Generalized Plane Strains: An Interesting and Efficient Extension of the Standard Plane Stress and Plane Strain Analysis

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Abstract

Very often when a problem is studied through a plane analysis, the solution is either too stiff in the case of plane strain analysis, or too soft in the case of plane stress. This situation is often encountered in the field of fracture mechanics. We give here a method to overcome this difficulty. This method is also very useful to the detailed analysis of the current section of a beam or of a straight pipe subjected to complex loading.

We shall call $S$ the section studied and shall suppose that this section is in the plane $(x,y)$. We shall call $\sigma_z$ the stress normal to the plane, and $\varepsilon_z$ the corresponding deformation. Instead of solving a problem in which either $\varepsilon_z$ and $\sigma_z$ is imposed to zero we shall solve the problem in which one of the quantities $(M_y, Q_y)$ $(M_x, Q_x)$ and $(M_y, Q_y)$ is imposed to a prescribed value. The six quantities defined above are associated with $\varepsilon_x$ and $\sigma_x$ by the equations:

1. $\varepsilon_x = Q_x + x \theta_y + y \theta_x$

2. $M_y = \int_S \sigma_x \partial_y \partial_z M_x = \int_S \sigma_x \partial_y \partial_z梅\quad y \sigma_z \partial_z \partial_y$

The paper gives a very simple iterative method to solve the problem.

Two examples are presented in details. One gives a case of validation by comparison with an analytical solution. The other is a case of industrial interest. Both are developed in elasto-plastic configurations.

1. POSITION OF THE PROBLEM

1.1 - Aim of the study

We want to study the current cross section $S$ of a long straight beam. This cross section is supposed to be perpendicular to the $z$ axis. It obeys to ST VENANT's principle.

The beam (represented on Figure 1) is loaded at its extremities by either loads or imposed displacements and rotations.

The cross section has, in addition to the loads associated with the beam, the general efforts, its own loadings (Temperature field, weight, pressure...).
1.2 - Hypothesis and notations

The general efforts applied to the beam will be called:
- $N_Z$ for the axial load
- $M_X$ for the moment parallel to $x$ axis
- $M_Y$ for the moment parallel to $y$ axis

The generalized displacements will be called:
- $Q_Z$ for the axial displacement per unit of length
- $\theta_X$ for the rotation parallel to $x$ axis
- $\theta_Y$ for the rotation parallel to $y$ axis

The consequence of ST. VENANT's principle is that the cross section has a deformation along $z$ axis, $\epsilon_Z$, which is given for each point $(x, y)$ by the following expression:

\[ \epsilon_Z = Q_Z + \theta_X (y - y_0) + \theta_Y (x - x_0) \]

$(x_0, y_0)$ are the coordinates of the centre of inertia of the section.

Eq (3) means that the cross section remains plane when deformed. This is the usual hypothesis for the deformation of the current cross section of a beam.

1.3 - Relation between the stresses in the cross section and the loads on the beam

We have the following relations between the stress $\sigma_Z$ normal to the plane $(x, y)$ and the general efforts:

\[
\begin{align*}
N_Z &= \int_S \sigma_Z \, dS \\
M_X &= \int_S \sigma_Z (y - y_0) \, dS \\
M_Y &= \int_S \sigma_Z (x - x_0) \, dS
\end{align*}
\]

in this expression $dS$ is the area element.

1.4 - Partition of deformation and stresses

We observe that the deformations at any point of the cross section is the sum of a state of plane strain deformation and of the deformations associated with eq (1).

From that fact we deduce that:

- The stress $\sigma_Z$ in the cross section is the sum of two stresses $\sigma_Z^1$ which is the stress due to plane strain condition and $\sigma_Z^2$ which the stress associated with the deformation $\epsilon_Z$ given by eq (1). Moreover we have:

\[ \sigma_Z = \sigma_Z^1 + \sigma_Z^2 \]

\[ \sigma_Z^1 = \frac{E(1-v)}{(1+v)(1-2v)} \epsilon_Z = \alpha \epsilon_Z \]

where $E$ is YOUNG's Modulus

$v$ is POISSON's ratio

$\alpha = \frac{E(1-v)}{(1+v)(1-2v)}$

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We remark that the non zero $\gamma_2$ deformation induces also modifications in the others components, $(\sigma_x, \sigma_y, \sigma_{xy})$ of the stress tensor. 

Finally we can write:

(4) \[ \bar{\gamma}_2 = \gamma_0 + \bar{\gamma}_1 \]

where $\gamma_2$ is the stress tensor in plane strain and $\gamma_1$ is given by eq (5)

(5) \[ \bar{\gamma}_1 = D\text{\underline{\Delta}\xi}_1 \]

where $D$ is the HOOKE'S matrix (plane strain)

1.5 - Relation between general effort and global displacements

From eq (2) we deduce:

(6) \[
\begin{bmatrix}
N_z \\
M_x \\
M_y \\
Q_z \\
\end{bmatrix} =
\begin{bmatrix}
N_z^0 \\
M_x^0 \\
M_y^0 \\
Q_z^0 \\
\end{bmatrix} + [A]
\begin{bmatrix}
\xi_x \\
\xi_y \\
\end{bmatrix}
\]

$N_z^0, M_x^0, M_y^0$ are obtained by replacing $\epsilon_z$ by $\epsilon_z^0$ in eq (2).

$A$ is a $(3 \times 3)$ symmetric matrix whose coefficients are the inertia expressed in eq (7).

(7) \[
A(1,1) = \int_S a \, dS \\
A(1,2) = A(2,1) = \int_S a \, (y - y_o) \, dS \\
A(1,3) = A(3,1) = \int_S a \, (x - x_o) \, dS \\
A(2,2) = \int_S a \, (y - y_o)^2 \, dS \\
A(2,3) = A(3,2) = \int_S a \, (x - x_o)(y - y_o) \, dS \\
A(3,3) = \int_S a \, (x - x_o)^2 \, dS
\]

2. Method for the solution of the problem

2.1 - Linear Elastic Case

It has been supposed that we know:

- either $N_z$ or $Q_z$,
- either $M_x$ or $\theta_x$,
- either $M_y$ or $\theta_y$.

We shall call the given value $D_i$ ($i = 1, 3$) and the unknown one $X_i$ ($i = 1, 3$).

From eq (6) and the definition given just above we deduce that the unknown vector $X$ is given by the solution of the linear system.

(8) \[ BX = \text{\underline{\Delta}\xi}_0 + C D \]
where B is non symmetric matrix constructed directly from eq (6). It is rather evident that B matrix is invertible, so that the linear system (8) is solvable. If the initial stress \( \sigma_0 \) was known, the problem would be solved in one go. But the stress state \( \sigma_0 \) depends on the deformations \( \varepsilon_X, \varepsilon_Y \) which depends on the imposed generalized efforts or deformations.

The problem is then solved iteratively with the following method.

a) initialisation of \( Q_0, \theta_X, \theta_Y \) with the given values (or 0 if not given)
b) compute B and C matrix
c) compute \( B^{-1} \) matrix (inversion)
d) compute the stress state \( \sigma_1 \) associated with the given deformation \( Q_0, \theta_X \) or \( \theta_Y \)
e) compute the external forces associated with \( \sigma_1 \), \( \int B \sigma_0 \, d_0 = F_1 \f) compute the stresses \( \sigma_0 \) as if they were due to plane strain condition
g) compute the stress field \( \sigma = \sigma_0 + \sigma_1 \) (eq 1, 3, 4)
h) compute \( \sigma_0 \) and solve for a new \( X \)
i) test for convergence of solution \( X \) if convergence go to k)
j) update the values of \( Q_0, \theta_X, \theta_Y \) and go to c)
k) stop

2.1 - Elastoplastic case

In the case of elastoplasticity we give a variation with "pseudo time" of the generalized loads as well as loads in the section.

The computation is incremental and at each load step (increment of pseudo time) we have to face the problem of solving incrementally eq (8)

\[
(9) \quad \Delta X = \Delta S_0 + C \Delta D
\]

The solution of eq (9) is obtained by the same procedure as given before (i.e., iteratively) the iterations associated with plasticity being mixed with the generalized plane strain iterations.

At each load step the algorithm is the following:

a) compute B C once for all
b) invert B once for all
c) Initialise \( \Delta Q_0, \Delta \theta_Y, \Delta \theta_X \) with the given increment (or 0 if they are unknown)
d) compute \( \Delta \sigma_1 \) associated with \( \Delta Q_0, \Delta \theta_Y, \Delta \theta_X \)
e) compute \( \Delta F_1 = \int B \Delta \sigma_1 \, d_0 \f) solve for displacement field \( \Delta U \) associated with the increment of load:
\[
\Delta F = \Delta F_0 + \Delta F_1 + \Delta F_{NL} + \Delta F_{NL}(\text{associated with plasticity})
\]
\[
\Delta F_0 \quad (\text{associated with in plane loads})
g) compute \( \Delta \sigma_0 \) and if it were in plane strain
h) compute \( \Delta \sigma = \Delta \sigma_0 + \Delta \sigma_1 \)
i) tests for plasticity (gives \( \Delta \sigma^p \) increment of plastic strain)
j) compute \( \Delta F_{NL} = \int B^T \sigma D \Delta \sigma_0 \, d_0 \)
k) from admissible stress field \( \sigma' (\sigma_0 + \Delta \sigma - D \Delta \sigma_0) \) compute \( \Delta S_0 \) and solve for \( \Delta X \)
1) test for convergence of $\Delta X$ and of plastic strain; if convergence go to n)
m) update $\Delta Q_1$, $\Delta \theta$, $\Delta \phi$, and go to d)
n) end of load step

3. EXAMPLES

3.1 - The first example is the computation of the moment-curvature curve of a rectangular beam under flexion with a perfectly plastic material. In this case an analytical solution is available. The geometry of the cross section is defined on Figure 2. The mesh consist of 5 Q8 plane elements.
The comparison between computed and analytical solution is quite good as can be observed from figure 3.

3.2 - The second example is the computation of a wall of concrete under a shear force. Figure 4 shows the discretisation of the structure. Figure 5 shows the load displacement curve. The load applied at point B and the displacement is the vertical displacement of point B. We can observe on this curve the nonlinear behaviour due to the cracking in the concrete. (Modelled by a DRUCKER-PRAGER criterium.)
Figure 10 shows the deformed state at the maximum load. We observe in the right bottom of the structure a swelling showing the cracking of this part of the wall. We had imposed zero general efforts ($N_{z} = M_{x} = M_{y} = 0$).

4. CONCLUSION

The method of generalized plane strain is a useful tool for the computation of complex loadings on beam type structures and also gives an intermediate state between plane strain and plane stress solution.
Fig. 4 - MODEL OF THE STRUCTURE

Fig. 5 - TOTAL LOAD VS. DEFLECTION IN K (MM)

Fig. 6 - DEFORMED STRUCTURE

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