DIRECT OUTPUT FEEDBACK CONTROL OF DISCRETE-TIME SYSTEMS

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ABSTRACT

An optimal direct output feedback control algorithm is developed for discrete-time systems with the consideration of time delay in control force action. Optimal constant output feedback gains are obtained through variational process such that certain prescribed quadratic performance index is minimized. Discrete-time control forces are then calculated from the multiplication of output measurements by these pre-calculated feedback gains. According to the proposed algorithm, structural system is assured to remain stable even in the presence of time delay. The number of sensors and controllers may be very small as compared with the dimension of states. Numerical results show that direct velocity feedback control is more sensitive to time delay than state feedback but, is still quite effective in reducing the dynamic responses under earthquake excitation.

1 INTRODUCTION

Due to the usage of digital computer for on-line calculation in real-time control, information such as output measurements involved in the calculation of control forces is sampled once every cycle of data acquisition and discrete-time control forces are converted to analog signal in the form of piecewise step functions and applied to the structure. In a word, every information including input control forces and output measurements of the control system are basically discrete-time in nature. Therefore, it is more logical and more realistic to formulate the control system in the discrete-time fashion. In recent years, several discrete-time control algorithms have been developed and verified in laboratory experiments (Lin et al. 1987; Yang et al. 1987; Rodellar et al. 1987, 1989).

A structure usually possesses a large number of degrees of freedom. Economy, data processing and on-line calculation considerations make it impractical and impossible to acquire full state measurement and feedback. Thus, direct output feedback becomes necessary from a practical point of view (Chung et al. 1992). In addition, time delay in data processing, control force execution, and on-line calculations is unavoidable in real-time control. It not only can render the control ineffective, but also may cause the system instability. Hence, time delay effect must be considered in control design.

The purpose of this paper is to develop an optimal direct output feedback control algorithm to tackle above practical problems for discrete-time systems with the consideration of time delay in control force application. Optimal constant output feedback gains are obtained through variational process such that certain prescribed quadratic performance index is minimized. Discrete-time control forces are then calculated by multiplying these pre-calculated feedback gains by output measurements. According to the proposed algorithm, structural system is assured to remain stable even with the presence of time delay. The number of sensors and controllers may be very small as compared with the dimension of states. Small number of sensors and controllers, and simple on-line calculation make the proposed control algorithm favorable to real-time implementation.
2 CONTROL ALGORITHM

Suppose that time delay exists in the execution of control with an amount equal to $t_d$, the modified continuous ordinary differential equation of motion of an n-DOF discrete-parameter structural system under dynamic loading $w(t)$ and active control force $u(t-t_d)$ can be written as

$$M \ddot{x}_c(t) + C \dot{x}_c(t) + K x_c(t) = B_0 u(t-t_d) + E_0 w(t)$$

(1)

where $x_c(t)$ is n×1 displacement vector, $M$, $C$, and $K$ are n×n mass, viscous damping, and stiffness matrices, respectively, $B_0$ is n×q location matrix of control forces, and $E_0$ is n×r location matrix of external loadings. Represented in state-space form, eq.(1) can be rewritten as

$$\dot{x}(t) = A_c x(t) + B_c u(t-t_d) + E_c w(t)$$

(2)

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A_c = \begin{bmatrix} 0 & & & & \\ -M^{-1}K & I & & & \\ & -M^{-1}C & & & \\ & & & 0 & & \\ & & & & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ M^{-1}B_0 \end{bmatrix}, \quad E_c = \begin{bmatrix} 0 \\ M^{-1}E_0 \end{bmatrix}$$

are 2n×1 state vector, 2n×2n system matrix, 2n×q controller location matrix, and 2n×r external loading location matrix, respectively.

Because the complete set of states $x(t)$ may not be available, measured output vector $y(t)$ from limited number of sensors, say p and $p<2n$, is just some linear combination of state vector $x(t)$. They are related by

$$y(t) = D_c x(t)$$

(3)

where $D_c$ is the p×2n sensor location matrix.

**Discrete-Time System.** Since the control system is linear and time-invariant, the general solution to eq.(2) is given by

$$x(t_2) = e^{A_c(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{A_c(t_2-\tau)} [B_c u(\tau-t_d) + E_c w(\tau)] d\tau$$

(4)

Suppose all information for the on-line calculation of control forces is sampled with period $\Delta t$. Between two consecutive sampling instants, $k \Delta t$ and $(k+1) \Delta t$, the best available information about the excitation is $w(k \Delta t)$. Therefore, external loading is sampled as zero-order hold and is thus assumed to be constant between two consecutive sampling instants. If we let $t_1 = k \Delta t$, $t_2 = (k+1) \Delta t$, and substitute $\eta = (k+1) \Delta t - \tau$ for $\tau$ in the integral, then eq.(4) becomes

$$x[(k+1) \Delta t] = e^{A_c \Delta t} x(k \Delta t) + \int_0^{\Delta t} e^{A_c \eta} B_c u[(k+1) \Delta t - t_d - \eta] d\eta + \int_0^{\Delta t} e^{A_c \eta} d\eta E_c w(k \Delta t)$$

(5)

If we separate the system delay $t_d$ into an integral number of sampling periods plus a fraction, we can define

$$t_d = (l-m) \Delta t \quad ; \quad l \geq 1 \quad , \quad 0 \leq m < 1$$

(6)

With this substitution, the delay time $t_d$ is not necessary to be a multiple of sampling period $\Delta t$. It can be any number and such substitution can reduce the dimension of the modified state significantly. Meanwhile, eq.(5) is thus rewritten as

$$x[(k+1) \Delta t] = e^{A_c \Delta t} x(k \Delta t) + \int_0^{\Delta t} e^{A_c \eta} B_c u[(k+1-l-m) \Delta t - \eta] d\eta + \int_0^{\Delta t} e^{A_c \eta} d\eta E_c w(k \Delta t)$$

(7)

The first integral in eq.(7) runs for $\eta$ from 0 to $\Delta t$, which corresponds to $t$ from $(k+1-l-m) \Delta t$ backward to $(k-l+m) \Delta t$. Over this period, the control force, which we assume is piecewise
constant, takes on first the value $u[(k+1-l)\Delta t]$ and then the value $u[(k-l)\Delta t]$. Therefore, the first integral can be broken into two parts and the discrete system is described by

$$\bar{x}(k+1) = A\bar{x}(k) + B_1 u(k-l) + B_2 u(k-l+1) + Ew(k)$$  

(8)

and

$$y(k) = D\bar{x}(k)$$  

(9)

where $A = e^{A_1 \Delta t}$, $B_1 = A_1^{-1}(A - A^m)B_2$, $B_2 = A_1^{-1}(A^m - I)B_2$, $E = A_1^{-1}(A - I)E_0$, and $D = D_0$.

To eliminate the past controls up to $u(k)$, we introduce a new $(2n+q)\times 1$ state vector

$$\bar{z}(k) = \begin{bmatrix} x(k) \\ u(k-l) \\ u(k-l+1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

such that the modified state equation is given by

$$\bar{z}(k+1) = \bar{A}\bar{z}(k) + \bar{B}u(k) + Ew(k)$$  

(10)

where

$$\bar{A} = \begin{bmatrix} A & B_1 & B_2 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

and the output equation

$$y(k) = D\bar{x}(k)$$  

(11)

in which $D = \begin{bmatrix} D & 0 & \cdots & 0 \end{bmatrix}$ is a $p\times(2n+q)$ matrix.

**Output-Feedback.** In output feedback, control forces are directly generated from the multiplication of output measurements by constant feedback gains

$$u(k) = Gy(k)$$  

(12)

where $G$ is a $q\times p$ shift-invariant feedback gain matrix. Since output measurements are linear combination of states, control forces are in turn linearly related with system states

$$u(k) = GD\bar{z}(k)$$  

(13)

Substituting eq.(13) into eq.(10) and rearranging, the state equation (10) becomes

$$\bar{z}(k+1) = (\bar{A} + BGD)\bar{z}(k) + Ew(k)$$  

(14)

Under classical quadratic performance criterion, the active control force is found such that the summation

$$J = \sum_{k=0}^{\infty} \left\{ \bar{x}^T(k)Q\bar{x}(k) + u^T(k)Ru(k) \right\}$$  

(15)

is minimized. In above equation, $Q$ is the $(2n+q)\times(2n+q)$ symmetric positive semi-definite weighting matrix for the responses, $R$ is the $q\times q$ symmetric positive definite weighting matrix for the input control forces, and superscript $T$ denotes transpose of a matrix.

For any output measurement, there exists an optimal output feedback gain matrix $G$ such that the performance index $J$ given in eq.(15) is minimized. Under random initial
disturbance \( \bar{x}(0) = x_0 = [x(0) 0 \ldots 0]^T \) only, the equivalent feedback state equation of eq.(14) is solved as
\[
\bar{x}(k) = (\bar{A} + BGD)^k \bar{x}_0
\] (16)

After eqs.(13) and (16) are sequentially substituted into eq.(15), it gives
\[
J = \sum_{k=0}^{\infty} \bar{x}_0^T [(\bar{A} + BGD)^k]^T (Q + D^T G^T RGD)(\bar{A} + BGD)^k \bar{x}_0 = \bar{x}_0^T \bar{H} \bar{x}_0 = \text{tr}(\bar{H} \bar{x}_0)
\] (17)

where the constant matrix \( \bar{H} \) is defined as
\[
\bar{H} = \sum_{k=0}^{\infty} [(\bar{A} + BGD)^k]^T (Q + D^T G^T RGD)(\bar{A} + BGD)^k
\] (18)

\( \text{tr}(\cdot) \) denotes trace of a matrix, and \( \bar{x}_0 = \bar{x}_0 \bar{x}_0^T \).

The constant matrix \( \bar{H} \) is first pre-multiplied and post-multiplied by \((\bar{A} + BGD)^T \) and \((\bar{A} + BGD) \), respectively, and the constant matrix itself is then subtracted from the product. As \( k \) approaches infinity, we get
\[
(\bar{A} + BGD)^T \bar{H}(\bar{A} + BGD) - \bar{H} + (Q + D^T G^T RGD) = 0
\] (19)

The optimization problem is now converted to one that minimizes the performance index \( J \) subject to the constraint of eq.(19). Incorporated with this constraint equation, the Lagrangian \( J' \) can be expressed as
\[
J' = \text{tr}(\bar{H} \bar{x}_0) + \text{tr}\left\{ L [(\bar{A} + BGD)^T \bar{H}(\bar{A} + BGD) - \bar{H} + (Q + D^T G^T RGD) \right\}
\] (20)

where \( L \) is a \((2n+q)|\times(2n+q)|\) Lagrangian multiplier matrix. Because the performance index is quadratic, and \( Q \) and \( R \) are positive semi-definite and positive definite matrices, respectively, the necessary and sufficient conditions for minimization of \( J' \) are
\[
\frac{\partial J'}{\partial L} = (\bar{A} + BGD)^T \bar{H}(\bar{A} + BGD) - \bar{H} + (Q + D^T G^T RGD) = 0
\] (21)
\[
\frac{\partial J'}{\partial \bar{H}} = (\bar{A} + BGD)L(\bar{A} + BGD)^T - L + \bar{x}_0 = 0
\] (22)
\[
\frac{\partial J'}{\partial G} = 2B^T \bar{H}(\bar{A} + BGD)LD^T + 2RGLD = 0
\] (23)

Introducing special matrix operations (Soong 1977), the optimal output feedback gain matrix \( G \) can be obtained by solving the simultaneously linear algebraic equations (21–23) iteratively until relative error is acceptable.

Transfer Function. Taking \( z \)-transform on both sides of eq.(8), coupling with eq.(13), and rearranging it, we get
\[
\chi(z) = H(z) \omega(z) \quad \text{and} \quad H(z) = \left[zI - A - (z^{-1}B_1 + z^{-(l-1)}B_2)GD \right]^{-1} E
\] (24)

where \( H(z) \) is the transfer function of the control system. The frequency response functions of the system are obtained by substituting \( z = e^{j2\pi f \Delta t} \) (where \( j = \sqrt{-1} \)) into the corresponding transfer function \( H(z) \) given in eq.(24). The stability of the control system can be checked by examining whether the poles of the transfer function are located within unit circle of the complex plane or not. Moreover, system parameters such as natural frequencies and damping factors of the control system can be extracted from the poles of the transfer function. Frequency response functions, natural frequencies and damping factors determine the control effectiveness.
3 NUMERICAL VERIFICATIONS

A SDOF structure \((M=16.69 \text{ lb}-\text{sec}^2/\text{in}, C=9.03 \text{ lb}-\text{sec}/\text{in}, K=7934 \text{ lb}/\text{in}, T_0=0.288 \text{ sec}, \xi=1.24\%\) with control device in place shown in Fig. 1 is studied to verify the control effectiveness of proposed algorithm. The first 10 second acceleration of El Centro earthquake (N-S component, 1940), which includes the strong motion part, is used as the base excitation. The control force produced by adjusting actuator displacement \(U(t)\) is transmitted to the structure through four pretensioned tendons. They are related by

\[
u(t) = 4 \frac{k_c}{\cos \alpha} U(t) \tag{25}\]

where \(k_c = \frac{-2124 \text{ lb/in}}{\cos \alpha} \) and \(\alpha = 36^\circ\) are the stiffness and inclination angle of tendons. The weighting matrices are given as element \(Q(1,1) = K\), zeros for the other elements in matrix \(Q\), and \(R = \beta/(16k_c\cos^2 \alpha)\) to make performance index \(J\) be the summation of structural strain energy and applied control energy. The coefficient \(\beta\) determines the relative importance of response reduction and control force requirements.

As concluded from authors' study (Chung et al. 1992), direct displacement feedback is found to be ineffective. As \(\beta=50\), the control effectiveness of both direct velocity feedback and state feedback with and without time delay is investigated. As seen in Figs. 2 and 3, direct velocity feedback control is more sensitive to time delay than state feedback but, the system is not destabilized. Under active control, the system damping factor always increases as compared with that of uncontrolled case, and thus the system dynamic responses can be reduced. When \(t_d=0.03 \text{ sec.}\) and \(\Delta t=0.02 \text{ sec.}\) (i.e. \(i=1, m=0.5\)), the transfer functions of relative displacement with respect to ground acceleration for direct velocity feedback, state feedback, and the uncontrolled one are shown in Fig. 4. The peaks in transfer function are dramatically suppressed with control force execution. The time histories of relative displacement under El Centro earthquake with and without control are illustrated in Fig. 5. Both peak and root-mean-square responses decrease significantly.

4 CONCLUSIONS

The theoretical development and numerical simulation results of the proposed control algorithm indicate that direct velocity feedback is effective in reducing the structural responses as that of full state feedback under earthquake excitation even in the presence of time delay. The number of sensors and controllers may be very small as compared with the dimension of states. Small number of sensors and controllers, and simple on-line calculation make the proposed control algorithm favorable to real-time implementation.

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6 REFERENCES

Fig. 1. SDOF Model Structure

Fig. 2. Time-Delay Effect on Natural Frequency

Fig. 3. Time-Delay Effect on Damping Factor

Fig. 4. Relative Displacement Transfer Function (with and without Control)

Fig. 5. Relative Displacement under El Centro Earthquake without and with Time-Delay Control