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A PARAMETRIC COMPLEMENTARITY PROBLEM FOR THE INITIAL MODE OF DEFORMATION IN RIGID-PLASTIC DYNAMICS

C.L. Sahlit[1] and D.L. Smith[2]

[1]Departamento de Engenharia Civil, Universidade de Brasília, Brasília, DF, CEP 70910, Brazil
[2]Department of Civil Engineering, Imperial College of Sciences, Technology and Medicine, London SW7 2BU, UK

ABSTRACT

The paper introduces a systematic technique to the automatic determination of the initial deformation mechanism of a rigid-plastic framed structure subjected to a pulse loading, for any level of that loading. The frame is envisaged as a structural network, and the fundamental vectorial conditions characterising its behaviour at the initial instant of time are combined in a consistent manner. By considering the initial level of the pressure loading to be a variable parameter, the structural governing system at the initial instant may be regarded as a parametric linear complementarity problem (PLCP).

1. INTRODUCTION

The study of pipe whip resulting from rupture of a pipeline containing high energy fluid is essential to meet design requirements for safety of nuclear power plants. Recently, this subject has interested many researchers and important contributions which clarify the understanding of the dynamics of a whipping pipe have been made. Since in this phenomenon the imparted dynamic energy is significantly larger than the maximum amount of strain energy which can be absorbed elastically, the studies have usually been based on a rigid-plastic beam model for the pipe.

A fundamental result for investigations in this field is the dynamic response of a straight cantilever loaded transversely at its tip by a steady jet[1] or by an impulsive load[2]. As a natural extension, a circular quadrant cantilever struck in its own plane at its free end by a step load[3] or by a rigid mass[4] has been studied. In all these problems, the dynamic motion may be described in terms of a single-hinge mode and complete solutions have been presented.

Semi-circular[5] and bent[6] cantilevers subjected to an out-of-plane step load applied at the tip have been considered. The results have shown that, in some loading conditions, a complete solution cannot be provided by single-hinge modes, since yield criterion violation would occur in some portion of the cantilever, the term "unstable hinge" being used to describe this behaviour. The analysis of bent cantilevers loaded transversely to their plane by a step load at the free end has been extended to include deformation modes more complex than the single-hinge mode and which do not violate the yield criterion[7, 8]. A model with a double-hinge mode is introduced and it is shown that, in certain cases, a three-hinge mode may have to be considered.

The primary concern in all these works has been the heuristic determination of the shape of the velocity field with which motion is initiated. It seems therefore worthwhile to look for a mathematical model which is able to furnish automatically the initial deformation mechanism of a structural system subjected to a pulse loading, for any level of that loading.

A framed structure is modelled as a set of finite elements linked at node points, and the structural mass is considered to be lumped at some or all of these points. The development of plasticity is assumed to be confined to critical sections which are located at the ends of the structural members. A nodal description of the kinetic and kinematic laws is furnished under the restriction of infinitesimal displacements. The material is assumed to be rigid, perfectly plastic, and a piecewise linear approximation of the yield surface is considered. The combination of the fundamental structural relations at the initial instant of motion and the consideration of the initial value of the pulse loading as a variable parameter lead to a governing system in the form of a PLCP.

2. FUNDAMENTAL RELATIONS

The nodal description of dynamic equilibrium and compatibility for the problem is of the form given in (1) and (2) respectively.

$$\boldsymbol{Q} = \boldsymbol{0} = \begin{bmatrix} \boldsymbol{A}^T & \boldsymbol{A}_d^T & \boldsymbol{A}_o^T \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ -\boldsymbol{\mu} \\ -\boldsymbol{\lambda} \end{bmatrix}, \qquad \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{\delta}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{A}_d \\ \boldsymbol{A}_o \end{bmatrix} \dot{\boldsymbol{q}} . \qquad (1, 2)$$

In the law of kinetics (1), $X(t)$ are the s independent member forces at time t, and D'Alembert's principle is followed in that the current applied loads $\lambda(t)$ are supplemented by the current values of the γ inertia forces $\mu(t)$; for dynamic equilibrium, it is necessary that the β nodal constraint forces $Q(t)$ are null. In the kinematic relations (2), $\dot{q}(t)$ are the β velocities corresponding to the Lagrange or generalised coordinates, $\dot{x}(t)$ are the s independent member deformation rates, $\dot{\delta}(t)$ are the load-point velocities which are dual to the applied loads $\lambda(t)$, and $\dot{u}(t)$ are the velocities corresponding to the γ dynamic degrees of freedom associated with the centres of gravity of the structural masses. Relations (1) and (2) display an adjoint form that is usually referred to as kinetic-kinematic duality[9]; they are not independent since

$$\mu = -m\ddot{u}, \qquad (3)$$

where the dot notation indicates derivation with respect to time and $\boldsymbol{m}$ is a diagonal matrix of masses and mass moments of inertia.

The appropriate constitutive laws are given by Maier's[10] matrix representation of plasticity as

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{N}^T \\ \boldsymbol{N} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}_* \\ \boldsymbol{S} \end{bmatrix} + \begin{bmatrix} \boldsymbol{y}_* \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_* \\ \dot{\boldsymbol{s}} \end{bmatrix} \qquad \begin{matrix} \text{(a)} \\ \text{(b)} \end{matrix} \qquad (4)$$

$$\boldsymbol{y}_* \geq \boldsymbol{0} \qquad \boldsymbol{y}_*^T \dot{\boldsymbol{x}}_* = 0 \qquad \dot{\boldsymbol{x}}_* \geq \boldsymbol{0} \qquad \text{(c–e)}$$

where vectors $\boldsymbol{S}$ and $\dot{\boldsymbol{s}}$ contain respectively the stress resultants and the dual plastic strain resultant rates, which contribute to the plastic dissipative energy at the critical sections. The plastic potentials $\boldsymbol{y}_*$ are defined by requirement (4a) wherein $\boldsymbol{X}_*$ is the vector of plastic capacities and $\boldsymbol{N}$ is the matrix that collects the unit outward normal vectors to the piecewise linear yield surface. Associated flow laws are implicit in expression (4b) of the strain resultant rates in terms of the plastic multiplier rates $\dot{\boldsymbol{x}}_*$.

It may be remarked that the nonholonomic plasticity relations (4) are expressed in terms of the generalised stresses $\boldsymbol{S}$ and generalised strain rates $\dot{\boldsymbol{s}}$ at the critical sections, whereas the structural elements are described in terms of the independent member forces $\boldsymbol{X}$ and independent deformation rates $\dot{\boldsymbol{x}}$, defined at the element termini. It is therefore necessary to set up the following transformations:

$$\dot{x} = T\dot{s}, \qquad S = T^T X. \qquad (5, 6)$$

3. GOVERNING SYSTEM FOR INITIAL DEFORMATION MECHANISM

For the present needs, there is no restriction in assuming that the structure is subjected to a proportional dynamic pressure; i.e. the applied loading has a continuous force-time relationship and the loads maintain a constant ratio to one another. Hence, the dynamic loading may be specified in the separable form $\lambda(t) = \bar{\lambda}\lambda(t)$ where $\bar{\lambda}$ is a dimensionless vector which defines the proportions among the applied loads and $\lambda(t)$ is the function which describes the time variation of the pressure. Accordingly, the matrix product $A_o^T\bar{\lambda}$ which appears in the kinetic relation (1) may be rewritten as $a_o\lambda(t)$ where, clearly, a_o is a β-vector.

For a structure which is coerced into motion by a pressure pulse, the initial conditions are usually taken as stationary so that $q(t = 0) = q_0 = 0$ and $\dot{q}(t = 0) = \dot{q}_0 = 0$. It follows from the second of these that the vector of initial plastic multiplier rates $\dot{x}_*(t = 0) = \dot{x}_{*0} = 0$. From this, by time differentiation of the constitutive laws (4), it may be shown[11, 12] that

$$y_{*0}^T\ddot{x}_{*0} = 0, \qquad \ddot{x}_{*0} \geq 0. \tag{7, 8}$$

With the deduction of results (7) and (8), valid only for $t = 0$, the exact nodal governing system at the initial instant of motion may be obtained through the kinetic, kinematic and constitutive laws, relations (1-6), as follows:

$$\begin{bmatrix} -M_q & -A^T & 0 & 0 \\ -A & 0 & TN & 0 \\ 0 & N^TT^T & 0 & I \end{bmatrix} \begin{bmatrix} \ddot{q}_0 \\ X_0 \\ \ddot{x}_{*0} \\ y_{*0} \end{bmatrix} = \begin{bmatrix} -a_o\lambda(0) \\ 0 \\ X_* \end{bmatrix} \quad \begin{matrix} \text{(a)} \\ \text{(b)} \\ \text{(c)} \end{matrix} \tag{9}$$

$$y_{*0} \geq 0 \qquad y_{*0}^T\ddot{x}_{*0} = 0 \qquad \ddot{x}_{*0} \geq 0 \qquad \text{(d-f)}$$

$$\ddot{q}_0,\ X_0 \ \text{unrestricted}$$

Here, $M_q = A_d^TmA_d$ denotes the mass matrix referred to the accelerations $\ddot{q}$ of the generalised or Lagrange coordinates.

Consider now that the fixed value $\lambda(0)$ in formulation (9) is substituted by a single non-negative real parameter Λ which may be varied as required. Then the nodal formulation of the structural system at $t = 0$ may be expressed as

$$\begin{bmatrix} -M_q & -A^T & 0 & 0 & a_o \\ -A & 0 & TN & 0 & 0 \\ 0 & N^TT^T & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ X \\ \ddot{x}_* \\ y_* \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ X_* \end{bmatrix} \quad \begin{matrix} \text{(a)} \\ \text{(b)} \\ \text{(c)} \end{matrix} \tag{10}$$

$$y_* \geq 0 \qquad y_*^T\ddot{x}_* = 0 \qquad \ddot{x}_* \geq 0 \qquad \text{(d-f)}$$

where the temporal subscript 0 has been suppressed.

System (10) has a special mathematical structure known as a PLCP. The solution of the PLCP, which can be achieved by a variant of Wolfe's algorithm in its long form, defines the critical load levels at which the pattern of initial motion changes from one form to another, yielding also all possible initial fields of acceleration and corresponding generalised stresses as the parameter is increased monotonically from zero. From this information, the sequence of possible velocity profiles with which motion is commenced is immediately established. Further details of this PLCP formulation as well as the variant of Wolfe's algorithm in its long form may be found in Reference [13].

4. NUMERICAL APPLICATIONS

To illustrate the capability of the proposed mathematical model and associated numerical implementation, two examples have been selected. The first is that of a uniform cantilever represented in discrete form by twelve equispaced point masses, as shown in Figure 1(a), where the critical sections are consecutively numbered. The beam, which is excited into motion by a pulse load $\lambda(t)$ applied transversely at its tip, is assumed to be rigid-plastic, having plastic moment of resistance X_*.

The application of PLCP (10) reveals that there are twelve initial modes of deformation, each corresponding to a different interval of the initial load level $\lambda_0 \equiv \Lambda$. Figure 1(b) indicates those critical sections of the cantilever which are initially active for each value of the initial load level Λ. For the first interval $0.083 \leq \Lambda L/X_* \leq 0.252$, where the lower limit represents the static plastic collapse load, the initial mechanism of motion features a single plastic hinge at section 0 and therefore coincides with the pattern of deformation associated with static plastic collapse. For $\Lambda L/X_* \geq 0.252$, two adjacent plastic hinges will generally be active in the initial mechanism. However, at each transitional load level, one of the two hinges, say that at section i, will be only potentially active in the sense that the bending moment $X(i)$ will be fully plastic while the rotational acceleration $\ddot{x}(i)$ will be zero.

Numerical values for the transitional loads, at which the pattern of initial motion changes from one mode to another, are given in Table 1. For each such load, the active and potentially active sections are noted and the initial rotational acceleration in the active hinge is recorded. As a matter of interest, the initial transverse tip acceleration, which would be induced by each of the transitional loads, is also given in the table.

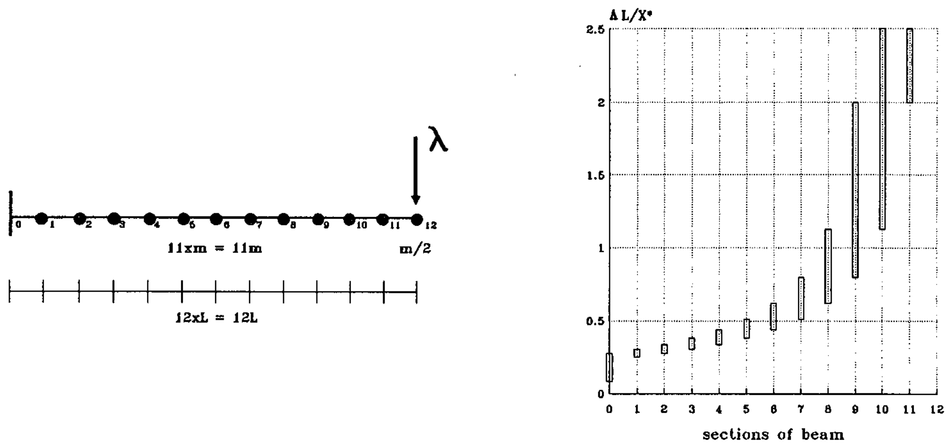
TABLE 1 Initial Accelerations Corresponding to Transitional Loads

Transitional load $\Lambda L / X_*$	Potentially active section	Active section	Hinge acceleration $\ddot{x}mL^2 / X_*$	Tip acceleration $\ddot{u}mL / X_*$
0.083	0			
0.252	1	0	0.003	0.042
0.275	2	1	0.005	0.050
0.303	3	2	0.006	0.061
0.337	4	3	0.008	0.075
0.381	5	4	0.012	0.095
0.438	6	5	0.018	0.125
0.514	7	6	0.029	0.171
0.625	8	7	0.050	0.250
0.800	9	8	0.100	0.400
1.125	10	9	0.250	0.750
2.000	11	10	1.000	2.000

It may be remarked that, as the discrete modelling of the cantilever approaches the continuum form, so Figure 1(b) approaches a continuous curve, becoming asymptotic to the vertical line at critical section 12. This curve may be imagined to give the initial position of a single travelling hinge as a function of the initial load level Λ. Therefore, as the initial load level approaches the impulsive case, the initial position of the travelling hinge approaches the tip of the cantilever.

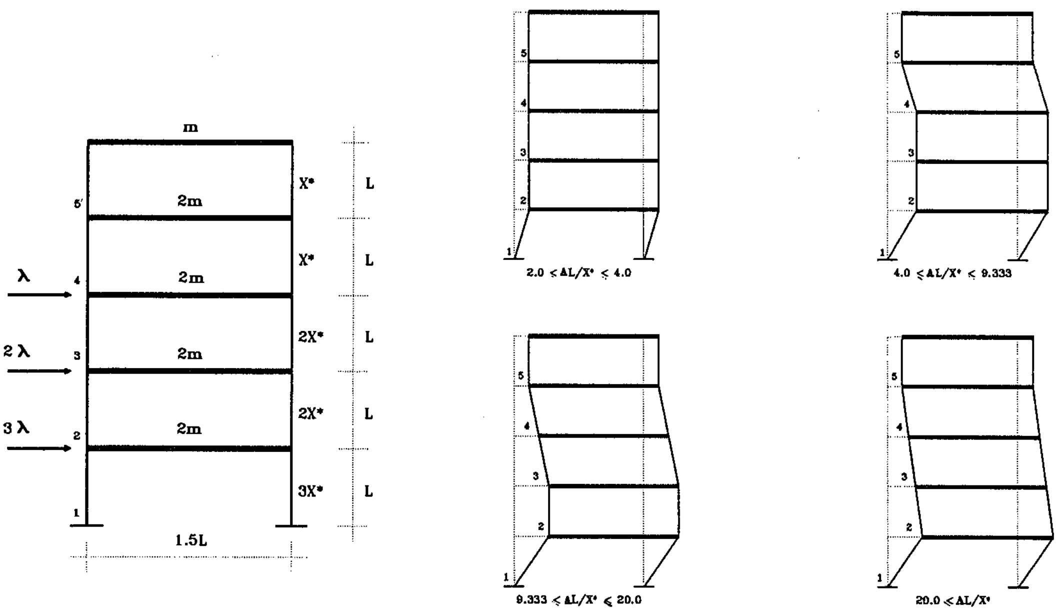
As a second example, the possible patterns of initial deformation of a five storey shear frame are investigated. The frame, which is indicated in Figure 2(a), is exposed to a pulse loading over the lower three floors. The immediate result of applying PLCP (10) to this problem is summarised by the sequence of velocity profiles of the initial motion shown in Figure 2(b). The solution of the PLCP once more defines the corresponding ranges of the initial load level Λ, beginning with the static plastic collapse load $\Lambda = 2X_*/L$.

It is worth stressing that the results just presented were independently confirmed by carrying out a full time-stepping, rigid-plastic dynamic analysis as described in References [14] and [15].



(a) (b)

FIGURE 1 Twelve Mass Cantilever — Transverse Tip Load: a) Geometrical and loading scheme, b) Initially active negative sections v. initial load level.



(a) (b)

FIGURE 2 Five Storey Shear Frame: a) Geometrical and loading scheme; b) Possible patterns of initial motion.

5. CLOSURE

By combining systematically the fundamental vectorial conditions which characterise the behaviour of a pulse-loaded rigid-plastic structure at the onset of motion, and by regarding the initial level of the pressure loading as a variable parameter, a governing system in the form of a PLCP has been presented. The PLCP solution yields complete information about the initial state of the structural system as the parameter is increased monotonically from zero.

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