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DETERMINATION OF NATURAL FREQUENCIES AND MODE SHAPES BY THE RAYLEIGH QUOTIENT

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ABSTRACT

The determination of natural frequencies and mode shapes is of general interest in linear and nonlinear dynamic analysis, particularly in the case of cable structures where geometric nonlinearity is predominant. An attempt is made in this paper to solve the problem of steady state free vibration by direct minimization of the Rayleigh quotient, which is carried out by the scaled conjugate gradients algorithm. Predictions by the proposed method compare well with digital simulation results.

1. INTRODUCTION

The work presented here deals with the problem of steady state vibrations. In the proposed method the inertia effect in the form of kinetic energy of the vibrating nodes is included in the energy function which, for static analysis, contains terms relating to the strain energy stored in the elements and the work done by the external loads in displacing the nodes. It is shown how the consideration of these energy terms leads to the derivation of the Rayleigh's quotient (RQ), which can be minimized to yield the natural frequencies and mode shapes of the structure. The method of conjugate gradients is then applied to the problem of direct minimization of the RQ. The suitability of the method with respect to cable structures together with details may be found in Ref. (1) and also in Refs. (2,3,4). Finally, where only a few of the lowest frequencies and mode shapes are required, and where the number of degrees of freedom is very large, then this approach is attractive, provided that reasonable approximations to the required mode shapes are avaible.

2. THEORETICAL ASPECTS OF THE METHOD

The total energy W of a freely vibrating structure, can be represented by the strain, the Kinetic and the potential energies, E_s, E_k, E_p.

$$w = E_s + E_k + E_p = \text{const} \quad \text{(ignoring damping)} \qquad \textbf{(1)}$$

For linear behaviour $w = \frac{1}{2}(q+\delta)^T K (q+\delta) + \frac{1}{2}\dot{q}^T M \dot{q} - (q+\delta)^T F$ **(2)**

M and K are the mass and stiffness matrices and q and δ are the dynamic and static displacement vectors.

For $F = K.\delta$, $w = \frac{1}{2} q^T K \dot{q} + \frac{1}{2}\dot{q}^T \dot{M} q - \frac{1}{2}\delta^T K \delta$ **(3)**

For steady state of vibration $\frac{\partial w}{\partial t} = \dot{q}^T K q + \ddot{q}^T \dot{M} q = 0$ **(4)**

and for sinusoidal with common frequency, the instantaneous displacement at a node j in the ith direction is:

$$q_{j,i} = x_{j,i} \sin\omega t \qquad (5)$$

$$\dot{q} = \omega x \cos\omega t, \; \ddot{q} = \omega^2 x \sin\omega t, \; \ddot{q} = \omega^2 q \qquad (6)$$

and from Eq. 4: $\omega^2 = f(x) = \frac{x^T K x}{x^T M x}$ (7)

Eq. 7 is the RQ whose stationary values correspond to the eigenvalues and vectors of matrix $M^{-1}K$. Differentiating Eq. 7 and premultiplyng by M^{-1} yields: $g = \frac{d\omega^2}{dx} = \frac{2}{x^T M x}(Kx - \omega^2 Mx) = 0$ (8)

or $Kx - \omega^2 Mx = 0$, and $M^{-1}Kx = \omega^2 x$ (9)

It can be shown (5) that the ith stationary point is a minimum in the subspace spanned by the eigen vector v_j, ($j = i, i+1, \dots n$). This shows that any eigen solution can be located by minimization of the RQ, in the appropriate subspace. It also indicates that any minimization algorithm may converge to an intermediate stationary point. Further more due to the homogeneous form of the RQ no minimum point exists and that the contours are straight lines through the origin. The value of the function on a line through the origin at any point remains constant.

at point A : $x_A^T = (x_1, x_2, \dots, x_n)$ and

at point B : $x_B^T = (\beta x_1, \beta x_2, \dots, \beta x_n) = \beta x_A^T$

Then $f(x_A) = \frac{x_A^T K x_A}{x_A^T M x_A}$ and $f(x_B) = \frac{x_B^T K x_B}{x_B^T M X_B} = \frac{\beta^2 x_A^T K x_A}{\beta^2 x_A^T M x_A} = f(x_A)$

The order of the problem may be reduced (6) by normalizing the trial vector. The normalization frequently used, restricts x such that:

$$x^T M x = I \qquad (10)$$

The orthogonalization procedure outlined here i due to Daniel (7),

(8) and has been applied to minimize the RQ.

3. EVALUATION OF STEP LENGTH S

The relationship between the nodal displacements and member strains can be approximated to a quadratic. This leads to cubic expression for Φ' which will have a real root, the solution of which can be obtained by Cardan's formula (2). Newtons method (9) has been used with low rate of convergence. Modern methods, based on minimization techniques proposed by Grant and Hitchins (10) can also be used. Laquerre's method (11) is also proposed because of its simplicity. From the condition : $\frac{d}{dS_k}(f(x_{k+1})) = 0$ $f(x_{k+1})$ is found by substituting into the RQ from $x_{k+1} = x_k + S_k V_k$ to yield:

$$f(x_{k+1} = \left(\frac{x^T K_x + 2Sx^T Kv + S^2 v^T Kv}{x^T Mx + 2Sx^T Mv + S^2 v^T Mv}\right) \text{ or } f(x_{k+1} = \frac{a+2Sb+c}{b+2Se+f} \tag{11}$$

where $a = x^T Kx$, $b = x^T Kv$, $c = v^T Kv$, $d = x^T Mx$, $e = x^T Mv$, $f = v^T Mv$. Differentiating Eq. 11 yields:

$$(ce-bf)S_k^2 + (cd+af)S_k + (db-ae) = 0 \tag{12}$$

and $S_k = \dfrac{-(cd-af) \pm \sqrt{(cd-af)^2 - 4(ce-bf)(bd-ae)}}{2(ce-bf)}$

4. MINIMIZATION ALGORITHM

The conjugate gradients method is applied to minize the RQ in terms of general symmetric matrices K and M. The problem reduces to supplying suitable routines for the function and its gradient from the expressions:

$$f(x) = \frac{x^T Kx}{x^T Mx}, \quad g(x) = \frac{2}{x^T Mx}(Kx - f(x)Mx) \tag{13}$$

The eigen value and eigen vector can then be tested from:

$$|(\omega_{k+1}^2 - \omega_k^2)/\omega_{k+1}^2| < \alpha|, (r_{k+1}^2 - r_k)/r_{k+1}^2| < \beta, r_k = x_k^T x_k \tag{14}$$

where α and β are preassigned small values and $\beta \approx \sqrt{x}$. Also:

$$a = \sum_m^{MC} (y^{C^T} K^C y^C)_m + \sum^{MB} (y^{B^T} K^B y^B)_m \tag{15}$$

$$d = \sum_m^{MC} (y^{C^T} M^C y^C)_m + \sum_m^{MB} (y^{B^T} M^B y^B)_m \tag{16}$$

where K_m^C and M_m^B = Stiffness and mass matrices of cable element m, K_N^F and M_n^B = matrices of element n, y^C and y^B are vectors of same order as K^C and K^B containing the appropriate elements of x.

4.1. Numerical Behaviour of the Algorithm.

Due to the homogeneous nature of the problem, the contours of the RQ. are straight lines passing through the origin. Fig. 1 shows the contours of the RQ. $\lambda = x^T K x / x^T x$. Where

$$K = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 6 & 2 \\ 0 & 2 & 4 \end{bmatrix} \text{ and } x^T = (1, x_1, x_2)$$

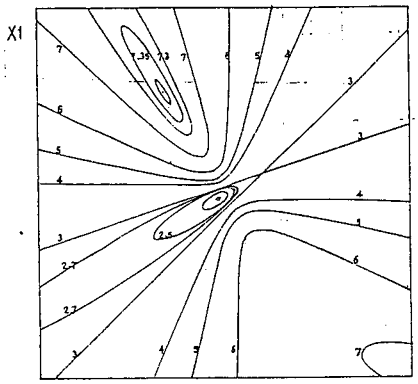

Fig. 1: Contours of Raleigh Quotient

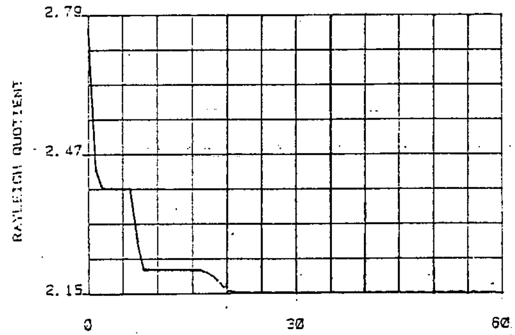

Fig.2: Progress towards the absolute minimum for the corresponding iteration

The highest eigen solution appears as a maximum while the lowest is a minimum. All intermediate solutions are saddle points of the RQ. Figure 2 illustrates the progress towards the absolute minimum for a circular two-way cable beam grid shown in Fig. 3, While Fig. 4 shows the norm of the out of balance forces for the corresponding iterations. An examination of the norm will show that the intermediate solutions are found to some degree of accuracy before any progress towards a lower solution can be made. The efficiency of the method will suffer if a large number of iterations are required to recover from the intermediate solutions.

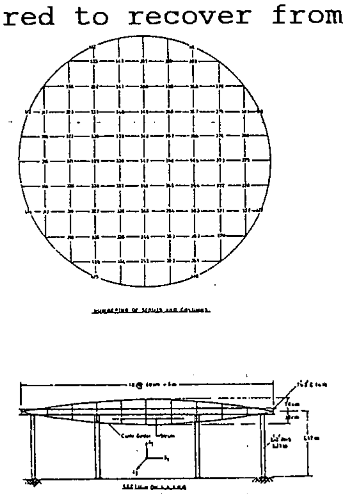

Fig. 3: Two way cable beam grid.

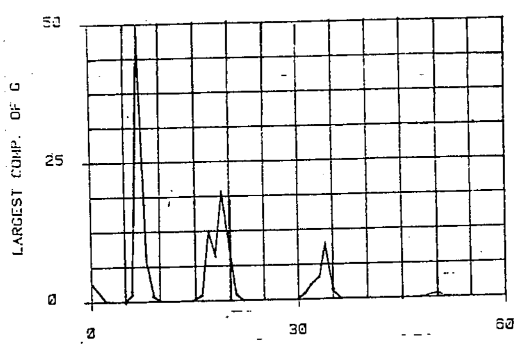

Fig. 4: The norm of the out of balance forces for corresponding iteration

From the above considerations, it can be seen that the initial estimate of the mode shape should be as accurate as possible. In the case of structures, the general form of the mode shapes can be found by inspection or by solution of a simplified model and if necessary can be improved upon by statically solving the structure for a loading regime corresponding to the mode shape. For any practical size cable structure, if a reasonable estimate of the nth mode is supplied, the algorithm will yield an accurate value for the nth frequency and mode shape and will not, in general, progress towards any other mode.

5. DAMPING

Damping values are very low for cable structures and damping in general does not affect the natural frequencies and mode shapes, except by variations in the stiffness due to clading (10). Damping terms, however, can be included in the algorithm by considering the RQ. : $\lambda = \frac{z^T S z}{z^T T z}$, where $\lambda = 1/\beta$ and

$$S = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \quad T = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}, \quad z = \begin{Bmatrix} \beta x \\ x \end{Bmatrix} \tag{17}$$

Stationary values of the RQ correspond to the solutions of the generalized eigen problem, (6) : $Sz = \lambda\, T\, z$ (18)
And from Eqs. 17 & 18 we obtain:

$$\begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \begin{Bmatrix} \beta x \\ x \end{Bmatrix} = \beta \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} \begin{Bmatrix} \beta x \\ x \end{Bmatrix} \quad \text{or} \quad \begin{bmatrix} Mx \\ \beta Mx + cx \end{bmatrix} = \frac{1}{\beta} \begin{Bmatrix} \beta Mx \\ -Kx \end{Bmatrix} \tag{19}$$

$$\text{and } \beta Mx + C\,x = -\frac{1}{8}\,K\,x, \quad \beta^2 Mx + \beta Cx + Kx = 0 \tag{20}$$

The RQ can be expanded to give:

$$f(x) = \lambda = \frac{\begin{Bmatrix} \beta x \\ x \end{Bmatrix}^T \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \begin{Bmatrix} \beta x \\ x \end{Bmatrix}}{\begin{Bmatrix} \beta x \\ x \end{Bmatrix}^T \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} \begin{Bmatrix} \beta x \\ x \end{Bmatrix}} \tag{21}$$

which reduces to

$$f(x) = \lambda = \frac{2\lambda x^T Mx + \lambda^2 x}{x^T Mx - \lambda^2 x^T Kx} \tag{22}$$

In typical cables structures, it is more effient to initially ignore the damping terms and use the previous algorithm to find an accurate solution for the undamped structure which can then be used as an initial approximation for an algorithm based on expression (11).

6. DISCUSSION AND CONCLUSIONS

Direct minimization of the RQ is an economic and useful tool for the determination of natural frequencies and mode shapes of structures with many degrees of freedom provided that reasonable approximations to the desired modes are known. Where the starting vector is chosen arbitrarily, there is in general no quarantee that the absolute minimum will be reached.

The algorithm presented here, should not be taken as the sole tool for calculation of frequencies and mode shapes. In practical design situations, the initial analysis is frequently performed on a reduced mathematical model having a fraction of the degrees of freedom of the detailed model.

Also, in practice, the mathematical model is amended frequently as the design progresses. It will be very costly to resolve the eigen problem using convensional methods every time a small change is made. In such circumstances the RQ algorithm makes an excellent tool.

Where the lumped mass approach is used for flexural elements, the rotational degrees of freedom of the stiffness matrix would need to be condenced out before convensional methods can be used. This is not necessary in the RQ algorithm and it is often an economic approach to solve the eigen value problem by convensional methods assuming the flexural elements to be infinitely rigid and then use the RQ to cater for the effects of the flexural elements with their true stiffness values.

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