Dynamic model of a simple supported RC rectangular plate for spreadsheet application – Part I: Motion in viscous-elastic domain

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1 ABSTRACT

The resolution of the equation of motion of a thin plate in elastic domain is based on the modal analysis: the displacements, bending moments and applied loadings are expressed in the eigenmode basis:

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin(m.\frac{\pi}{a}).\sin(n.\frac{\pi}{b}).$$

The pertinent variables for solving the equation of motion are the velocity $V$ and the structural response $F_s$:

$$F_s = \frac{\partial^2 Mx}{\partial x^2} + 2.\frac{\partial^2 Mxy}{\partial x\partial y} + \frac{\partial^2 My}{\partial y^2}.$$ In the modal basis, the equation of motion is superseded by infinity of equations:

$$Q_{mn}^t \dot{V}_{mn}^t + 2.\dot{\omega}.V_{mn}^t + \frac{1}{\bar{n}.h} F_{s_{mn}}^t = \frac{1}{\bar{n}.h} Q_{mn}^t, \ m \ and \ n \in N^*.$$

After classical finite difference approximation, the values of $V_{mn}^t$ and $F_{s_{mn}}^t$ are computed at time step $t+\Delta t$ from their value and from $Q_{mn}^t$ loading value at time step $t$ according to iterative relations:

$$\begin{align*}
(1+\dot{\omega}.\Delta t + \frac{\Delta t^2}{4.\rho.h}.D_{mn}).V_{mn}^{t+\Delta t} &= (1-\dot{\omega}.\Delta t - \frac{\Delta t^2}{4.\rho.h}.D_{mn}).V_{mn}^t - \frac{\Delta t}{\bar{n}.h} \cdot (F_{s_{mn}}^t - \frac{Q_{mn}^{t+\Delta t} + Q_{mn}^t}{2}) \\
(1+\dot{\omega}.\Delta t + \frac{\Delta t^2}{4.\rho.h}.D_{mn}).F_{s_{mn}}^{t+\Delta t} &= (1-\dot{\omega}.\Delta t - \frac{\Delta t^2}{4.\rho.h}.D_{mn}).F_{s_{mn}}^t + \Delta t.D_{mn}.V_{mn}^t + \frac{\Delta t^2}{4.\rho.h}.Q_{mn}^{t+\Delta t} + \frac{Q_{mn}^t}{2}
\end{align*}$$

The bending moment tensor is deduced from the curvature. The rectangular plate is discretized into 30x30 elements.

The changes of basis from physical space into modes space and back are operated through matrix products. The value of the projections of the loading $Q$ on the modal basis involves integrals that can be computed by Simpson’s rule and can be operated through the same matrices of basis change.

The implementation on current spreadsheet software (like Excel®) is easy and allows calculating the structural response of the plate within few seconds on any current personal computer.

2 INTRODUCTION

This paper is intended to provide structural engineers with a simple and efficient calculation code, on a current spreadsheet software (of EXCEL® type), for the study of the dynamic motion of a simply supported rectangular plate submitted to any variable loading. Civil engineers in the nuclear industry often have to evaluate the effect of a loading on a plate: impact of drop weight, impact of internal or external missile on wall, impact of military or commercial airplane for instance. They have at their disposal either tables of results (Bares) or a FEM computation code. The use of these methods can be laborious. For the static loading application, the method is derived from the decomposition of the plate loading and resulting displacement on an eigenmode basis. Such a procedure is well known. For the dynamic loading and resulting plate movement, the method presents a resolution way based on the extension of eigenmode basis decomposition. The motion of the plate beyond its elastic domain is considered in the companion paper [Rambach, 2009].
3 BEHAVIOR OF AN ELASTIC PLATE UNDER STATIC LOADING

3.1 Fundamental equations

The expression of the deformation of an elastic, homogeneous and isotropic plate, when neglecting the shear deformation, is given by the bending moment – curvature law:

\[
\begin{align*}
M_x &= D \left( \frac{\partial^2 w}{\partial x^2} + \frac{i}{1 - \nu} \frac{\partial^2 w}{\partial y^2} \right) \\
M_y &= D \left( \frac{\partial^2 w}{\partial y^2} + \frac{i}{1 - \nu} \frac{\partial^2 w}{\partial x^2} \right) \\
M_{xy} &= D (1 - i) \frac{\partial^2 w}{\partial x \partial y} \\
\end{align*}
\]  

(1)

and the equilibrium relation:

\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - Q = 0
\]

(2)

with the usual notation:

\[\begin{align*}
w: & \text{ displacement of the point (x,y) of the mean surface of the plate,} \\
D &= \frac{E h^3}{12(1 - \nu^2)}, \text{ stiffness modulus of the plate} \\
E: & \text{ Young’s modulus and } \nu: \text{ Poisson’s ratio of plate material} \\
h: & \text{ plate thickness, supposed constant.} \\
M_x, M_y \text{ and } M_{xy}: & \text{ components of the bending moment tensor,} \\
Q: & \text{ external loading density applied transversally and supposed constant.}
\end{align*}\]

The association of the elastic bending moment – curvature law (1) with the equilibrium relation (2) gives the fundamental differential equation:

\[
\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{Q}{D} = 0
\]

(3)

Equation (3) is valid whatever the contour of the plate and its liaison mode and whatever the distribution of the external loading.

3.2 Decomposition of displacement and loading on the eigenmode basis

The resolution of the above equation of motion in elastic domain is based on the modal analysis of a simply supported rectangular plate (Navier’s method). It can be demonstrated that the eigenmodes of deformation of such a rectangular plate can be characterized by the following normalized modal surfaces:

\[
S_{mn}(x, y) = \sin (m \cdot \frac{x}{a}) \cdot \sin (n \cdot \frac{y}{b})
\]

(4)

where \(a\) and \(b\) are the length of the adjacent sides of the plate, along the x and y directions (see Fig. 1 and Fig. 2) and where \(m\) and \(n\) are integers that characterize the number of extrema along x and y directions.

\textbf{Figure 1.} Co-ordinate of a rectangular plate \(a \times b\)
It can be demonstrated that the eigen deformation modes constitute a basis on which any continuous deformation \( W(x, y) \) can be decomposed as follows:

\[
W(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b})
\]

(5)

where \( W_{mn} \) is the weight of the real displacement \( W \) of the plate of the mode \( S_{mn}(x, y) \).

The loading \( Q(x, y) \) can be expressed in the same modal basis under the following notation:

\[
Q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b})
\]

(6)

where the coefficient \( Q_{mn} \) is the weight of \( Q \) for the mode \( S_{mn} \).

Owing that

\[
\int_{0}^{a} \sin(i \cdot \frac{x}{a}) \cdot \sin(j \cdot \frac{x}{a}) \cdot dx = (a/2) \cdot \bar{a}_i \quad \text{with} \quad \bar{a}_i = 0 \quad \text{when} \quad i \neq j \quad \text{and} \quad \bar{a}_i = 1 \quad \text{when} \quad i = j
\]

(and symmetrically for the b side), it can be checked that the coefficients \( Q_{mn} \) of the decomposition (6) are given by:

\[
Q_{mn} = \frac{4}{a \cdot b} \int_{0}^{a} \int_{0}^{b} Q(x, y) \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b}) \cdot dx \cdot dy
\]

(7)

3.3 Resolution method

The resolution of the ODE (3) is made in the modal space, where the derivation is translated in a multiplication, namely:

\[
\frac{\partial^4 W}{\partial x^4} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m \cdot \hat{\partial}}{a} \right)^4 \cdot W_{mn} \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b})
\]

(8a)

\[
\frac{\partial^4 W}{\partial y^4} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{n \cdot \hat{\partial}}{b} \right)^4 \cdot W_{mn} \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b})
\]

(8b)

\[
2 \cdot \frac{\partial^2 W}{\partial x^2 \partial y^2} = 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m \cdot \hat{\partial}}{a} \right)^2 \cdot \left( \frac{n \cdot \hat{\partial}}{b} \right)^2 \cdot W_{mn} \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b})
\]

(8c)
After introducing these expressions of the second derivatives of \( W \) into the ODE (3), it can be easily established, due to the unicity of the solution, that the relationship between \( W_{mn} \) and \( Q_{mn} \) is obtained by:

\[
\left( \frac{m\alpha}{a} \right)^2 + \left( \frac{n\beta}{b} \right)^2 \right) \cdot W_{mn} = \left( \frac{1}{D} \right) \cdot Q_{mn} \quad \Rightarrow \quad W_{mn} = \frac{Q_{mn}}{D_{mn}} \tag{9}
\]

with \( D_{mn} = D \left( \frac{m\alpha}{a} \right)^2 + \left( \frac{n\beta}{b} \right)^2 \) : modal stiffness \tag{10}

The principle of the resolution is summarized in following actions:

- to transform the loading \( Q \) into coefficients of \( Q_{mn} \) type, by projection on the modal basis, by using relation (7)
- to divide each modal coefficient \( Q_{mn} \) by the modal stiffness \( D_{mn} \) in order to get the modal coefficients of displacement \( W_{mn} \), by using relations (9) and (10),
- to come back into the physical space by modal recombination of all \( W_{mn} \), by using relation (5), in order to get the physical displacement \( W(x,y) \).

3.4 Practical algorithms for the resolution on a spreadsheet

**Discretization**

The modal basis dimension is theoretically infinite, but it is sufficient to limit the summation for \( n \) and \( m \) as follows: \( 1 \leq n \leq N \) and \( 1 \leq m \leq M \), \( N \) and \( M \) having sufficiently high values. The spreadsheet presentation, with cells organized in rows and columns, is well adapted for the representation of the spatial distribution of the variable characteristics of any rectangular plate.

The characteristics of the plate are considered with the spatial discretization of \( N \times M \) unit cells, the plate characteristics are then naturally described by a \( N \times M \) matrix.

It is convenient to consider \( N = M = 30 \): the 900 first modes can capture with sufficient accuracy the current loading distribution and resulting plate deformation.

**Transfer matrix**

Two types of spaces have to be considered in the computation: the real physical space \((x,y)\) coincident with the mean surface of the plate and the modal space \((m,n)\). It is necessary to transform the loading in terms of \( Q_{mn} \) coefficients, belonging to the modal space, so that the modal displacement \( W_{mn} \) can be easily computed and it is necessary to come back in the real physical space, in order to obtain the physical displacement \( W(x,y) \).

The transformation from the physical space \((x,y)\) into the modal space \((m,n)\) and reciprocally is allowed by the \( N \)-order transfer matrix \([H] = [h_{mn}]\) with:

\[
h_{mn} = \sin\left( \frac{m\pi}{N} \right) \quad \text{with } m,n = 1, 2, \ldots N \tag{11}
\]

Let us define the loading \( Q \) by the \( N \)-order matrix \([Q] = [Q_{ij}]\) with:

\[
Q_{ij} = Q(x,y) \quad \text{with } x = i.a/N \text{ and } y = j.b/N \text{ on each cell and } i,j = 1, 2, \ldots N. \tag{12}
\]

\([Q]\) is the loading matrix expressed in the physical space.

The double integral (7), necessary for the transformation of the loading from the physical space in the loading in the modal space, may be approximated by using the Simpson rule. It can be noted that the double summation of (7) for the computation of the \( Q_{mn} \) coefficients of the loading matrix in the modal space can be expressed as a result of the following matrices product:

\[
[Q^*] = (4/N^2).[H].[Q].[H] \tag{13}
\]
where \[^{t}H\] is the transposed matrix of \([H]\) (\([H]\) being symmetric, \[^{t}H\] = \([H]\) )

An asterisk has been introduced in the expression of the loading in modal space, in order to distinguish it when expressed in the physical space: \([Q^*]\) is the matrix of the loading when expressed in modal space.

**Resolution**

The loading is transposed from physical space into modal space: \([Q^*]\) is deduced from \([Q]\) by (13). The matrix of the displacement \([W^*]\), in the modal space, is easily deduced from \([Q^*]\) by (8); each term \(q_{mn}\) of the matrix \([Q^*]\) has to be divided by \(D_{mn}\) expressed by (10).

The return of the displacement matrix \([W^*]\) from the modal space into the physical space is obtained by (5) whose translation in terms of matrices product is:

\[
[W] = \[^{t}[H]\].[W^*].[H] \tag{14}
\]

**Bending moment tensor**

The components (\(M_x, M_y, M_{xy}\)) of the bending moment tensor are deduced from the expressions of the bending moment curvature law (1) which need the computation of the second derivatives of \(W\): \(\frac{\partial^2 W}{\partial x^2}\), \(\frac{\partial^2 W}{\partial x \partial y}\) and \(\frac{\partial^2 W}{\partial y^2}\). The components of the bending moment tensor can be obtained by summation of products of \(W_{mn}\) and sin and cos functions of \(x\) and \(y\) and/or \(m\) and \(n\), but it is more convenient, and not less accurate, to approximate the second derivatives of \(W\) by the finite centred difference method as follows:

\[
\begin{align*}
\frac{\partial^2 W}{\partial x^2} &= W(x + Ax, y) - 2W(x, y) + W(x - Ax, y), \\
\frac{\partial^2 W}{\partial x \partial y} &= W(x + Ax, y - Ay) - W(x + Ax, y + Ay) - W(x - Ax, y + Ay) + W(x - Ax, y - Ay), \\
\frac{\partial^2 W}{\partial y^2} &= 4 \cdot Ax \cdot Ay
\end{align*} \tag{15}
\]

The shearing forces \(T_x\) and \(T_y\) are deduced from the expressions of the components of the bending moment tensor, according to the classical relations of the strength of materials and according to the finite centred differences classical approximation:

\[
\begin{align*}
T_x &= \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = \frac{My(x, y + Ay) - My(x, y - Ay)}{2 \cdot Ay} + \frac{Mxy(x + Ax, y) - Mxy(x - Ax, y)}{2 \cdot Ax}, \\
T_y &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = \frac{Mx(x + Ax, y) - Mx(x - Ax, y)}{2 \cdot Ax} + \frac{Mxy(x, y + Ay) - Mxy(x, y - Ay)}{2 \cdot Ay}
\end{align*} \tag{16}
\]

The above expressions are adapted (forward or backward instead of centred finite differences) for the cells adjacent to the edges.

**Checking**

As a feedback control, the following relation has to be verified:

\[
\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} = \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = Q \tag{17}
\]

Another verification can be performed by using the energy theorem: the work done by the external loading on the plate \(E_Q\) has to be equal to the elastic energy stored in the plate \(E_{elastic}\).

The work done by the external loading is given by:

\[
E_Q = \frac{1}{2} \int_{a}^{b} \int_{0}^{b} Q(x, y) W(x, y) dx dy \tag{18}
\]

The scalar \(E_Q\) is computed by using the Simpson rule, practically by adding all terms of the matrix \([E_Q]\) whose generic term is defined by \(E_{Qij} = \frac{1}{2} Q_i W^0 \Delta x. \Delta y\), with \(W^0 = W(i \Delta x, j \Delta y)\), for \(1 \leq i, j \leq N\) and
\[ \Delta x = a/N \text{ and } \Delta y = b/N. \] The work done by the external loading can also be expressed by the scalar product of the loading matrix by the displacement matrix (in physical space): \[ E_{Q} = \frac{1}{2} [Q][W]a.b/N^2. \]

The energy stored in the plate is given by:

\[ E_{\text{Elastic}} = \frac{1}{2} \int_{0}^{a} \int_{0}^{b} \left( Mx \frac{\partial^2 w}{\partial x^2} + My \frac{\partial^2 w}{\partial y^2} + 2Mxy \frac{\partial^2 w}{\partial x \partial y} \right) dx \, dy \]  (19)

The scalar \( E_{\text{Elastic}} \) is computed by a similar way as above.

3.5 Examples and order of magnitude of the accuracy of the resolution

Two simple examples are considered: a square plate loaded by a uniform pressure and a rectangular plate (a x 2a) loaded by a punctual force centred on the middle of the plate. The results, in terms of displacement, bending moment and shearing forces, are compared to those proposed by Bares. The relative differences are found around or less than 1%.

3.6 Conclusion for static loading results

The method can be easily encoded on current spreadsheet program, with data easily introduced and controlled. The method promptly gives reliable results with good accuracy and easy to be presented.

4 BEHAVIOR OF A PLATE UNDER DYNAMIC LOADING

4.1 Fundamental equation

The fundamental equation for the dynamic motion of a plate is similar to that obtained for static motion, by considering the local equilibrium of the loading force by the sum of the inertial force, the resisting force of viscous type (proportional to the velocity) and the elastic structural force. The membrane forces are neglected, due to the possible slippage of the plate along its supports. The directions of all these forces being normal to the plate, it is therefore convenient to simply consider their projection on the normal to the plate and to consider their algebraic value.

The inertial force \( F_i \) is given by:

\[ F_i = \bar{n} \cdot h \cdot \frac{\partial^2 w}{\partial t^2} \]

The inertial rotation effect is neglected, due to the fact that the plate is supposed relatively thin (\( h < 5.\text{Min}(a,b) \)), the displacement being essentially due to the bending moment.

The resisting force of viscous type \( F_v \) is given by:

\[ F_v = \lambda \cdot \frac{\partial w}{\partial t} \]

It is convenient to decompose the proportionality parameter \( \lambda \) into the following product of scalars:

\[ \lambda = 2.\rho.h.\xi.\omega, \] with the classical notation:

\( \xi \): damping ratio \( \omega \): spatial pulsation of the 1st modal deformation

The structural response force \( F_s \) is given by:

\[ F_s = \frac{\partial^2 Mx}{\partial x^2} + 2\frac{\partial^2 Mxy}{\partial x \partial y} \frac{\partial^2 My}{\partial y^2} \]

The structural response force only depends on the bending moments.

The dynamic equilibrium of all these forces leads to:

\[ \bar{F}_i + \bar{F}_v + \bar{F}_s - \bar{Q} = 0 \]
\( \frac{\partial^2 w}{\partial t^2} + 2. \ddot{u} \cdot \frac{\partial w}{\partial t} + \frac{\partial^2 M_x}{\partial x^2} + 2 \cdot \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - Q = 0 \) \hspace{1cm} (20)

This equation may be written as follows:

\( \frac{\partial V(x,y,t)}{\partial t} + 2. \ddot{u} \cdot \dot{V}(x,y,t) + F_{s}(x,y,t) - Q(x,y,t) = 0 \) \hspace{1cm} (21)

with: \( V = \frac{\partial w}{\partial t} \) \hspace{1cm} (22)

The relation (20) suggests that the pertinent variables for the resolution of the dynamic equilibrium are the velocity \( V \) and the structural response force \( F_s \).

4.2 Eigenmodes basis for displacement and loading

The displacement \( W \) depends on the location \( (x,y) \) on the plate and is varying with the time \( t \). It may be decomposed, like for the static loading, on the eigenmode basis according to:

\[ w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(t) \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b}) \] \hspace{1cm} (23)

The velocity \( V \) is given by:

\[ V(x,y,t) = \frac{\partial w(x,y,t)}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{dW_{mn}(t)}{dt} \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b}) \] \hspace{1cm} (24)

\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn}(t) \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b}) \]

The acceleration is given by:

\[ \frac{\partial V(x,y,t)}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{d^2W_{mn}(t)}{dt^2} \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b}) \] \hspace{1cm} (25)

Equations (8) show that the structural response force \( F_s \) is also decomposable into the eigenmode basis according to:

\[ F_{s}(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{s_{mn}}(t) \cdot \sin(m \cdot \frac{x}{a}) \cdot \sin(n \cdot \frac{y}{b}) \] \hspace{1cm} (26)

Relations (24), (25) and (26) imply that the relation (21) is also valid in the modal space and can be written, at each time \( t \), in the following form:

\[ \frac{\partial V_{mn}'}{\partial t} + 2. \ddot{u} \cdot \dot{V}_{mn}' + \frac{F_{s_{mn}'}'}{n.h} - \frac{Q_{mn}'}{n.h} = 0 \] \hspace{1cm} (27)

In the above equation, in the modal space, the acceleration (or velocity variation) \( \frac{\partial V_{mn}'}{\partial t} \) at the time \( t \) is given by the velocity \( V \) at the time \( t \), the structural response \( F_s \) at the time \( t \) and the loading \( Q \) at the time \( t \).

4.3 Resolution method

The acceleration/velocity variation \( \frac{\partial V_{mn}'}{\partial t} \) of equation (27) is approximated by the finite difference method by introducing the following semi-implicit hypothesis: the velocity variation depends on the mean value, on the interval \([t, t+\Delta t]\), of the other variable terms.
\[
\frac{V_{mn}^{t+\Delta t} - V_{mn}^t}{\Delta t} + 2i.\dot{u} \left( \frac{V_{mn}^{t+\Delta t} + V_{mn}^t}{2} \right) + \frac{1}{\hat{n}.h} \left( \frac{F_{s_{mn}}^{t+\Delta t} + F_{s_{mn}}^t}{2} \right) - \frac{1}{\hat{n}.h} \left( \frac{Q_{mn}^{t+\Delta t} + Q_{mn}^t}{2} \right) = 0 \tag{28}
\]

For similar reason, the relation between velocity \(V\) and displacement \(W\), in the modal space, is given by:

\[
W_{mn}^{t+\Delta t} = W_{mn}^t + \Delta t \left( \frac{V_{mn}^{t+\Delta t} + V_{mn}^t}{2} \right) \tag{29a}
\]

It can be noted that this relation is still valid in the physical space:

\[
W(x,y,t+\Delta t) = W(x,y,t) + \Delta t \left( \frac{V(x,y,t+\Delta t) + V(x,y,t)}{2} \right) \tag{29b}
\]

A second relation, which is necessary for the determination of the 2 unknowns \(V_{mn}\) and \(F_{s_{mn}}\), is given by the bending-curvature law.

In the physical space the structural response \(F_s\) is given by:

\[
F_s = \frac{\partial^2 M_x}{\partial x^2} + 2 \cdot \frac{\partial^2 M_y}{\partial x \partial y} + \frac{\partial^2 M_{xy}}{\partial y^2} = D \cdot \left( \frac{\partial^4 W}{\partial x^4} + 2 \cdot \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) \tag{30a}
\]

In the modal space, the above relation becomes:

\[
F_{s_{mn}} = D \cdot \left( \left( \frac{m\dot{\theta}}{a} \right)^2 + \left( \frac{n\dot{\theta}}{b} \right)^2 \right) \cdot W_{mn} = D_{mn} \cdot W_{mn} \tag{30b}
\]

When introducing the relation (30b) in the relation (29a), we obtain:

\[
F_{s_{mn}}^{t+\Delta t} = F_{s_{mn}}^t + D_{mn} \cdot \Delta t \cdot \left( \frac{V_{mn}^{t+\Delta t} + V_{mn}^t}{2} \right) \tag{31}
\]

After rearranging equations (28) and (31), the relation between \(F_s\) and \(V\) in the modal space is given by:

\[
\begin{align*}
V_{mn}^{t+\Delta t} - V_{mn}^t - 2i.\omega.\Delta t \left( \frac{V_{mn}^{t+\Delta t} + V_{mn}^t}{2} \right) + \frac{\Delta t}{\hat{n}.h} \cdot \left( \frac{F_{s_{mn}}^{t+\Delta t} + F_{s_{mn}}^t}{2} \right) &= \Delta t \cdot \left( \frac{Q_{mn}^{t+\Delta t} + Q_{mn}^t}{2} \right) \\
F_{s_{mn}}^{t+\Delta t} &= F_{s_{mn}}^t + D_{mn} \cdot \Delta t \cdot \left( \frac{V_{mn}^{t+\Delta t} + V_{mn}^t}{2} \right)
\end{align*} \tag{32}
\]

This system of the two above equations with the two unknowns \(V^{t+\Delta t}_{mn}\) and \(F_{s_{mn}}^{t+\Delta t}\) is resolved in:

\[
\begin{align*}
\left\{ \begin{array}{l}
(1+i.\dot{u} \cdot \Delta t) \cdot D_{mn} \cdot V_{mn}^{t+\Delta t} = (1-i.\dot{u} \cdot \Delta t) \cdot D_{mn} \cdot V_{mn}^t - \frac{\Delta t}{\hat{n}.h} \cdot \left( F_{s_{mn}}^t - \frac{Q_{mn}^t}{2} \right) \\
(1+i.\dot{u} \cdot \Delta t) \cdot D_{mn} \cdot F_{s_{mn}}^{t+\Delta t} = (1+i.\dot{u} \cdot \Delta t) \cdot D_{mn} \cdot F_{s_{mn}}^t + \Delta t \cdot D_{mn} \cdot V_{mn}^t - \frac{\Delta t}{\hat{n}.h} \cdot \left( F_{s_{mn}}^{t+\Delta t} - \frac{Q_{mn}^{t+\Delta t}}{2} \right)
\end{array} \right. \tag{33}
\]

Equations (33) allow the computing, by recurrence, of \(V\) and \(F_s\), in the modal space and at the time \(t+\Delta t\) when knowing \(V\), \(F_s\) and \(Q\), in the modal space at the time \(t\).

The values of the velocity \(V\) and of the structural response \(F_s\) in the physical space are obtained by modal recombination allowed by the formulae (24) and (26).

The displacement \(W\) in the physical space is obtained from the velocity \(V\) by the relation (29a). The components of the bending moment tensor (\(M_x, M_y\) and \(M_{xy}\)) can be directly obtained, as in the static case, by the bending moment-curvature law that needs to compute the second order differentiations of the displacement \(W\): this can be given by the centred second order finite difference method.
4.4 Practical algorithms for resolution

Discretization
The discretization of the plate is the same as that proposed for the static loading, in order to handle 30x30 matrix products.

Transfer matrix
The transfer matrix is the same as that for static loading, \([H]\) defined by (11).

Resolution
The resolution method is based on the computing of the coefficients \(Q_{tn}^l\) at each step by using the procedure described for the static loading through equation (13) and on the computing of \(V_{tn}\) and \(F_{tn}\) by the recurrence formulae (33). The initial conditions, at \(t=0\), are supposed known and generally \(V_{tn}=0\) and \(F_{tn}=0\), \(\forall \ m, n \in \mathbb{N}^*\). It can be noted that this hypothesis is not necessary: an initially deformed shape of the plate may be at the origin of the motion and can then be taken into account, with \(F_{tn}^0 \neq 0\).

The value of \(W_{tn}\) at each time step is deduced from the value of \(V_{tn}\), the coming back from the modal space into the physical space is also obtained by the same procedure developed for the static loading, by using the relation (14), at each step time.

It is then necessary to proceed with an iterative computation, which is generally allowed by the spreadsheet code (at least for EXCEL\textsuperscript{®}) and to store some results in terms of displacement, velocity, bending moment, in the physical space, in order to issue their variation with the time.

Bending moment tensor
The components of the bending moment tensor are issued from the spatial second order derivatives of the displacement field \(W\), at each time step, in a similar way proposed for the static loading computation.

Checking
A feedback control may be performed with the energy theorem: when neglecting the loss of energy due to the viscous dissipation, the total mechanical energy, equal to the sum of the kinetic energy \(E_{\text{Kinetic}}\) and the stored elastic energy \(E_{\text{Elastic}}\), is equal to the work of the external loading \(Q\).

The expressions of these different kinds of energy are given below:

\[
\begin{align*}
E_{\text{Elastic}}(t) &= \frac{1}{2} \int_{a}^{b} \int_{0}^{b} \left( M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) dx \cdot dy \\
E_{\text{Kinetic}}(t) &= \frac{1}{2} \cdot \mathbf{n} \cdot \mathbf{\hat{n}} \cdot \int_{a}^{b} \int_{0}^{b} V^2(x, y, t) \cdot dx \cdot dy \\
E_{\hat{Q}}(t) &= \int_{a}^{b} \int_{0}^{b} Q(x, y, t) \cdot W(x, y, t) \cdot dx \cdot dy
\end{align*}
\]

With, at any time,

\[
E_{\text{Elastic}}(t) + E_{\text{Kinetic}}(t) = E_{\hat{Q}}(t)
\]

5 CONCLUSION
The proposed method allows computing - on a spreadsheet - the dynamic motion of a simply resting elastic rectangular plate submitted to any set of transverse forces varying with the time and located on any part of the plate. The accuracy of the results is around the percent and the computation time is a few seconds. Such performances are sufficient for engineering use and a program based on this method results in a useful tool for the verification of the results from more sophisticated codes. In particular, it is convenient for testing the sensitivity of any mechanical parameters.

REFERENCES
Rambach, J.-M. Dynamic model of a simple supported RC rectangular plate for spreadsheet application – Part II: Motion in elasto-plastic domain, in SMiRT 20 Proceedings, Espoo, Finland, 9-14 August 2009.