8W \dot{a}^2 + \int \left( t(x)^2 - 2t \dot{a}(x) \right) dx \right\}
\]

(10)

\[
m_L = \frac{M_L}{M_0} = \frac{2M_L}{t^2 W \dot{a} y}
\]

(11)

\[
m_L = \frac{\delta y a - S x}{2} \left\{ \int - \frac{\dot{a}^2}{t^2} + \int \frac{a(x)^2}{2Wt^2 - \dot{a}(x)} dx \right\}
\]

(12)
3.3 Constant Depth Crack

For an edge cracked plate, or for an extended circumferential crack when applied to a cylinder, the integrals within Equations (9) and (12) can be evaluated to provide the solutions in Equation (13); where \( a_0 \) is the crack depth.

\[
\begin{align*}
\eta_L &= \frac{\hat{\alpha}}{t} \left( \hat{\xi}_i^+ - \hat{\xi}_i^- \right) \left( \frac{\hat{\xi}_i^+ + \hat{\xi}_i^-}{2} \right) \left( 1 - \frac{a_0}{t} \right) \\
m_L &= \frac{\left( \hat{\xi}_i^+ - \hat{\xi}_i^- \right)}{2t^2} \left[ (t - a)^2 - 4 \hat{\alpha}^2 \right] 
\end{align*}
\]

3.4 Semi-Elliptical Crack

If the crack is assumed to be semi-elliptical, with the geometry shown in Figure 2, the crack depth will change in the x axis as in Equation (14), so that the integral for the crack depth can be approximated to Equation (15) (with Gauss’ Hyper-geometric Function, \( _2F_1 \), defined in Equation (16)).

\[
a(x) = \begin{cases} 
  a_0 \sqrt{1 - \left( \frac{x}{c} \right)^2} & \text{for } x \leq c \\
  0 & \text{for } x > c 
\end{cases}
\]

\[
W \int_0^c a(x) dx = a_0 x_2F_1 \left[ \frac{1 - 1}{2} \cdot \left( \frac{x}{c} \right)^2 \right] 
\]

\[
_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} = 1 + \frac{abz}{c} + \frac{a(a+1)b(b+1)z^2}{c(c+1)} + \ldots 
\]

When the crack is fully enclosed within the width of the plate the limits of the integral are 0 to \( c \) (where \( c \) is the semi crack length), which allows \( _2F_1 \) to be evaluated to 0.78648. However, under conditions where the crack breaches two edges of the plate (i.e. is upon a corner) and the limits of the integral are no longer 0 and \( c \), the value of \( _2F_1 \) would have to be re-evaluated. This allows the integral within Equations (9) and (12) to be evaluated so that the normalised limit loads for a semi-elliptical crack become:
\[
\begin{align*}
n_L &= \frac{\hat{\alpha}}{t} \left( \sigma_i^+ - \sigma_j^- \right) \frac{t^2}{2} \left( \frac{a_0}{2W} \right) F_j \\
m_L &= \frac{\left( \sigma_i^+ - \sigma_j^- \right)}{2} \left[ 1 - \frac{\hat{\alpha}}{t^2} \right] + \frac{a_0^2}{3} \frac{1}{W} - \frac{a_0}{t} \frac{c}{W^2} F_j
\end{align*}
\] (17)

Figure 2 – Geometry used to define the Semi-Elliptically Cracked Plate

3.5 Means of Solving

Within the solutions presented there are 3 unknowns, \( n_n \), \( m_n \) and \( \hat{\alpha} \). Estimates of \( n_n \) and \( m_n \) can be found by simultaneously solving Equation (5), Equation (6), Equation (13) and Equation (19) which assumed proportional loading. This also follows the example shown in R5 (2003) and R6 (2007) where;

\[
L_y = \frac{\sigma_y}{\sigma_y} = \frac{M}{M_L} = \frac{N}{N_L}
\] (18)

Therefore, in the work presented the ratio of the stresses has been held constant, as determined by the parameter \( \hat{\varepsilon} \), where \( \hat{\varepsilon} \) is defined (for a plate) as:

\[
\hat{\varepsilon} = \frac{M}{tN} = \frac{M_L}{tN_L} = \frac{\sigma_y}{6\varepsilon_m}
\] (19)

Therefore, by maintaining \( \hat{\varepsilon} \), a value for \( \hat{\alpha} \) can be found which can be used to find \( n_L \), \( m_L \). This can be easily achieved within a spreadsheet by using the “goal seek” (within Excel) or other similar functions.

4 FINITE ELEMENT APPROXIMATION

To validate the derived solutions, finite element analyses (within ABAQUS 6.8.1 (2008)) were used with three different out of plane loading combinations. Initially, to validate Equation (13), an edge cracked plate was used with only 1 element (C3D20RH elements) thickness, thus ensuring plane strain conditions with the ability to apply out of plane loads. Where the geometry was modified to a cylinder the same mesh was adapted to a 2D axi-symmetric (by using CAX8R elements) case to simulate a fully circumferentially cracked plate. The effect of the cylinder radius was also investigated to estimate where the solutions break down. It is worth noting that an in plane, through wall, bending was applied as a primary stress upon the cylinder in the finite element analyses as a primary load even though this would only be found in a real component with secondary stresses. Finally two semi-elliptical crack depths were used within a plate model to consider the solutions presented by Equation (17). The material properties used for all models represented
an elastic-perfectly-plastic material with a yield stress of 300 MPa, a Young’s Modulus of 3,000,000 MPa.mm$^{-1}$ and a poisons ratio of 0.3.

4.1 Geometry

4.1.1 Edge Cracked Plate
The three dimensional model had a wall thickness of 10 mm, a height of 15 mm and a depth of 0.01 mm. The depth is only 0.01 mm as only 1 element was used in thickness so that plane strain conditions were enforced and an out of plane stress could be applied. Symmetry conditions were applied along the 10 mm thickness so the total length of the model assumed was 30 mm. The crack was also positioned upon this through thickness symmetry. Therefore, a range of different crack lengths ($a/t = 0, 0.1, 0.2, 0.4, 0.6$ and $0.8$) can be obtained by adjusting the symmetry conditions.

4.1.2 Fully Circumferential, Externally Cracked Cylinder
The plate geometry was adopted for the edge crack plate as used for the cylinder geometry by using axi-symmetric elements. Therefore, the cylinder had a wall thickness of 10 mm, a height of 15 mm and an internal radius of 100 mm making the cylinder approximately thin shelled ($r_m/t = 10$). Note that the short height of the model was beneficial in the cylinder as it allowed more of the bending stress to be transferred to the crack. This would not happen if the height was much larger as a through thickness bending stresses can only act over a short range in a cylinder because of the self-constraint effects. However, if the height was too small it would affect the resultant stress field. The same range of crack lengths as used in the plate was modelled by modifying the symmetry conditions. When investigating the effect of the cylinder radius on the accuracy of the results, one crack depth corresponding to $a/t = 0.2$ was used and the internal radius changed to values of $r_i = 10, 30, 50, 75, 100, 200, 400$ and $600$ mm.

4.1.3 Semi-Elliptical Crack in a Plate
To model a semi-elliptical crack in a plate, the same 10 mm thick and 15 mm high model was used but the thickness was increased to 5 mm to allow the crack to be introduced. The crack was again introduced by modifying the through thickness symmetry condition. Because of the symmetry condition in the depth direction, only half the crack was modelled. Two crack depths were used, $a_0/t = c/2W = 0.1$ and $0.4$ (note that as a semi-elliptical crack, $a_0 = c$), where $a_0$ is the crack depth at the deepest point and $c$ is the half crack length.

4.2 Loading

4.2.1 In Plane Loads
In all cases, the applied loads were those that would be applied to the un-cracked geometry to result in the desired tensile or bending stress. Therefore, the applied tensile loads, and applied moment, are:

$$N_I = \dot{\sigma}_{ml} t 2W$$

$$M_I = \frac{1}{3} \dot{\sigma}_{bl} t^2 W$$

where, $2W$ is the model width, $\dot{\sigma}_{ml}$ is the in plane tensile stress and $\dot{\sigma}_{bl}$ is the in plane bending stress. Note that, for the cylinder, the circumference of the cylinder, at the mean radius, was used as $2W$. To ensure that the tensile load remained tensile and did not induce any bending, the loading surface was held from rotating. Where, this condition had to be relaxed because of additional bending, a counter moment was applied equal to the reaction moment seen within the fixed run (for that crack depth) where the surface was constrained from bending. Note that this was not required for the cylinder because of the self constraining effects. Generally 5 different possible combinations of tensile and bending loading were used. These corresponded to tensile stress to bending stress ($\dot{\sigma}_{ml}:\dot{\sigma}_{bl}$) ratios of $1:0, 2:1, 1:1, 1:2$ and $0:1$. These then correspond to values for the loading ratio, $\bar{\varepsilon}$, of $0, 1/12, 1/6, 1/3$ and $>1000$. 

6
The in plane loads were applied as a static step with a minimum permissible increment size of 0.001 MPa and allowed up to 300 increments per step to reach failure (note that only 1 run required above 200 increments). The bound limit load was taken at the maximum load which could be solved.

4.2.2 Out of Plane Loads

Three different out of plane stresses were applied to the plate model corresponding to: (a) no out of plane load, (b) an out of plane stress of 50 MPa and (c) an out of plane stress of 50 MPa tensile and 100 MPa bend. The cylinder only had a hoop stress of 50 MPa produced by an internal pressure (which was adjusted when changing the cylinder radius to ensure the right hoop stress).

4.2.3 Application of the Loads

The in plane loads were applied as a concentrated load to a reference node upon a rigid surface. This was to ensure that the load was distributed evenly across the rigid surface before being transferred to the material and to allow the direction of the force to be easily controlled. This was necessary as a pressure on the loading surface would have additional components of stress when a bending stress is also applied. The presence of the rigid body contact also ensured that the surface remained flat during yielding, thus prevented excessive deformations at the edges, preventing the solution failing too early and giving erroneous results.

The out of plane loads were applied by a pressure in all cases. The plates had a pressure applied directly to the surface, and where the bending plus membrane out of plane stress was applied, a user defined analytical field was used to enforce a non uniform pressure. The cylinder hoop stress was applied through applying an internal pressure to the cylinder.

5 COMPARISON FINITE ELEMENT RESULTS WITH LIMIT LOAD SOLUTION

5.1 Edge Cracked Plate

The results for the edge cracked plate shown in Figure 3 to Figure 5 show that, in general, Equation (13) provides good estimates for the limit load, for any combination of tensile and bending loads, and for all out of plane loads. The plots show that the best results are for the cases where the membrane stress dominates, and for the un-cracked plate. Conversely, for the cracked geometry, the results estimated by Equation (13) appear to be a constant factor of 1.15 to 1.2 conservative. This factor is approximately that by which the Tresca and Mises solutions differ. A potential explanation for this would be the additional out of plane stresses induced in the third axis through crack-tip triaxiality. However, this effect would be expected to be most prominent at small crack depths and to diminish as the crack length gets larger. Another possible explanation is that the equation used to normalise $M_L$ (Equation (11)) and introduce $m_L$ is based upon a Tresca solution for an uncracked structure. This would mean that, for the cracked cases, an additional factor of $2/\sqrt{3}$ can be applied to Equation (11), to provide the limit load solution for an edge cracked plate shown in Equation (22). It is worth noting that this is similar to the case presented in the Miller Compendium (1988), where, for deeper cracks, the Equation (11) is adjusted by a factor of between 1.072 (plane stress Mises) and 1.455 (plain strain Mises) but has to merge with Equation (11) for the uncracked case.

$$n_L = \frac{\hat{a}}{t} \left( \sigma_i^+ - \sigma_i^- \right) \left[ \frac{\sigma_i^+ + \sigma_i^-}{2} \left( I - \frac{a}{t} \right) \right]$$

$$m_L = \frac{(\sigma^+ - \sigma^-)}{4\sqrt{3}t^2} \left[ I - a^2 - 4\hat{a}^2 \right]$$

(22)

When compared to the Miller solutions, the results provided by Equation (13) are seen to be very similar. That the Miller Equation and the derived solution provide similar results provides additional validation to the derived solutions. The advantage that the derived solution holds is that it can also be applied to cases where out of plane bending has a strong influence and for non-uniform cracks.
5.2 Circumferentially Cracked Cylinder

The results for the circumferentially cracked cylinder shown in Figure 6 show that Equation (13) provides good estimates of the limit load for the un-cracked cylinder and for small crack lengths. Again, Equation (13) provides the best results are for the cases where the membrane stress dominates, and for crack lengths less than $2.0 \frac{a}{t}$. However, at larger crack depths, and for the cases where bending dominates, the difference becomes very significant. For the worst case the conservatism can be seen to be as high as a factor of six on the finite element result.
The significant difference at larger crack depths can be explained, at least partly, by the self constraining effect of the cylinder geometry. Effectively, the bending moment applied to the cylinder will induce a compressive hoop stress at the point where the load is applied. This hoop stress will constrain the bend of the pipe and effectively remove a proportion of the applied load before it can affect the crack. As the magnitude the cylinder wants to deform is related to the magnitude of applied bending load and the depth of the crack, the bending and crack depth will also change the apparent limit load. However, the solutions can be seen to remain conservative.

It is also worth noting that (as this solution is most likely to be used for a displacement induced secondary stress field), with a displacement induced stress field, the bending stress is already present at the crack tip. This will remove a limited amount of the apparent conservatism, and, in effect, will bring the solution closer in line with the plate solution. Therefore, it is proposed that the derived solutions are still valid for a through thickness displacement controlled secondary stress.

A short sensitivity study of the results when changing the internal radius showed a change in results as the internal radius of the cylinder approached 100 mm. This is the point which represents the boundary between a thick shell and thin shell cylinder. It is therefore the point at which the radial stress becomes significant and the hoop stress can no longer be approximated to be a constant. This thus provides the limit to which a cylinder can be confidently approximated by a plate solution. It is worth noting that including a constant radial stress into Equation (5) did not improve the results significantly.
6 CONCLUSIONS

A net stress limit load solution for a cracked plate under combined, multiaxial tension and bending, is presented, with the two cases of an edge cracked plate and a semi-elliptical cracked plate highlighted. The solution has been derived specifically for obtaining the secondary reference stress for cases where an out of plane secondary stress is present. Common examples of this may be a bi-axial thermally induced bending stress or weld residual stress fields. Also considered is how the derived solution can be adapted to a cylinder. This is considered an important comparison as current plate solutions (namely the Miller equation) are being used as an approximation to cylinder geometry. The comparisons performed within this report therefore provide guidance to the limits of this assumption. The following conclusions were obtained through the course of the investigation:

(a) The derived solutions provide very good limit load comparisons with finite element results for the plate geometries.

(b) Application to the cylinder results is a good approximation to the limit load for small crack depths and where the tensile stress dominates.

(c) Under load controlled analyses, the self constraint effect of the cylinder makes the derived solutions increasingly conservative. However, where the stress is residual in nature, the self constraint effect is reduced, and it is believed that the plate solutions would provide better agreement.

(d) An improvement to the solution is possible by taking Mises plane strain or plane stress into account into the equation detailing the normalised moment. This is a case by case consideration, and is probably only required if Equation (13) or (17) are unsuitable.

(e) The application of the derived plate solution to a cylinder is seen to break down as the cylinder becomes thick shelled; as the radial stress and varying hoop stress become more significant.

REFERENCES


