Numerical Simulation of Elastic-Visco-Plastic Large Strains of Metal at High Temperature

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ABSTRACT

The derivation of the tangent stiffness matrix for the finite element simulation of large elastic-visco-plastic strains is studied. It is shown that, in general, it is non symmetric and depends on the integration scheme used to compute the stress increments during a configuration change. The use of sub-intervals integration schemes allows to perform strain increments up to 5 % in one step with good convergence and accuracy.

1. INTRODUCTION

The use of elastic-visco-plastic constitutive equations for the finite element simulation of metal forming processes requires a careful derivation of the tangent stiffness matrix in order to ensure convergence of large deformation steps. The aim of this paper is to study in what manner this stiffness matrix is influenced by the choice of an objective stress derivative in the constitutive equation and by the choice of an integration scheme to compute the stresses during a deformation step.

2. ELASTIC-VISCO-PLASTIC (EVP) CONSTITUTIVE EQUATION

EVP constitutive equations are known to model fairly well the behaviour of metals in the high temperature range [1]. The authors are presently interested in the simulation of steel hot rolling. The systematic exploitation of torsion test results [2] on steel in conditions similar to hot-rolling has shown [3] that the following set of equations gives satisfactory results : let

- x_i, v_i, σ_{ij} be the coordinates, velocities and CAUCHY stresses of a particle of a deformed solid;
- $L_{ij} = \partial v_i/\partial x_j$; $D_{ij} = \frac{1}{2}(L_{ij} + L_{ji})$; $W_{ij} = \frac{1}{2}(L_{ij} - L_{ji})$;
- $\sigma_m = \frac{1}{3}\sigma_{ll}$; $\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$; $J_2 = \frac{1}{2}\hat{\sigma}_{ij}\hat{\sigma}_{ij}$;
- $D_m = \frac{1}{3} D_{ll}$; $\hat{D}_{ij} = D_{ij} - D_m \delta_{ij}$;
- E, ν, $2G = E/(1+\nu)$; $\chi = E/(1-2\nu)$ the elastic material parameters (at the considered temperature)
- $\hat{D}^n_{ij}$ the inelastic strain rate;
- κ a scalar state variable for isotropic hardening.

Then the constitutive equations are :

$$\overset{\triangledown}{\hat{\sigma}}_{ij} = 2G\,(\hat{D}_{ij} - \hat{D}^n_{ij}) \; ; \; \dot{\sigma}_m = \chi\, D_m \qquad (1)$$

$$\hat{D}^n_{ij} = \Phi(F) \cdot \frac{\hat{\sigma}_{ij}}{\sqrt{J_2}} \quad ; \quad \kappa = \frac{2}{\sqrt{3}} \; \Phi(F) \cdot h_\kappa \tag{2}$$

$$F = \frac{\sqrt{J_2}}{\kappa} \quad ; \quad \Phi(F) = B\,(F)^n \quad ; \quad h_\kappa = \frac{H_1}{(\kappa)^m} \tag{3}$$

where B, n, H_1, and m are inelastic material parameters (at the considered temperature).

In (1), $\overset{\circ}{\hat{\sigma}}_{ij}$ is an objective stress derivative. In the present case, the problem of choosing this derivative is avoided since an isotropic hardening model is used. Hence, the classical JAUMANN rate will be adopted ($\overset{\circ}{\hat{\sigma}}_{ij} \equiv \overset{\nabla}{\hat{\sigma}}_{ij}$) although many other possibilities exist.

Then, $\frac{D\hat{\sigma}_{ij}}{Dt} = \overset{\nabla}{\hat{\sigma}}_{ij} + W_{ik}\,\hat{\sigma}_{kj} + W_{jk}\,\hat{\sigma}_{ki}$, where $D\hat{\sigma}_{ij}/Dt$ is the material derivative of $\hat{\sigma}_{ij}$. (4)

3. INTEGRATION OF THE CONSTITUTIVE LAW

3.1. Configurations of the solid

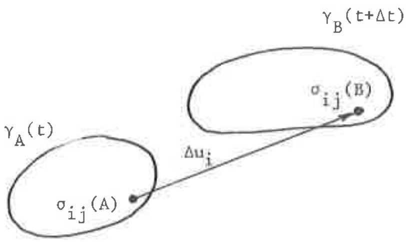

Figure 1.

Let γ_A be a known configuration of the solid at time t. The CAUCHY stresses in γ_A are denoted $\sigma_{ij}(A)$. Let γ_B be a configuration of the solid at time t+Δt, defined the coordinates

$$x_i(B) = x_i(A) + \Delta u_i \tag{5}$$

We will examine hereafter the problem of computing the CAUCHY stresses $\sigma_{ij}(B)$ in γ_B. We will notice, first, that the solution is not unique since $\sigma_{ij}(B)$ depend on the strain history between γ_A and γ_B. For example, if it is assumed that the velocity v_i remains constant between γ_A and γ_B ($v_i = \Delta u_i/\Delta t$), then the velocity gradient $L_{ij} = \partial v_i/\partial x_j$ is not constant since x_i changes. However, this problem will not be studied here and we will assume that $L_{ij} = \partial v_i/\partial x_i(A) = \frac{1}{\Delta t}\,(\Delta u_i)/\partial x_i(A)$ remains constant during the step from γ_A to γ_B.

Hereafter, we recall three possible integration schemes to compute $\sigma_{ij}(B)$. They will be applied to the plane strain state for the sake of simplicity.

3.2. Explicit integration

$$\Delta\sigma_{ij} = \sigma_{ij}(B) - \sigma_{ij}(A) = \frac{D\sigma_{ij}(A)}{Dt} \cdot \Delta t \tag{6}$$

In plane strain state, this gives :

$$\begin{aligned}
\Delta\hat{\sigma}_{11} &= [\,2G\,(\hat{D}_{11} - \hat{D}^n_{11}(A)) + 2\,W_{12} \cdot \hat{\sigma}_{12}(A)\,] \cdot \Delta t \\
\Delta\hat{\sigma}_{22} &= [\,2G\,(\hat{D}_{22} - \hat{D}^n_{22}(A)) - 2\,W_{12} \cdot \hat{\sigma}_{12}(A)\,] \cdot \Delta t \\
\Delta\hat{\sigma}_{12} &= [\,2G\,(D_{12} - D^n_{12}(A)) + W_{12}\,(\hat{\sigma}_{22}(A) - \hat{\sigma}_{11}(A))\,] \cdot \Delta t \\
\Delta\sigma_m &= \chi \cdot D_m \cdot \Delta t \\
\Delta\kappa &= \dot{\kappa}\,(A) \cdot \Delta t
\end{aligned} \tag{7}$$

3.3. Implicit integration

$$\Delta\sigma_{ij} = \sigma_{ij}(B) - \sigma_{ij}(A) = [\,(1-\beta)\,\frac{D\sigma_{ij}(A)}{Dt} + \beta\,\frac{D\sigma_{ij}(B)}{Dt}\,] \cdot \Delta t \tag{8}$$

This equation must be solved iteratively since $D\sigma_{ij}(B)/Dt$ is unknown a priori. In plane strain state, this gives :

$$\Delta\hat{\sigma}_{11} = 2G\,\hat{D}_{11}.\Delta t - 2G\,[\,(1-\beta)\,\hat{D}^n_{11}(A) + \beta\,\hat{D}^n_{11}(B)\,] \cdot \Delta t + 2W_{12}\,[\,(1-\beta)\,\hat{\sigma}_{12}(A) + \beta\,\hat{\sigma}_{12}(B)\,].\Delta t$$

$$\Delta\hat{\sigma}_{22} = 2G\,\hat{D}_{22}.\Delta t - 2G\,[\,(1-\beta)\,\hat{D}^n_{22}(A) + \beta\,\hat{D}^n_{22}(B)\,] \cdot \Delta t - 2W_{12}\,[\,(1-\beta)\,\hat{\sigma}_{12}(A) + \beta\,\hat{\sigma}_{12}(B)\,].\Delta t$$

$$\Delta\hat{\sigma}_{12} = 2G\,\hat{D}_{12}.\Delta t - 2G\,[\,(1-\beta)\,\hat{D}^n_{12}(A) + \beta\,\hat{D}^n_{12}(B)\,]\,.\,\Delta t$$
$$+ W_{12}\,[\,(1-\beta)(\hat{\sigma}_{22}(A) - \hat{\sigma}_{11}(A)) + \beta\,(\hat{\sigma}_{22}(B) - \hat{\sigma}_{11}(B))\,]\,.\,\Delta t \qquad (9)$$
$$\Delta\sigma_m = \chi\,.\,D_m\,\Delta t$$
$$\Delta\kappa = [\,(1-\beta)\,\dot{\kappa}\,(A) + \beta\,\dot{\kappa}\,(B)\,]\,.\,\Delta t$$

This is the complete implicit scheme. In [4] it is simplified by explicit integration of the rotation terms in JAUMANN derivative (the coefficients of W_{ij} above are computed in γ_A).

3.4. Integration with sub-intervals

This method is developed in [5]. Only the basic principle and the results are recalled here. The time interval Δt between γ_A and γ_B is divided into N equal sub-intervals $dt = \Delta t/N$. If $\sigma_{ij}(I)$, $\kappa(I)$ are known at the end of sub-interval number I, then, at the end of sub-interval I+1, one has in plane strain state :

$$s_{11} = [\,1 + c_2(I)\,]\,\hat{\sigma}_{11}(I) + c_1(I)\,.\,\hat{D}_{11}$$
$$s_{22} = [\,1 + c_2(I)\,]\,\hat{\sigma}_{22}(I) + c_1(I)\,.\,\hat{D}_{22} \qquad (10)$$
$$s_{12} = [\,1 + c_2(I)\,]\,\hat{\sigma}_{12}(I) + c_1(I)\,.\,\hat{D}_{12}$$

$$\hat{\sigma}_{11}(I+1) = s_{11} + 2\,W_{12}\,s_{12}\,dt$$
$$\hat{\sigma}_{22}(I+1) = s_{22} - 2\,W_{12}\,s_{12}\,dt \qquad (11)$$
$$\hat{\sigma}_{12}(I+1) = s_{12} + W_{12}\,(s_2 - s_1)\,dt$$

$$\kappa(I+1) = \kappa(I) + c_5(I) + c_6(I)\;[\,a_{11}(I).\hat{D}_{11} + a_{22}(I).\hat{D}_{22} + a_{12}(I).\hat{D}_{12}\,] \qquad (12)$$

where $c_1(I)$, $c_2(I)$, $c_5(I)$, $c_6(I)$,

$$a_{11}(I) = 2\,\hat{\sigma}_{11}(I) + \hat{\sigma}_{22}(I)$$
$$a_{22}(I) = 2\,\hat{\sigma}_{22}(I) + \hat{\sigma}_{11}(I) \qquad (13)$$
$$a_{12}(I) = 2\,\hat{\sigma}_{12}(I)$$

are functions of $\sigma_{ij}(I)$, $\kappa(I)$ and dt.

4. TANGENT STIFFNESS MATRIX

When the stresses in γ_B are known, the corresponding out-of-balance forces can be computed from the virtual work principle. If the body is deformed by imposed displacements only, one has :

$$\int_V \sigma_{ij}\;\delta\varepsilon_{ij}\;dv = \langle F\rangle\;\{\delta u\} \qquad (14)$$

where $\{\delta u\}$ are virtual nodal displacements of the discretized body and $\langle F\rangle$ are the corresponding out-of-balance forces.

All quantities in (14) are evaluated in γ_B. Generally γ_B is not at equilibrium and one has to find a displacement increment du_i that will be added to Δu_i in order to reach a new configuration γ_B^* closer to equilibrium (figure 2).

Coordinates in γ_B^* are :

$$x_i^*\,(B) = x_i(A) + \Delta u_i^* = x_i(B) + du_i \qquad (15)$$

Obviously, the stresses in γ_B^* are computed by integrating the constitutive equation, starting from γ_A, as indicated in paragraph 3. The displacement increment du_i creates an increment of the out-of-balance forces $\{dF\}$ related to $\{du\}$ by :

$$\{dF\} = [\,K_T\,]\,\{du\} \qquad (16)$$

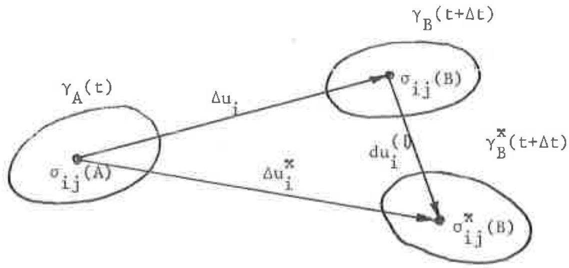

Figure 2.

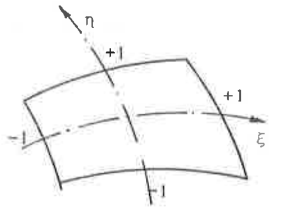

Figure 3.

This means that the tangent stiffness matrix is the derivative of {F} with respect to {u} . Its expression can be obtained by differentiating (14) :

$$d\left[\int_v \sigma_{ij}\,\delta\varepsilon_{ij}\,dv\right] = \langle\delta u\rangle\,\{dF\} = \langle\delta u\rangle\,[K_T]\,\{du\} \tag{17}$$

Care must be taken when evaluating the first member of (17) because the integral is taken over γ_B, a changing configuration. For a 2-dimensional isoparametric element (figure 3), this term writes :

$$d\left[\int_v \sigma_{ij}\,\delta\varepsilon_{ij}\,dv\right] = \iint_{-1}^{+1} d\,(\sigma_{ij}\,\delta\varepsilon_{ij}\,|J|\,)\,d\xi d\eta = \int_v d\sigma_{ij}\,\delta\varepsilon_{ij}\,dv + \iint_{-1}^{+1} \sigma_{ij}\,d(\delta\varepsilon_{ij}\,|J|)\,d\xi d\eta \tag{18}$$

where $|J|$ is the jacobian of the isoparametric transformation. In plane strain state, it can be shown that [5]

$$\iint_{-1}^{+1} \sigma_{ij}\,d\,(\delta\varepsilon_{ij}\,|J|\,)\,d\xi\,d\eta = (\Delta t)^2 \int_v \langle\delta L\rangle\,[\sigma_o]\,\{dL\}\,dv \tag{19}$$

$$\text{with } \Delta t\ \langle\delta L\rangle = \left\langle \frac{\partial\delta u_1}{\partial x_1}\ ;\ \frac{\partial\delta u_1}{\partial x_2}\ ;\ \frac{\partial\delta u_2}{\partial x_1}\ ;\ \frac{\partial\delta u_2}{\partial x_2} \right\rangle \tag{20}$$

$$[\sigma_o] = \begin{bmatrix} 0 & 0 & -\sigma_{12} & \sigma_{11} \\ 0 & 0 & -\sigma_{22} & \sigma_{12} \\ \sigma_{12} & -\sigma_{11} & 0 & 0 \\ \sigma_{22} & -\sigma_{12} & 0 & 0 \end{bmatrix} \tag{21}$$

Remember that all these terms are computed in γ_B ($\sigma_{ij} = \sigma_{ij}(B)$; $x_i = x_i(B)$; $v = v(B)$ ...). The remaining term of (18) writes, in plane strain state :

$$\int_v d\sigma_{ij}\,\delta\varepsilon_{ij}\,dv = \Delta t \int_v \langle\delta L\rangle\,\{d\sigma\}\,dv = (\Delta t)^2 \int_v \langle\delta L\rangle\,[C_T]\,\{dL\}\,dv \tag{22}$$

with $\langle d\sigma\rangle = \langle d\sigma_{11}\ ;\ d\sigma_{12}\ ;\ d\sigma_{21}\ ;\ d\sigma_{22}\rangle$

$$\text{and}\quad \{d\sigma\} = [C_T]\,\{dL\}\ .\ \Delta t \tag{23}$$

Matrix $[C_T]$ represents the influence of a perturbation of the velocity gradients between γ_A and γ_B on the stresses in γ_B. Therefore $[C_T]$ depends on the integration scheme used to compute $\sigma_{ij}(B)$! Matrix $[C_T]$ will be called "incremental compliance matrix". Finally, from (17), (19) and (22),

$$\langle\delta u\rangle\,[K_T]\,\{du\} = (\Delta t)^2 \int_v \langle\delta L\rangle\,\{[C_T] + [\sigma_o]\}\,\{dL\}\,dv \tag{24}$$

$$\text{Let}\quad \Delta t\,\{dL\} = [B]\,\{du\}\ ;\quad [C] = [C_T] + [\sigma_o] \tag{25}$$

$$\text{Then}\quad [K_T] = \int_v [B]^T\,[C]\,[B]\,dv \tag{26}$$

where all the terms are calculated in γ_B. It can be shown that (24) is equivalent to the linearized incremental equilibrium equation derived by NAGTEGAAL and VELDPAUS in [6] .

5. INCREMENTAL COMPLIANCE MATRIX

5.1. Explicit integration

If the stresses in γ_B are computed by (6), (7), it is easily seen that :

$$[C_T] = [C_G] + [C_X] + [C_\sigma(A)] \tag{27}$$

with

$$[C_G] = \begin{bmatrix} \frac{4G}{3} & 0 & 0 & -\frac{2G}{3} \\ 0 & G & G & 0 \\ 0 & G & G & 0 \\ -\frac{2G}{3} & 0 & 0 & \frac{4G}{3} \end{bmatrix} \qquad [C_X] = \begin{bmatrix} \frac{X}{3} & 0 & 0 & \frac{X}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{X}{3} & 0 & 0 & \frac{X}{3} \end{bmatrix}$$

$$[C_\sigma(A)] = \begin{bmatrix} 0 & \hat\sigma_{12}(A) & -\hat\sigma_{12}(A) & 0 \\ 0 & \frac{\hat\sigma_{22}(A) - \hat\sigma_{11}(A)}{2} & -\frac{\hat\sigma_{22}(A) - \hat\sigma_{11}(A)}{2} & 0 \\ 0 & \frac{\hat\sigma_{22}(A) - \hat\sigma_{11}(A)}{2} & -\frac{\hat\sigma_{22}(A) - \hat\sigma_{11}(A)}{2} & 0 \\ 0 & -\hat\sigma_{12}(A) & \hat\sigma_{12}(A) & 0 \end{bmatrix}$$

$[C_G]$ and $[C_X]$ are elastic deviatoric and hydrostatic contributions respectively. $[C_\sigma(A)]$ is the contribution of the rotation terms associated with JAUMANN derivative. Hence $[C_T]$ does not contain any inelastic contribution because, in (6), the inelastic strain rates $\hat{D}^n_{ij}(A)$ are computed in γ_A. It must also be emphasized that $[C_\sigma(A)]$ contains the stresses in γ_A while $[\sigma_o]$ contains the stresses in γ_B. Neither matrix is symmetric and, consequently, $[K_T]$ is not. However, the unsymmetrical contributions are of the order of magnitude of stresses while the symmetrical ones are of the order of the elastic modulus. If the former are neglected, $[K_T]$ becomes symmetric. On the other hand, if TRUESDELL derivative (denoted $\overset{T}{\sigma}_{ij}$ hereafter) were used instead of JAUMANN in formulating the constitutive equation, and if the approximation $\sigma_{ij}(B) \simeq \sigma_{ij}(A) = \sigma_{ij}$ is accepted, one gets :

$$[C_\sigma(A)] = \begin{bmatrix} \sigma_{11}(A) & 0 & 2\sigma_{12}(A) & -\sigma_{11}(A) \\ 0 & \sigma_{11}(A) & \sigma_{22}(A) & 0 \\ 0 & \sigma_{11}(A) & \sigma_{22}(A) & 0 \\ -\sigma_{22}(A) & 2\sigma_{12}(A) & 0 & \sigma_{22}(A) \end{bmatrix} ; \quad [C^*_\sigma] = [\sigma_o] + [C_\sigma] = \begin{bmatrix} \sigma_{11} & 0 & \sigma_{12} & 0 \\ 0 & {}_{11} & 0 & \sigma_{12} \\ \sigma_{12} & 0 & \sigma_{22} & 0 \\ 0 & \sigma_{12} & 0 & \sigma_{22} \end{bmatrix}$$

In that case, $[K_T]$ becomes also symmetric :

$$[K_T] = \int_V [B]^T \; [C_G] + [C_X] + [C^*_\sigma] \; [B] \, dv$$

The latter form coincides with the classical one for small strains [7] .

5.2. Implicit integration

One gets from (8, (9)

$$\{d\sigma\} = [\, [C_G] + [C_X] + [C_\sigma(\beta)] \,] \; \{dL\} \; \Delta t - 2\, G\, \beta\, \Delta t \; \{d\hat{D}^n\} + \beta\, W_{12} \; \Delta t \, [M] \; \{d\hat\sigma\} \tag{28}$$

$$<d\hat{D}^n> = <d\hat{D}^n_{11} \;\; d\hat{D}^n_{12} \;\; d\hat{D}^n_{21} \;\; d\hat{D}^n_{22}>$$

$$<d\hat\sigma> = <d\hat\sigma_{11} \;\; d\hat\sigma_{12} \;\; d\hat\sigma_{21} \;\; d\hat\sigma_{22}> \qquad [M] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

$$[C_\sigma(\beta)] = (1-\beta) \; [C_\sigma(A)] + \beta \, [C_\sigma(B)]$$

It can be shown [8] that $\{d\hat{D}^n\} = [R(B)]\{d\hat\sigma\}$ where $[R(B)]$ is computed in γ_B.
Since $\{d\hat\sigma\} = \{d\sigma\} - [C_X] \; \{dL\} \, \Delta t$, one gets $[P]\{d\sigma\} = [Q]\{dL\} \; \Delta t$
with $[P] = [I] + 2\, G\, \beta \;\; \Delta t \, [R(B)] \; - W_{12}\, \beta . \, \Delta t \; [M]$; $[Q] = [C_G] + [C_\sigma(A)] + [P] \; [C_X]$

Hence $[C_T] = [P]^{-1} [Q] = [C_X] + [P]^{-1} \{ [C_G] + [C_\sigma(\beta)] \}$ (29)

This result shows that the tangent stiffness matrix depends on the integration scheme used to compute the stresses and, in particular, of the integration parameter β. If the simplified implicit integration of [4] is used, (29) is still valid provided the last term of [P] is dropped and $[C_\sigma(\beta)]$ is replaced by $[C_\sigma(A)]$.

5.3. Integration with sub-intervals

In this case, the simplest way to compute $[C_T]$ is to use a numerical perturbation technique : each component of {L} is successively given in a small increment and the corresponding increment of $\{\sigma\}$ are computed numerically. This allows to construct $[C_T]$ column by column. However, it is also possible to compute $[C_T]$ analytically from a recurrence formula developed in [8]. For shortness, this method is not exposed here. Obviously, $[C_T]$ depends on the number of sub-intervals (N).

6. CONCLUSION

For large strain EVP analysis, the tangent stiffness matrix is generally unsymmetric and depends on the integration scheme used to compute stresses as well as of the choice of an objective stress rate. For practical application, the integration scheme with sub-intervals is preferred because it is more accurate for large time steps. With this method strain increments of 5 % can be reached in one step with a good convergence and a good accuracy on the stresses.

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