SMiRT-12 / K. Kussmaul (Editor)

BG11/2

AN ARBITRARY LAGRANGIAN DESCRIPTION OF THE MOVING CRACK PROBLEM

G. Debruyne

Electricité de France, Département Mécanique et Modèles Numériques, 1 av. du Général de Gaulle, F-92141 Clamart, France

ABSTRACT.

An arbitrary Lagrangian description is proposed for the moving crack analysis. The method is based on the use of a mapping of the moving domain to the initial stationary one (boundaries shape and conditions may be very general). In the framework of elasticity, the energy release rate and the tearing modulus are expressed as surface integrals on the fixed domain, explicitely related to the current crack growth parameter. Numerical results are presented for a large range of crack extension. The advantages of this method are firstly to avoid remeshing or introducing special numerical parameters such as relaxation forces to simulate the crack growth, secondly to obtain exact derivatives of the potential energy.

1 INTRODUCTION.

The moving crack problem is generally treated with specific numerical techniques, such as remeshing or nodal forces relaxation, directly in the discretized model. Nevertheless, in some particular cases, the moving crack problem has already been treated in the framework of a continuum model. For instance, in the case of the steady-state crack growth in an infinite tensile strip, the complete analysis of the continuum problem has been achieved for an elastoplastic material [Nguyen, Rahimian 1981]. For the elastodynamic case, a moving finite element method based on a mixed Eulerian-Lagrangian description has been performed [Koh et al. 1988]. To avoid the difficulties of the numerical techniques used to describe the crack kinematics (in particular remeshing), we propose an arbitrary Lagrangian description of the continuum problem, for any kind of two dimension cracked domain and for a large class of boundary conditions, which leads to a variational problem on the initial cracked domain. For the quasi-static elastic case, we can express in the arbitrary Lagrangian description, an exact form of the first and second derivative of the energy as a function of the current crack length, represented by integrals defined on a subdomain containing the initial crack tip.The value of these integrals is independant of the choice of the subdomain.

2 STATEMENT OF THE MOVING CRACK PROBLEM.

2.1 Definition of the problem on the current moving cracked domain.

We consider an homogeneous 2D continuum medium, with a straight moving crack , represented by a family of domains Ω_α of $\mathbb{R}^2$, with a moving line $\overline{AP}$ of length L+α, as presented in figure 1. The propagation parameter of the medium is the crack extension $\alpha(t)$, and we note $\lambda(t)$ the loading parameter. The stationary part of the boundary of Ω_α is subjected to prescribed forces $T^d(\lambda)$ and displacements $U^d(\lambda)$, and the moving crack is free of stress. For each couple (α,λ), the variational form of the problem leads to the search of the displacement field **u**, among the admissible fields **v**, which minimizes the potential energy :

$$W(\mathbf{v},\alpha,\lambda) = \int_{\Omega_\alpha} \Phi\,(\nabla \mathbf{v})_S\, dV - \int_{S_T} T^d(\lambda).\mathbf{v}\; dS\,. \qquad (1)$$

where Φ is the elastic potential. The key problem of this formulation is the dependance of the integral domain with respect to the crack extension α. We assume that $\alpha(t)$ is known all along the evolution process, the determination of $\alpha(t)$ is a problem we will not discuss in this paper. The following method stays available when only the evolution law of $\alpha(t)$ is given.

2.2 Mapping of the moving domain Ω_α to the initial domain

We introduce a one-to-one regular geometrical mapping f_α, which connects any point M(X,Y) of Ω_0 to a current material point m(x,y) of Ω_α, such that f_α^{-1} maps the family of cracks $\overline{AP}$ (length L+α) into the initial crack $\overline{AO}$ (length L), for any current domain Ω_α , as presented in the figure 1.

Formally, this mapping , which becomes identity function out of a sub-domain $\mathcal{D}$ of Ω_0 surrounding the initial crack tip, has the following properties:

$$\begin{cases} f_\alpha(\mathcal{D}) = \mathcal{D},\ f_\alpha(M) = M \ \text{ in } \ \Omega_0\text{-}\mathcal{D}, \\ f_\alpha(\overline{AO}) = \overline{AP},\ f_\alpha(O) = P, \\ f_0(M) = M, \text{in} \ \ \Omega_0. \end{cases} \qquad (2)$$

2.3 Definition of the problem on the initial cracked domain.

With the help of the previous mapping, we can change the spacial variable in the displacement field:

$$\forall M,\ M \in \Omega_0,\ \mathbf{v}_\alpha(M) \equiv \mathbf{v}(f_\alpha(M)). \quad (3)$$

The new displacement field $\mathbf{v}_\alpha$ associated to the point M does not describe a specific material particle as in classical Lagrangian representation but a family of material particles $f_\alpha(M)$, determined by the choice of the mapping. At each step of the evolution, the set of points M of Ω_0 represents an arbitrary Lagrangian configuration. If we note $\mathbf{F}_\alpha$ the gradient of f_α, and with the variable switching (3), the new problem, defined on the fixed domain Ω_0, lies in searching, among the admissible fields **v**, the solution $\mathbf{u}_\alpha$ which minimizes the new expression of the potential energy:

$$\widehat{W}(\mathbf{v},\alpha,\lambda) = \int_{\Omega_0} \Phi(\nabla \mathbf{v}.\mathbf{F}_\alpha^{-1})_S \det\mathbf{F}_\alpha\, dV - \int_{S_T} T^d.\mathbf{v}\; dS. \quad (4)$$

The variable parameter of the problem is now switching from the domain to the integral operator.

3 CASE OF LINEAR ELASTICITY.

3.1 Weak form of the elastic problem in Arbitrary Lagrangian Description.

In the case of linear elasticity, we can develop the elastic potential as following: $\Phi(\varepsilon) = \frac{1}{2}\mathbf{A}:\varepsilon:\varepsilon$, where $\mathbf{A}$ is a fourth-order positive symetric tensor. To pull off, in the mapping f_α, the crack growth parameter α from the specific geometrical function, we choose the mapping as following:

$$\forall M, M\in\Omega_0,\ f_\alpha(M) = M + \alpha\overrightarrow{\xi(M)}\ ,$$

where ξ is a vector field, which becomes zero outside of the domain $\mathbf{D}$. Then, we can develop the inverse of the transformation gradient $\mathbf{F}_\alpha^{-1}$:

$$\mathbf{F}_\alpha^{-1} = \frac{1}{\det\mathbf{F}_\alpha}(\mathbf{I}^d + \alpha\mathbf{C}),\ \text{with } \mathbf{C} = -\nabla\vec{\xi} + \mathrm{div}\vec{\xi}\ \mathbf{I}^d,\ \det\mathbf{F}_\alpha = 1 + \alpha\,\mathrm{div}\vec{\xi} + \alpha^2\det\nabla\vec{\xi}.$$

The minimization problem of the functional $\widehat{W}$ leads to the following variational expression :

$$\left\{\begin{aligned} &\int_{\Omega_0}\frac{1}{\det\mathbf{F}_\alpha}\mathbf{A}:(\nabla\mathbf{u}_\alpha):\nabla\mathbf{v}\,dV + \alpha\int_{\mathbf{D}}\frac{1}{\det\mathbf{F}_\alpha}\mathbf{A}:(\nabla\mathbf{u}_\alpha):(\nabla\mathbf{v}.\mathbf{C})\,dV \\ &+\alpha\int_{\mathbf{D}}\frac{1}{\det\mathbf{F}_\alpha}\mathbf{A}:(\nabla\mathbf{v}):(\nabla\mathbf{u}_\alpha.\mathbf{C})dV + \alpha^2\int_{\mathbf{D}}\frac{1}{\det\mathbf{F}_\alpha}\mathbf{A}:(\nabla\mathbf{u}_\alpha.\mathbf{C}):(\nabla\mathbf{v}.\mathbf{C})\,dV = \int_{S_T}\mathbf{T}^d.\mathbf{v}\,dV, \end{aligned}\right. \tag{5}$$

which means that the finite element formulation, with the constant mesh of Ω_0 ,is for each α:

$$\left[(K_0 + \alpha\ K_1(\alpha,\xi) + \alpha^2\ K_2(\alpha,\xi)\right]\mathbf{u}_\alpha = \mathbf{F},$$

where K_0 is the classical stiffness matrix of the initial problem (α=0), K_1 and K_2 are pseudo stiffness matrices depending on the mapping.

3.2 Expression of the energy release rate in arbitrary Lagrangian description.

Remarking that the solution of (5) $\mathbf{u}_\alpha$ is optimal for the functional $\widehat{W}$, and developing in the expression (4) the quantities $\mathbf{F}_\alpha^{-1}$ and $\det\mathbf{F}_\alpha$, the expression of $G(\alpha,\lambda) = -\dfrac{\partial\widehat{W}(\mathbf{u}_\alpha,\alpha,\lambda)}{\partial\alpha}$ holds:

$$\left\{\begin{aligned} G(\alpha,\lambda) = \frac{1}{2}&\int_{\mathbf{D}}\frac{\alpha^2(-\mathrm{div}\vec{\xi}^{\,3} + 2\mathrm{div}\vec{\xi}\det\nabla\vec{\xi}) + 2\alpha(-\mathrm{div}\vec{\xi}^{\,2} + \det\nabla\vec{\xi}) - \mathrm{div}\vec{\xi}}{(1+\alpha\,\mathrm{div}\vec{\xi} + \alpha^2\det\nabla\vec{\xi})^2}\,A:\nabla u_\alpha:\nabla u_\alpha\,dV \\ &+\int_{\mathbf{D}}\frac{(1 + \alpha^2(-\det\nabla\vec{\xi} + \mathrm{div}\vec{\xi}^{\,2}) + 2\alpha\,\mathrm{div}\vec{\xi}\,)}{(1+\alpha\,\mathrm{div}\vec{\xi} + \alpha^2\det\nabla\vec{\xi})^2}\,A:\nabla u_\alpha:(\nabla u_\alpha.\nabla\vec{\xi})\,dV \\ &-\frac{1}{2}\int_{\mathbf{D}}\frac{\alpha^2\mathrm{div}\vec{\xi} + 2\alpha}{(1+\alpha\,\mathrm{div}\vec{\xi} + \alpha^2\det\nabla\vec{\xi})^2}\,A:(\nabla u_\alpha.\nabla\vec{\xi}):(\nabla u_\alpha.\nabla\vec{\xi})\,dV\ . \end{aligned}\right. \tag{6}$$

For $\alpha=0$, the expression (6) leads to a result obtained by a domain perturbation method called theta method for a stationary crack [Destuynder-Djaoua 1981]. Our result (6) is an extension of the theta method for finite crack growth.

3.3 Expression of the tearing modulus in arbitrary Lagrangian description.

The tearing modulus, defined as the derivative of G with respect of α, is very classical in stability analysis of one crack or a system of cracks [Nemat-Nassers et al. 1980]. This quantity depends on the solution $\frac{\partial \mathbf{u}_\alpha}{\partial \alpha}$ of the rate displacement problem.This problem leads to a rather complicate formal expression for a general mapping form, though it is easy to introduce it into a finite element code. Nevertheless, a special selection of the mapping can simplify the resolution of the displacement and the rate displacement problems and the formula of the energy derivatives. For instance, for an incompressible mapping, the geometrical parameters are very simple ($\det\mathbf{F}_\alpha = 1$, $\mathbf{F}_\alpha^{-1} = \mathbf{I}^d - \alpha\nabla\vec{\xi}$), and the elastic problem may be written in the simple following form for each (α,λ):

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega_0} A:\nabla\mathbf{u}_\alpha:\nabla\mathbf{v}\, dV - \alpha\int_D A:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):\nabla\mathbf{v}\, dV - \alpha\int_D A:(\nabla\mathbf{u}_\alpha):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV \\ \displaystyle +\alpha^2\int_D A:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV = \int_{S_T} T^d.\mathbf{v}\, dS, \forall\ \mathbf{v} \text{ admissible displacement field.} \end{array}\right. \quad (7)$$

An incompressible mapping is not easy to determine in an analytical way but may be set up in a numerical way, solving for instance by FEM an auxiliary incompressible elastic problem in finite strain where $\mathbf{u}=\xi$. Anyway, an incompressible mapping is not essential for the method (the further applications do not use it), but presents some advantages. The linear system (7) has constant coefficients (that means that in the discretized problem, if we use the Finite Element Method, the "element stiffness matrix" corresponding to those coefficients are computed once only during the problem evolution). The rate displacement problem has the following form:

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega_0} A:\nabla\frac{\partial\mathbf{u}_\alpha}{\partial\alpha}:\nabla\mathbf{v}\, dV - \alpha\int_D A:(\nabla\frac{\partial\mathbf{u}_\alpha}{\partial\alpha}.\nabla\vec{\xi}):\nabla\mathbf{v}\, dV - \alpha\int_D A:(\nabla\frac{\partial\mathbf{u}_\alpha}{\partial\alpha}):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV \\ \displaystyle +\alpha^2\int_D A:(\nabla\frac{\partial\mathbf{u}_\alpha}{\partial\alpha}.\nabla\vec{\xi}):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV = \int_D A:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):\nabla\mathbf{v}\, dV + \int_D A:(\nabla\mathbf{u}_\alpha):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV \quad (8) \\ \displaystyle - 2\alpha\int_D A:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV, \forall\ \mathbf{v} \text{ admissible displacement field.} \end{array}\right.$$

From the numerical point of view, this formulation is convenient, because the rate displacement problem (8) is the same problem as the displacement problem (7) with another right hand side. Deriving G and using expression (8) with $\mathbf{v} = \frac{\partial\mathbf{u}_\alpha}{\partial\alpha}$, we get the tearing modulus for each (α,λ):

$$\begin{cases} T(\alpha,\lambda) = \dfrac{\partial G(\alpha,\lambda)}{\partial \alpha} = -\displaystyle\int_{D} A:(\nabla u_\alpha.\nabla\vec{\xi}):(\nabla u_\alpha.\nabla\vec{\xi})\, dV + \int_{\Omega_0} A:\nabla\frac{\partial u_\alpha}{\partial\alpha}:\nabla\frac{\partial u_\alpha}{\partial\alpha}\, dV \\ -2\alpha\displaystyle\int_{D} A:(\nabla\frac{\partial u_\alpha}{\partial\alpha}.\nabla\vec{\xi}):\nabla\frac{\partial u_\alpha}{\partial\alpha}\, dV + \alpha^2\int_{D} A:(\nabla\frac{\partial u_\alpha}{\partial\alpha}.\nabla\vec{\xi}):(\frac{\partial u_\alpha}{\partial\alpha}.\nabla\vec{\xi})\, dV \end{cases} \tag{9}$$

4 SOME NUMERICAL RESULTS AND PERSPECTIVES FOR THE METHOD.

Fig. 2 shows a very long strip of finite width W, with a single-edge crack, subjected to a tensile load σ=1MPa. The variable crack length is L+α and the constant mesh models the initial crack L=W/2. The domain D is a ring (cf fig.2), in which $\xi_1=\pm1$ in the vicinity of the crack following the sign of α, and linear from ±1 to 0 in the remaining part of the domain, $\xi_2=0$ everywhere. We compute G(α) for a range of the ratio L+α/W from 0.025 to 0.975, and compare the corresponding KI to analytical results obtained by Tada (cf. fig. 3). For a ratio from 0.025 to 0.85, the difference between the two results is about 1% . For L+α > 0.85W, the divergence increases very quickly but the theoretical values are not valid for such a large ratio. Others numerical examples show a similar concordance with theoretical results or numerical methods with remeshing such as theta method.Others mappings have also been used with success, but the mapping described on fig. 2 is very convenient: in the vicinity of the crack tip where the computed fiels are not very accurate, $\nabla\xi=0$, so that the integral terms of the formulation depending on the mapping vanish in this area.The conclusion is that the present method,using the same mesh all along the evolution of the crack, gives a good accuracy even with a large crack extension occurs . This is very useful to carry out parametric analysis.We only present here a numerical application in the frame of quasi-static elasticity, but this formulation may be extended to non-elastic materials with internal variables "transported" on the fixed domain, where the propagation has a physical meaning. These internal variables and their evolution law are then described on the arbitrary Lagrangian configuration. In this case, the crack extension α may be considered as a supplementary internal variable of the problem.

REFERENCES.

Ph. DESTUYNDER, M. DJAOUA, 1981: Sur une interprétation mathématique de l'intégrale de Rice en théorie de la rupture fragile. Mathematic Methods in the Applied Science, Vol. 3, pp. 70-87.

H.M. KOH, H.-S. LEE, R.B. HABER, 1988: Dynamic crack propagation analysis using Eulerian-Lagrangian kinematic descriptions. Computational Mechanics, 3 ,141-155.

S. NEMAT-NASSERS, Y. SUMI, L.M. KEER, 1980: Unstable Growth of Tensile Crack in Brittle Solids, Int. J. Solids Struct. 16, 1017-1035.

Q.S. NGUYEN , M. RAHIMIAN, 1981: Mouvement permanent d'une fissure en milieu élastoplastique. Journal de Mécanique appliquée, Vol. 5, n°1, pp. 95-120.

H. TADA, P.C. PARIS, G.R. IRWIN: The stress analysis handbook. Del Research Corp., Hellertown, PA, 1973.

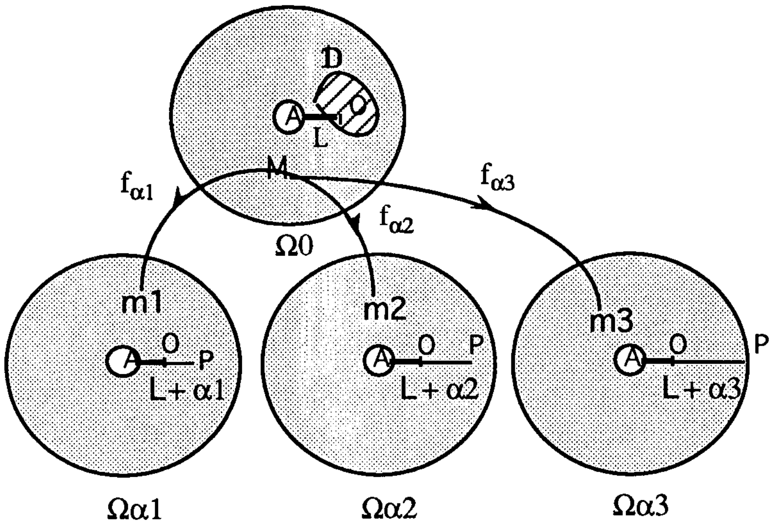

Fig. 1. Family of domains with a moving crack and mapping to the initial domain.

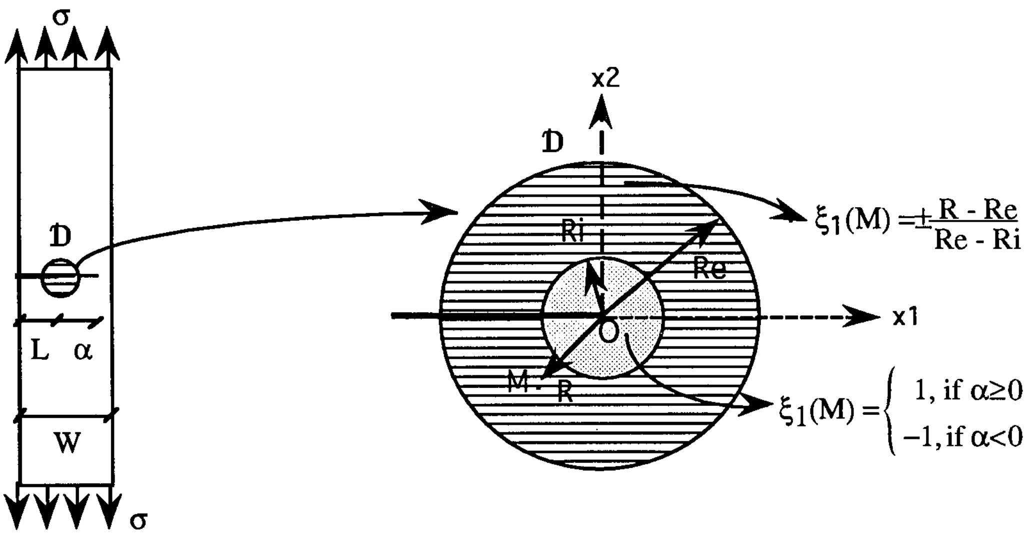

Fig. 2 Cracked strip under tensile load and definition of the mapping.defined on subdomain D

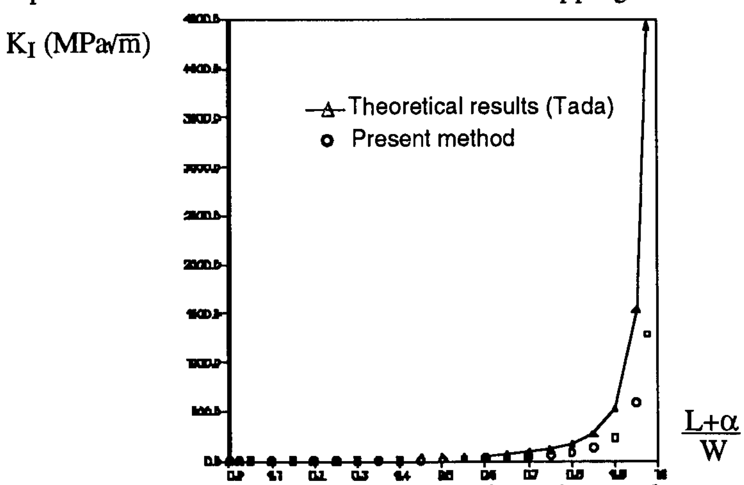

Fig. 3 Theoretical and numerical stress intensity factors against edge-crack extension ratio