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BRITTLE - RUPTURE MECHANISMS OF AXISYMMETRIC PLATES SUBJECT TO CREEP UNDER SURFACE AND THERMAL LOADINGS

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ABSTRACT

The problem deals with analysis of the damage mechanisms of thermally prestressed plates subject to brittle rupture. The rheological properties of the structure are described by the similarity of deviators according to the flow theory, and the time hardening theory coupled with the Kachanov orthotropic brittle rupture equations. Two boundary problems are formulated: the plate prestressed by the elastic ring or by the cylindrical shell. Depending on the values of prestressing parameters two basic failure mechanisms may by distinguished: 1) first macrocracks appear in the middle part of the plate, 2) first macrocracks appear along the periphery of the plate. Optimization of these parameters may lead to the uniform creep strength (double failure mechanism: macrocracks appear in the middle and at the periphery of the plate simultaneously).

1 BASIC EQUATIONS

Let us consider an axisymmetric sandwich plate of constant thickness. The substitutive sandwich section, which obeys the Love-Kirchhoff's hypothesis, is composed of three layers: the two working layers of constant thickness g and a core of constant depth h (Fig.1). The plate is loaded simulteneously by the uniform pressure q, and the field of constant temperature t which produces both the effect of brittle rupture and the prestressing.

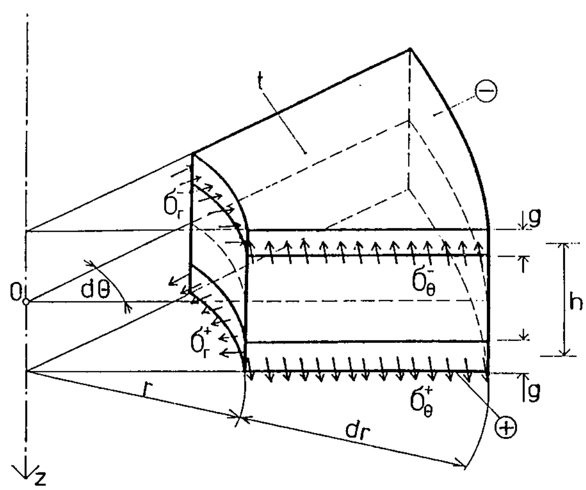

Fig.1. Substitutive sandwich section for plate element

The prestressing is produced as the result of interaction between the plate and either the elastic prestressing ring or the elastic cylindrical shell, characterized by the different coefficients of thermal expansion (Fig.2).

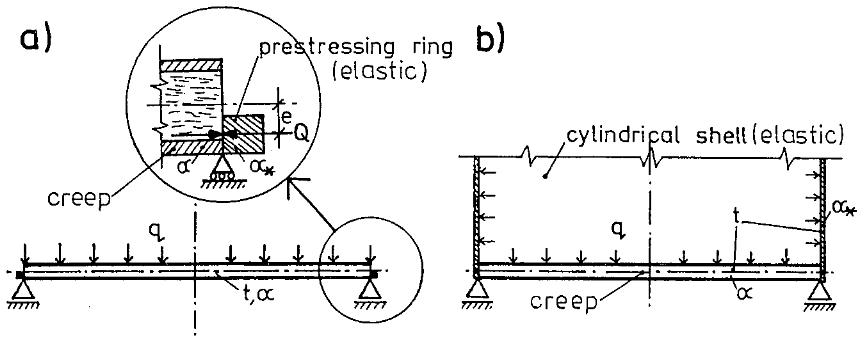

Fig.2 a. A simply supported plate prestressed by the elastic ring, b. An built-in plate prestressed by the elastic cylindrical shell

Applying the geometrically linear theory of small displacements, but with the geometry changes introduced, the problem can be formulated as "quasi coupled" [†], when von Kármán equation system extended to visco-thermo-elasticity is used:

$$\left.\begin{aligned} &\mathcal{B}\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}\right)=(1+\nu)\,\alpha\mathcal{B}\frac{dt}{dr}\\ &\mathcal{D}\nabla^4 w-\frac{1}{r}\frac{d\left(n_r r\frac{dw}{dr}\right)}{dr}=q \end{aligned}\right\}\quad \text{for}\quad \tau=0,$$

$$\left.\begin{aligned} &\mathcal{B}\left(\frac{d^2\dot u}{dr^2}+\frac{1}{r}\frac{d\dot u}{dr}-\frac{\dot u}{r^2}\right)=\frac{d\dot n_r^c}{dr}+\frac{\dot n_r^c-\dot n_\theta^c}{r}\\ &\mathcal{D}\nabla^4 \dot w-\frac{1}{r}\left[\frac{d\left(n_r r\frac{dw}{dr}\right)}{dr}\right]^{\cdot}=-\frac{d^2\dot m_r^c}{dr^2}-\frac{1}{r}\frac{d(2\dot m_r^c-\dot m_\theta^c)}{dr} \end{aligned}\right\}\quad \text{for}\quad \tau>0. \tag{1}$$

The equations of the membrane state of Eqs. (1) have the elementary solutions:

$$\left.\begin{aligned} &u=c_1 r+(1+\nu)\alpha t r/2\\ &n_{r/\theta}=\mathcal{B}(1+\nu)c_1-\mathcal{B}(1-\nu^2)\alpha t/2 \end{aligned}\right\}\quad \text{for}\quad \tau=0,$$

$$\left.\begin{aligned} &\dot u=c_2 r+\frac{r}{\mathcal{B}}\left[\frac{i_1}{1-\nu}+\frac{i_2}{1+\nu}\right]\\ &\dot n_r=\mathcal{B}(1+\nu)c_2-i_1+i_2\\ &\dot n_\theta=\mathcal{B}(1+\nu)c_2+i_1+i_2-\dot n_\theta^c+\nu\dot n_r^c\\ &i_1=\frac{1-\nu}{2r^2}\int_0^r \xi(\dot n_r^c+\dot n_\theta^c)d\xi\,,\qquad i_2=\frac{1+\nu}{2}\int_0^r\frac{\dot n_r^c+\dot n_\theta^c}{\xi}d\xi \end{aligned}\right\}\quad \text{for}\quad \tau>0, \tag{2}$$

whereas the equations of the bending state have to be numerically solved. The stresses and its rates are defined as follows:

$$\sigma_{r/\theta}^{\pm}=\pm\frac{m_{r/\theta}}{hg}+\frac{n_{r/\theta}}{2g}\,,$$

$$\dot\sigma_{r/\theta}^{\pm}=\pm\frac{\dot m_{r/\theta}+\dot m_{r/\theta}^c}{hg}+\frac{\dot n_{r/\theta}+\dot n_{r/\theta}^c}{2g}-\frac{\mathcal{B}}{2g}(\dot\varepsilon_{r/\theta}^{c\pm}+\nu\dot\varepsilon_{\theta/r}^{c\pm})\,. \tag{3}$$

[†]terms associated with the Gaussian curvature are disregarded

The generalized inelastic forces $n^c_{r/\theta}$, $m^c_{r/\theta}$, and the sandwich stiffnesses, are:

$$n^c_{r/\theta} = \frac{\mathcal{B}}{2}\left[\varepsilon^{c+}_{r/\theta} + \varepsilon^{c-}_{r/\theta} + \nu\left(\varepsilon^{c+}_{\theta/r} + \varepsilon^{c-}_{\theta/r}\right)\right] , \qquad \mathcal{B} = 2\frac{Eg}{(1-\nu^2)} ,$$
$$m^c_{r/\theta} = \frac{\mathcal{D}}{h}\left[\varepsilon^{c+}_{r/\theta} - \varepsilon^{c-}_{r/\theta} + \nu\left(\varepsilon^{c+}_{\theta/r} - \varepsilon^{c-}_{\theta/r}\right)\right] , \qquad \mathcal{D} = \frac{Eh^2 g}{2\,(1-\nu^2)} . \tag{4}$$

The total strains are decomposed into the elastic (superscript e), the creep (superscript c) and the thermal parts: $\varepsilon^{\pm}_{r/\theta} = \varepsilon^{e\pm}_{r/\theta} + \varepsilon^{c\pm}_{r/\theta} + \alpha t$, where superscripts + or − refer to the lower (exterior) or the upper (interior) sandwich layers, respectively. The similarity of deviators, based on the flow theory, and the time hardening hypothesis associated with the Kachanov's orthotropic brittle rupture theory, are taken as the constitutive relationships for creep (cf. Boyle and Spence (1983), Kachanov (1986))

$$\dot{\varepsilon}^{c\pm}_{kl} = \frac{3}{2}\frac{\dot{\varepsilon}^{c\pm}_e}{\sigma^{\pm}_e} s^{\pm}_{kl} , \quad \dot{\varepsilon}^{c\pm}_e = (\sigma^{net\pm}_e)^m \dot{f}(\tau) , \quad \dot{\psi}^{\pm}_k = -C\left\langle\frac{\sigma^{\pm}_k}{\psi^{\pm}_k}\right\rangle^n , \quad k,l = r,\theta , \tag{5}$$

where $\langle\rangle$ denote the Macauley brackets. For the plane stress state the intensities of the stress, the net stress, and the strain rates are defined by the following formulae (cf. Ganczarski and Skrzypek (1991)):

$$\sigma^{\pm}_e = \sqrt{\frac{3}{2}s^{\pm}_{kl}s^{\pm}_{kl}} , \quad \sigma^{net\pm}_e = \sqrt{\left(\frac{\sigma^{\pm}_r}{\psi^{\pm}_r}\right)^2 + \left(\frac{\sigma^{\pm}_\theta}{\psi^{\pm}_\theta}\right)^2 - \frac{\sigma^{\pm}_r}{\psi^{\pm}_r}\frac{\sigma^{\pm}_\theta}{\psi^{\pm}_\theta}} , \quad \dot{\varepsilon}^{c\pm}_e = \sqrt{\frac{2}{3}\dot{e}^{c\pm}_{kl}\dot{e}^{c\pm}_{kl}} , \tag{6}$$

respectively, where $k,l = r,\theta$, the symbols C, n, and m denote the temperature dependent material constants determined from experiments. Finally, assuming the creep incompressibility, the principal components of creep strain rates may be written as:

$$\dot{\varepsilon}^{c\pm}_{r/\theta} = \frac{(\sigma^{net\pm}_e)^m}{\sigma^{\pm}_e}\left(\sigma^{\pm}_{r/\theta} - \frac{\sigma^{\pm}_{\theta/r}}{2}\right)\dot{f}(\tau) , \quad \dot{\varepsilon}^{c\pm}_z = -\left(\dot{\varepsilon}^{c\pm}_r + \dot{\varepsilon}^{c\pm}_\theta\right) . \tag{7}$$

The elastic ring is thin enough to assume the following relation between the displacement and the radial force:

$$u_* = \frac{n_* R^2}{E_* A} + \alpha_* t R , \tag{8}$$

whereas the cylindrical shell is described by the classical equation:

$$\frac{d^4 w_*}{dx^4} + 4k^4 w_* = \frac{2-\nu}{2}\frac{q}{\mathcal{D}_*} + \frac{E_* h_*}{R\mathcal{D}_*}\alpha_* t , \tag{9}$$

the solutions of which (for the half-infinite stucture) take the form:

$$w_* = \frac{M}{2\mathcal{D}_* k^2}e^{-kx}\left[\cos(kx) - \sin(kx)\right] + \frac{Q}{2\mathcal{D}_* k^3}e^{-kx}\cos(kx) + \frac{2-\nu}{2}\frac{R^2}{E_* h_*}q + R\alpha_* t ,$$
$$\vartheta_* = -\frac{M}{\mathcal{D}_* k}e^{-kx}\cos(kx) - \frac{Q}{2\mathcal{D}_* k^2}e^{-kx}\left[\cos(kx) - \sin(kx)\right] . \tag{10}$$

The auxiliary shell parameter k and its stiffness $\mathcal{D}_*$ are:

$$k^4 = \frac{3(1-\nu^2)}{R^2 h_*^2} , \qquad \mathcal{D}_* = \frac{E_* h_*^3}{12(1-\nu^2)} . \tag{11}$$

2 FORMULATION OF THE BOUNDARY PROBLEMS AND OPTIMIZATION OF THE INITIAL PRESTRESSING

Two boundary problems are considered:

1 A simply-supported plate prestressed by the elastic ring, imposed with the initial fit δ [†)] and the eccentric e, which produce the initial radial force Q and the moment Qe:

$$\left.\begin{array}{l} n_r(R) = -Q \\ m_r(R) = -Qe \\ u(R) - u_* = \delta \\ w(R) = 0 \end{array}\right\} \quad \text{for} \quad \tau = 0\,, \qquad \left.\begin{array}{l} \dot{n}_r(R)d\tau = -dQ \\ \dot{m}_r(R)d\tau = -dQe \\ \dot{u}(R)d\tau - du_* = 0 \\ \dot{w}(R) = 0 \end{array}\right\} \quad \text{for} \quad \tau > 0\,. \tag{12}$$

2 An elastically built-in plate fitted into the cylindrical shell, with the inital fit δ imposed:

$$\left.\begin{array}{l} n_r(R) = -Q \\ m_r(R) = -M \\ u(R) - w_*(0) = \delta \\ \frac{dw(R)}{dr} = \vartheta(0) \end{array}\right\} \quad \text{for} \quad \tau = 0\,, \qquad \left.\begin{array}{l} \dot{n}_r(R)d\tau = -dQ \\ \dot{m}_r(R)d\tau = -dM \\ \dot{u}(R)d\tau - dw_*(0) = 0 \\ \frac{d\dot{w}(R)}{dr}d\tau = d\vartheta(0) \end{array}\right\} \quad \text{for} \quad \tau > 0\,. \tag{13}$$

Following the above boundary problems two optimization problems are formulated:

1 Plate prestressed by the ring:

The initial fit δ (or the peripheral prestressing force Q), and the eccentric e (or the prestressing radial moment Qe), which maximize the lifetime τ_I of the plate, under the stability and the geometric constraints, are sought for (two-parameter optimization):

$$\tau_I(Q,e) \longrightarrow \max\,, \qquad Q \ll n_E\,, \qquad e_{\max} \leq h/2\,. \tag{14}$$

Symbol n_E denotes the basic Eulerian force for the plate, and the eccentric e cannot exceed the half thickness of the plate.

2 Plate fitted into the cylindrical shell:

The initial fit δ (or the peripheral prestressing force Q and the corresponding prestressing radial moment M), such that the uniform creep strength is achieved (in sense of two-point failure), is sought for (one-parameter optimization):

$$\inf\left\{\psi^{\pm}_{r/\theta}(\delta)\right\}\Big|_{r\cong 0} = \inf\left\{\psi^{\pm}_{r/\theta}(\delta)\right\}\Big|_{r=R} = 0 \qquad \text{at} \qquad \tau = \tau_I\,. \tag{15}$$

The infimum of the vector continuity functions $\psi^{\pm}_r$ and $\psi^{\pm}_\theta$ is found when both sandwich layers are examined (cf. Ganczarski (1992)). Note that, in this case, the prestressing force Q and the moment M are the dependent quantities, since they both are equivalent to one prestressing parameter δ. Hence, when the transient creep process is solved, one of these quantities, say Q, is to be determined by the additional interaction loop in order to satisfy the current plate-shell interation.

[†)] $\delta \overset{def}{=} R_{ring/shell} - R_{plate}$ therefore $\delta > 0$ and $\delta < 0$ denote too loose and too narrow fits, respectively

The optimization requires the solution of the subsequent creep rupture problems. The numerical solution of the creep problem starts from the elastic solution with $\psi^{\pm}_{r/\theta} \equiv 1$. Then the "new" values of strain $\varepsilon^{\pm}_{r/\theta}$, stresses $\sigma^{\pm}_{r/\theta}$, inelastic generalized forces $n^{c}_{r/\theta}$, $m^{c}_{r/\theta}$, and continuity functions are computed for the subsequent time step. The procedure is repeated until the first macrocrack is initiated. In all cases under consideration and each step of time the equations of bending state (1) are solved numerically when the finite difference method (FDM) is used, whereas for the time increments the Runge - Kutta II method is applied.

3 RESULTS

The numerical examples are presented for the plates made of the ASTM 321 stainless steel: $E = 1.77 \times 10^5 MPa$, $\sigma_0 = 1.18 \times 10^2 MPa$, $\nu = 0.3$, $\alpha = 1.8 \times 10^{-5} 1/K$, $R = 0.5m$, $h = 0.025m$, $g = 0.005m$, $q = 118kPa$; the temperature (783 K) dependent material constants for creep rupture are (cf. Odqvist (1966)): $C = 2.13 \times 10^{-42} Pa^{-n}/s, n = 3.9, m = 5.6$, whereas material constants for both the prestressing ring and the cylindrical shell made of ASTM 310 stainless steel are as follows: $E_* = E$, $\nu_* = \nu$, $\alpha_* = 1.7 \times 10^{-5} 1/K$, $h_* = \sqrt[3]{\frac{6}{5}}\frac{h}{2.5}$, $A = 2g \times 2g = 4g^2$.

The dimensionless lifetime of the plate prestressed by the ring, versus the fit δ with the eccentric e taken as a parameter, is shown in Fig.3. The lifetime of the plate fitted into the cylindrical shell, versus the fit δ, is shown in Fig.5. The corresponding distributions of continuity the components $\psi^{\pm}_{r/\theta}$ (at the instant of failure) for the optimal solutions for each case, respectively, are presented in Fig.4, and Fig.6.

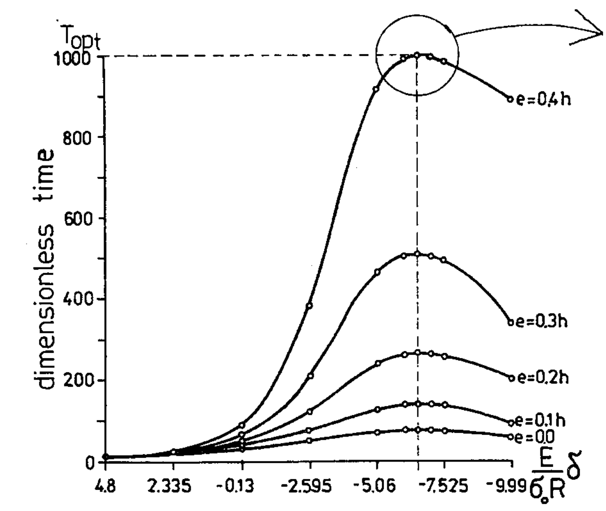

Fig.3. Lifetime of the simply supported plate prestressed by the ring, versus the initial fit δ with the eccentric e as parameter

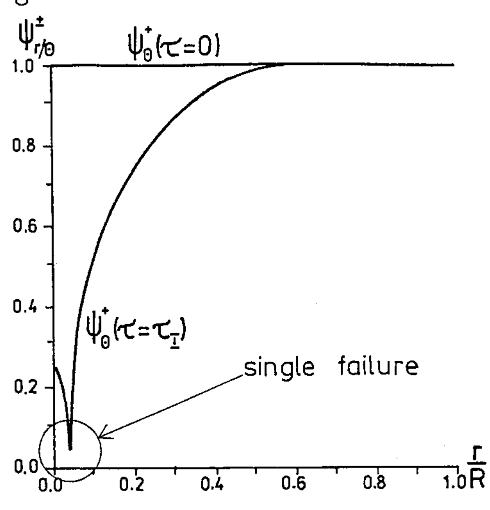

Fig.4. Distribution of the continuity functions at the instant of failure (single failure)

The time to rupture for both discussed cases, compared to the lifetime of simply-supported plate in pure bending state (without initial prestressing), is presented in Table 1.

Table 1. Comparison of lifetime for optimally prestressed plates

mode of prestressing	pure bending state (simply-supported plate)	optimal prestressing	
		$\delta = \delta_{opt}$, $e = 0$	$\delta = \delta_{opt}$, $e = e_{opt}$
elastic ring	τ_I	$5.76\tau_I$	$75.78\tau_I$
cylindrical shell		$2.97\tau_I$	—

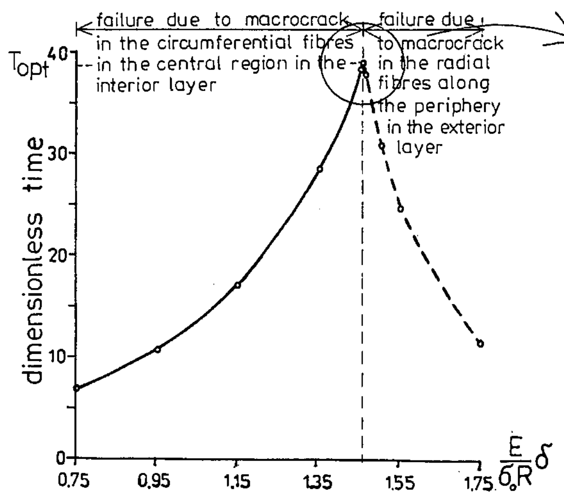

Fig.5. Lifetime of the plate fitted into the cylindrical shell, versus the initial fit δ

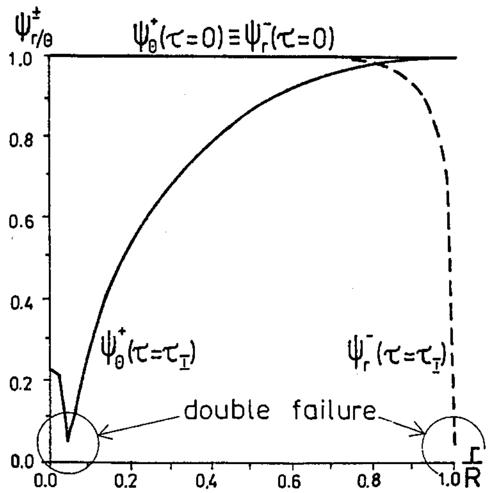

Fig.6. Distribution of the continuity functions at the instant of failure (double failure)

4 CONCLUSIONS

1 Both optimization problems: 1) a simply-supported plate prestressed by the ring (two-parameter optimization) and 2) plate fitted into the cylidrical shell (one-parameter optimization) lead to a significant increase of the plate lifetime, when compared to the simply-supported plate without prestressing.

2 In case of a simply-supported plate prestressed by the elastic ring the maximum of lifetime is reached at the smooth point, and, in consequence, only single failure mechanism appears (first macrocrack in the central part of the plate), though the double failure mechanism is hypothetically possible when the geometric constraint is exceeded. In case of a plate fitted into the cylindrical shell the maximum of lifetime occurs at the "switch" point, where the intersection of the curves that correspond to two different failure mechanisms (failure due to macrocracks in the circumferential fibres in the central region, in the interior layer, and failure due to macrocracks in the radial fibres along the periphery, in the exterior layer) hold.

3 In both cases under consideration the maximum of lifetime is reached when the prestressing moment causes unloading of the structure subject to working loading, and it requires: the negative fit $\delta < 0$ (too narrow) in case of prestressing by the ring, and the positive fit $\delta > 0$ (too loose) in case of fitting into the shell.

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