A Two-Dimensional Simultaneous Grid Interaction Algorithm
in Lagrangian Frame of Reference

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Abstract
We propose in the present paper a new method to calculate restoring forces
among interacting lagrangian grids by the momentum balance condition at intersected
positions using extrapolated values on the boundary segment. With the assumption
to neglect compressibility in the time increment, the problem of continuum can be
simplified to the problem of interacting mass points. The effective and accurate
algorithm has been developed and verified against missile impact loadings and a
broken dam problem.

1. Introduction
Grid interactions in two dimensions take place in either way: gridpoints vs
gridpoints or segments vs gridpoints. We can regard the latter case as the
collisions between the particles by generating material points at the impact
positions under the assumption that the material is incompressible at the moment.
It is evident that so far as internal forces are concerned, the total momentum of
all the particles stay constant /1/. Hence, the positions and velocities of the
gridpoints in the next computational cycle are determined uniquely and simulta-
neously from those in the present cycle as long as the interaction is independent
of history effect. Such adjustments are not required as to move "slave points" to
lie along "master-lines" after the completion of interface calculations.

The present scheme performs the following three steps,

1) Check that any interactions occur by the next cycle.
2) Generate a fictitious gridpoint at the intersected position where a gridpoint
    impacts a segment.
3) Evaluate an impulsive force under the condition that a perfectly inelastic
    collision takes place between those points during a given time step.

The details of the above steps will be presented in the next chapter. The section
numbers correspond to them.
2. Numerical algorithm

The material of interest is covered by a Lagrangian grid of quadrilaterals. For its momentum equations, material velocities are defined at zone corners, which are called gridpoints; one-fourth of the mass of the zone is associated with each of the four gridpoints via the mass of the gridpoint is equal to one-fourth of the sum of the mass of surrounding zones (see Figure 1).

2.1 Preliminary check on interactions

The purposes of this section are to find the possibilities of interactions along boundary segments and to obtain the coordinates of these positions. When no impact is observed, the remaining calculations are omitted; free surface conditions are applied to the boundary segments.

An example of the impact between two Lagrangian grids is shown in Figure 2. The material lies on the left if we travel from point A to B or P to Q. Points Q and R are in contact with segment A-B; point A with segment R-S; point B with point P. Accordingly three typical interactions are at a time included in the example.

- a gridpoint collides with one of the other grid.
- several gridpoints are in contact with one segment.
- an end point of an impacted segment interacts with a segment of the other grid.

Segments A-B and R-S are named "buckler segments"; the gridpoints except point S "halberd points" (see Figure 3).

2.2 Fictitious points

In order to obtain the masses of the fictitious points (F1, F2 and F3), it is useful to think of the two buckler zones, denoted as $\mathcal{O}$ and $\mathcal{C}$ in Figure 2, as being divided into two and three "subzones", respectively, proportional to the corresponding terms $k_i$ ($i=1,2$) and $l_i$ ($i=1,2,3$). They are, therefore, evaluated as the sum of one-fourth of the masses of the two subzones surrounding the fictitious point. Figure 4 shows the subzones in zone $\mathcal{O}$. The masses of $l_1 \ m_\mathcal{O}$ and $l_2 \ m_\mathcal{O}$ contribute to the mass of point F1. In plane geometry it is given by the expression

$$m_{F1} = \left(\frac{3}{4}\right)l_1 \ m_\mathcal{O} + \left(\frac{1}{4}\right)l_2 \ m_\mathcal{O}.$$  \hspace{1cm} (1)

The momentum of point F1 is evaluated from the conservation of angular momentum of segment A-B:

$$m_{\mathcal{O}B} = \left(\frac{3}{4}\right)l_1 \ m_\mathcal{O} u_B + \left(\frac{1}{4}\right)l_2 \ m_\mathcal{C} u_A.$$  \hspace{1cm} (2)
These procedures permit us to conserve the total amount of masses and momenta of the interfaces. Those values of all fictitious points and halberd points are given, respectively, in Table I and II.

2.3 Impulsive forces

Both grids are required to maintain contact with each other during a time step by the preliminary checks mentioned in the section 2.1. The impact between a pair of the points is, therefore, treated as a perfectly inelastic collision. The loss of kinetic energy here is converted into internal energy, which is propagated into the material below the interface. The impulsive force on point R exerted by point F1 is expressed by the equation

$$F_{R-F1} = \frac{m_R}{m_{F1} + m_R} \frac{V_{F1}}{m_{F1} + m_R} \frac{U_R}{at} .$$

(3)

If no fictitious point is produced, this becomes equivalent to an interior momentum equation.

On the other hand, the forces on the two ends of the buckler segment A-B are evaluated from eq.(3) by considering, similarly in eq.(2), the conservation of angular momentum of the segment:

$$F_{A-R} = -\frac{1}{2} F_{R-V1} ,$$

$$F_{B-R} = -\frac{1}{2} F_{R-V1} .$$

(4)

Frictional forces are calculated after the decomposition of the above forces into components normal and tangential to segment A-B. The impulsive forces of all the rest pairs are computed in accordance with eq.(3) and eq.(4), and shown in Table III.

Thus, external boundary forces loaded on the gridpoints are given by the sum of the impulsive forces obtained. For the points A and B they are expressed in the forms

$$F_A = F_{A-R} + F_{A-C} + F_{A-T3} ,$$

$$F_B = F_{B-R} + F_{B-C} + F_{B-T3} .$$

(5)

3. Test calculations

A two-dimensional Lagrangian computer program "HANJUSHT-2D" is devised, which implements the present "halberd-buckler" scheme and utilizes modern finite difference techniques. ICPL-IACM /2/. The material model employed includes an equation of state, a deviatoric constitutive relation for elastic and plastic deformation, and a failure criterion /3,4/. Once distortions become critical, automatic reasoning routines are triggered. One of the powerful features of this code enables us to free from specifying "slide-lines" with input data. The sliding interfaces are searched and consequently determined by the program.
3.1 Missile impact problem

A numerical analysis of missile impact loadings was carried out and calculated results were compared with those given by experiments /5/.
Initial configurations of a projectile and a target plate are shown in Figure 5; material properties are listed on Table IV. Figure 6 presents the deformed shape of the two grids at 10 milliseconds. The good agreement was obtained about the maximum deformation at the center of the plate between the calculation and the experiment as indicated in Figure 7. The total CPU time of 620 seconds was required to complete the analysis by CRAY-1.

3.2 Broken dam /6/

The other sample calculation was performed of two impacting incompressible fluids which were free to flow outward under the effect of gravity. It was run implicitly in plane geometry for 54 computational cycles. Figure 8 presents a series of mesh configurations, which reveals that the grids on the two sides of the interface are symmetrical.

4. Conclusion

A numerical algorithm has been developed to solve a problem of Lagrangian grid interaction. Upon application of the present method, the conservation of momentum is satisfied even in such a complex case as several grids interact with each other at the same position on a two-dimensional space, so that the overlap of grids is prevented. While it might be induced by the violation of "mass balance" – the large difference between $m_p$ and $m_R$ in eq.(3) – and the adoption of a large time step in a implicit calculation.

5. Acknowledgments

The authors would like to thank Dr. Chiba of Hitachi Ltd. for his useful discussions, who willingly offered us the experimental results cited in this paper.

References

### Table I: masses and moments of the fictitious points

<table>
<thead>
<tr>
<th></th>
<th>mass</th>
<th>momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>( m_{i1} = (\alpha_i + \beta_i l_i) m ) / 4</td>
<td>( M_{i1} = (\alpha_i + \beta_i l_i) m ) / 4</td>
</tr>
<tr>
<td>F2</td>
<td>( m_{i2} = (\alpha_i + \beta_i l_i) m ) / 4</td>
<td>( M_{i2} = (\alpha_i + \beta_i l_i) m ) / 4</td>
</tr>
<tr>
<td>F3</td>
<td>( m_{i3} = (\alpha_i + \beta_i l_i + \lambda_i k_i) m ) / 4</td>
<td>( M_{i3} = (\alpha_i + \beta_i l_i + \lambda_i k_i) m ) / 4</td>
</tr>
</tbody>
</table>

* for plane (Cartesian) coordinates, \( \alpha_i = \beta_i = 1 \), \( i = 1, 2 \) \( \mu = \lambda = 1 \).

* for cylindrical coordinates; radial weighing is necessary for consistency with momentum equations,

\[
\alpha_i = \left( x_i + \sum \frac{1}{2} l_i (x_i - x) \right) \left( x_i + \frac{x_i + x}{2} \right),
\]

\[
\beta_i = \left( x_i + \sum \frac{1}{2} l_i (x_i - x) \right) \left( x_i + \frac{x_i + x}{2} \right),
\]

\[
\mu = \left( x_i + \sum \frac{1}{2} k_i (x_i - x) \right) \left( x_i + \frac{x_i + x}{2} \right),
\]

\[
\lambda = \left( x_i + \sum \frac{1}{2} k_i (x_i - x) \right) \left( x_i + \frac{x_i + x}{2} \right).
\]

The radial coordinate is denoted as \( x \).

### Table II: masses and moments of the halberd points

<table>
<thead>
<tr>
<th></th>
<th>mass</th>
<th>momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( m_a = m_0 + \mu )</td>
<td>( M_a = m_0 )</td>
</tr>
<tr>
<td>B</td>
<td>( m_a = m_0 + \mu )</td>
<td>( M_a = m_0 )</td>
</tr>
<tr>
<td>P</td>
<td>( m_a = m_0 + \mu )</td>
<td>( M_a = m_0 )</td>
</tr>
<tr>
<td>Q</td>
<td>( m_a = m_0 + \mu )</td>
<td>( M_a = m_0 )</td>
</tr>
<tr>
<td>R</td>
<td>( m_a = m_0 + \mu )</td>
<td>( M_a = m_0 )</td>
</tr>
</tbody>
</table>

### Table III: impulsive forces exerted on the halberd points and the fictitious ones

<table>
<thead>
<tr>
<th>Grid</th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>F3</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>( f_{x,1} = -\mu k_1 )</td>
<td>—</td>
</tr>
<tr>
<td>F1</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>( f_{x,2} = -\mu k_2 )</td>
<td>—</td>
</tr>
<tr>
<td>F2</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>( f_{x,3} = -\mu k_3 )</td>
<td>—</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>—</td>
<td>—</td>
<td>( f_{x,4} = -\mu k_4 )</td>
<td>—</td>
</tr>
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</table>

### Table IV: material properties of SUS304 and SGV49

<table>
<thead>
<tr>
<th></th>
<th>density ((g/cm^3))</th>
<th>Young's modulus ((M_p))</th>
<th>Poisson's ratio</th>
<th>true strain (%)</th>
<th>true stress ((M_p))</th>
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<tr>
<td>SUS 304</td>
<td>8.03</td>
<td>1.94 x 10^5</td>
<td>0.26</td>
<td>0</td>
<td>294.0</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>40</td>
<td>980.0</td>
</tr>
<tr>
<td>SGV 49</td>
<td>8.00</td>
<td>2.06 x 10^5</td>
<td>0.30</td>
<td>0</td>
<td>343.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>17.3</td>
<td>656.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>40</td>
<td>784.0</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>100</td>
<td>983.3</td>
</tr>
</tbody>
</table>

\( (M_p):\) megapascals

### Figure 1: Lagrangian Mesh

### Figure 2: Two Impacting Lagrangian Grids and Zones ①-⑦

### Figure 3: An Interface: Halberd Points: Buckler Segments: Fictitious Points
FIGURE 4. THREE SUBZONES IN ZONE $\mathcal{O}$

FIGURE 5. INITIAL CONFIGURATION OF MISSILE IMPACT PROBLEM

FIGURE 6. THE DEFORMED SHAPE OF MISSILE IMPACT PROBLEM AT 10 MILLISECONDS

FIGURE 7. THICKNESS DISTRIBUTION OF THE TARGET PLATE

FIGURE 8. MESH CONFIGURATIONS OF BROKEN DAM PROBLEM