A New Approach of Shell Displacements in a Beam—Type Pipe Element

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Abstract

In this paper a new beam-type pipe Element is presented. For the shell displacements of the pipe, splines interpolation is used in the circumferential direction, while strip theory interpolation is used along pipe length

The element has been tested and as a result it seems to be more effective than beam-type pipe elements already known in the literature available.

1. Introduction

Basically, three methods [1] of modelling piping systems are known:

(a) First, methods based on a beam element analysis which utilizes v.Karman flexibility, and stress-intensification factors.

(b) Second, methods based on a shell discretization of the pipe,

(c) and third, combined methods based on a superposition of beam and shell displacements according to v.Karman's hypothesis.

The scope of the present paper is the development of a new pipe (c)-methods-element, which, instead of the v.Karman generalized displacements [2], uses, strip theory interpolation, and discrete degrees of freedom on the middle circumference of the shell. This new type element seems to offer a better approximation of the pipe displacements and flexibility.

2. Curved Beam Element Formulation

For the geometry of Fig.1a,1b using curvilinear coordinates (r,θ,φ) the following strain-displacement relations (linear elasticity) are obtained

$$\varepsilon_{rr} = \frac{\partial w}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left[\frac{\partial v}{\partial \theta} + w\right]$$

$$\varepsilon_{\varphi\varphi} = \frac{1}{Rn}\left[\frac{\partial u}{\partial \varphi} + v\sin\theta - w\cos\theta\right]$$

$$\gamma_{\varphi\theta} = \frac{1}{Rn}\left[\frac{\partial v}{\partial \varphi} - u\sin\theta\right] + \frac{1}{r}\cdot\frac{\partial u}{\partial \theta}$$

$$\gamma_{r\theta} = \frac{1}{r}\left[\frac{\partial w}{\partial \theta} - v\right] + \frac{\partial v}{\partial r}$$

$$\gamma_{r\varphi} = \frac{1}{Rn}\left(\frac{\partial w}{\partial \varphi} + u\cos\theta\right) + \frac{\partial u}{\partial r} \qquad (1)$$

where
$$n = 1 - \frac{r}{R}\cos\theta$$

Eqs.(1) are confirmed by BICKFORD and STROM [3]. It is simple to express the components u_1,u_2,u_3 of u,v,w in coordinates system (x,y,z). Then, u_1,u_2,u_3 are - in terms of centerline displacements and rotations:

$$u_1 = u_x-\theta_z r\cos\theta-\theta_y r\sin\theta$$
$$u_2 = u_y+\theta_x r\sin\theta$$
$$u_3 = u_z+\theta_x r\cos\theta \qquad (2)$$

Finally, using the exact shape functions $S(\varphi)$ for a 2-node element, $\underline{U}^T=[u_x,u_y,u_z,\theta_x,\theta_y,\theta_z]$ is expressed in terms of $\underline{U}^{kT}$ (k=1,2, node number). These functions (which allow the use of large element angle φ_o, e.g. $\varphi_o=90^0$) are computed either (i) numerically, using straight beam elements to approximate the curved beam, or (ii) analytically, using the differential equations. The unknowns $\underline{U}_G^k$ are the components of $\underline{U}^k$ in the global system (x_G,y_G,z_G) Strain $\underline{\varepsilon}_b$ is

$$\underline{\varepsilon}_b = \sum_{k=1}^{2} \underline{B}_b^k \underline{U}_G^k \qquad (3)$$

3. Shell Formulation for Pipe Element

3.1. Strain-Displacement Relations

Setting $r=r_o$ in eqs.(1) and using for u,v,w the expressions for the shells as they are found in KRAUS [4] (ch.2, §4), the relations for a thin curved pipe are obtained (R=const., Fig.1a,1c). If u_s,v_s,w_s are the displacements of middle surface, then it is valid

$$\varepsilon_{\theta\theta}(\zeta) = \frac{1}{r_o}\left(\frac{\partial v_s}{\partial \theta} + w_s\right) + \frac{\zeta}{r_o^2}\left(\frac{\partial v_s}{\partial \theta} - \frac{\partial^2 w_s}{\partial \theta^2}\right)$$

$$\varepsilon_{\varphi\varphi}(\zeta) = \frac{1}{Rn_o}\left(\frac{\partial u_s}{\partial \varphi} + v_s\sin\theta - w_s\cos\theta\right) + $$
$$+ \zeta\left[-\frac{1}{R^2 n_o^2}\left(\cos\theta\frac{\partial u_s}{\partial \varphi} + \frac{\partial^2 w_s}{\partial \varphi^2}\right) + \frac{1}{Rn_o r_o}\left(v_s - \frac{\partial w_s}{\partial \theta}\right)\sin\theta\right]$$

$$\gamma_{\varphi\theta}(\zeta) = \frac{1}{Rn_o}\left(\frac{\partial v_s}{\partial \varphi} - u_s\sin\theta\right) + \frac{1}{r_o}\frac{\partial u_s}{\partial \theta} +$$
$$+ \zeta\left[\frac{1}{Rn_o r_o}\left(\frac{\partial v_s}{\partial \theta} - 2\frac{\partial^2 w_s}{\partial \theta \partial \varphi} - \cos\theta\frac{\partial u_s}{\partial \theta} + \frac{u_s}{n_o}\sin\theta\right) + \right.$$
$$\left.\frac{1}{R^2 n_o^2}\sin\theta\left(2\frac{\partial w_s}{\partial \varphi} + u_s\cos\theta\right)\right] \qquad (4)$$

where $n_o=1-r_o\cos\theta/R$.

The above relations are in full agreement with MILLARD and HOFFMANN [5] and KRAUS [3] (ch.6, §1, application for pipe geometry). Following thin shells hypothesis, these lead to a simple expression for $\underline{\varepsilon}_s^T=[\varepsilon_{\theta\theta},\varepsilon_{\varphi\varphi},\gamma_{\varphi\theta}]$ (keeping appropriate terms to secure continuity along $x=R\varphi$)

$$\varepsilon_{\theta\theta} = \frac{1}{r_o}\left(\frac{\partial v_s}{\partial\theta}+w_s\right)+\frac{\zeta}{r_o^2}\left(\frac{\partial v_s}{\partial\theta}-\frac{\partial^2 w_s}{\partial\theta^2}\right)$$

$$\varepsilon_{\varphi\varphi} = \frac{1}{Rn_o}(v_s\sin\theta-w_s\cos\theta)-\frac{\zeta}{R^2n_o^2}\frac{\partial^2 w_s}{\partial\varphi^2}$$

$$\gamma_{\varphi\theta} = \frac{1}{Rn_o}\frac{\partial v_s}{\partial\varphi} \tag{5}$$

while the inextinsibility hypothesis is

$$w_s = -\partial v_s/\partial\theta \tag{6}$$

3.2. Shell Shape Functions

According to von Karman's original idea, v_s is expanded in Fourier series (variable θ). The choice of these types of functions was based on the necessity of periodic function and the remark: $\int_o^{2\pi} w_s\partial\theta=-\int_o^{2\pi}\partial v_s=0$. MILLARD [5], BATHE and ALMEIDA [6] and other used the same approximation for v_s, to construct such a beam-type element. This paper introduces splines interpolation functions around angle θ and discrete degrees of freedom at four points (step=90^o) of middle circumference of the pipe

$$v_s(x,\theta) = \sum_{i=1}^{4} N_i(\xi)v_{si}(x) \quad \text{with } \xi=\theta/2\pi \tag{7}$$

$N_1(\xi)$ is shown in Fig.2. The rest $N_i(\xi)$ are obtained with rotation of $N_1(\xi)$.

Shape functions $N_i(\xi)$ provide among others the possibility of a non-symmetrical deformation pattern of pipe cross section, of a better approximation of internal pressure effects, etc.

Under the hypothesis of in-plane bending only and $\theta_z=\partial u_y/R\partial\varphi$, eqs.(1) and instructions of §2 lead to

$$\varepsilon_{\varphi\varphi} = \frac{\partial u_x}{nR\partial\varphi}-\frac{r\cos\theta}{R^2n}\frac{\partial^2 u_y}{\partial\varphi^2}-\frac{u_y}{Rn} \tag{8}$$

Let it be, $n\simeq1$, $\varphi\to\theta$, $R\to r_o$, $u_y\to -w_s$, $u_x\to v_s$, $y=r\cos\theta\to-\zeta=-(r-r_o)$, eq.(8) gives

$$\varepsilon_{\theta\theta} = \frac{1}{r_o}\frac{\partial v_s}{\partial\theta}-\frac{\zeta}{r_o^2}\frac{\partial^2 w_s}{\partial\theta^2}+\frac{w_s}{r_o} \tag{9}$$

Eq.(9) is the first of eqs.(4) if the term $\zeta\partial v_s/r_o^2\partial\theta$ is neglected. Hence, the exact shape functions for the curved beam can be used to approximate $N_i(\theta)$. This is especially useful in case reinforcements are used along pipe length, which means irregularity on cross section thickness.

The interpolation functions along the pipe length are,

$$v_{si}(x) = (1-3\eta^2+2\eta^3)v_{si}^1+(\eta-2\eta^2+\eta^3)R\varphi_o v_{si}^{1'}+$$
$$+(3\eta^2-2\eta^3)v_{si}^2+(-\eta^2+\eta^3)R\varphi_o v_{si}^{2'} \tag{10}$$

where $\eta=\varphi/\varphi_o$, $v'_{si}=dv_{si}/dx$ and upper subscript (k=1,2) means the number of node. The selection of (10) was based on "Strip theory" hypothesis. A pipe element based on finite strips is f.e. the MARK element 17 [7].

Using eqs.(5), (6), (7) and (10) we find

$$\underline{\varepsilon}_s = \sum_{k=1}^{2} \underline{B}_s^k \underline{U}_s^k \tag{11}$$

where $\underline{U}_s^{kT} = \left[v_{s1}^k, v_{s2}^k, v_{s3}^k, v_{s4}^k, v_{s1}^{k'}, v_{s2}^{k'}, v_{s3}^{k'}, v_{s4}^{k'}\right]$

4. Stress-Strain Matrix and Stiffness Matrix

The following $\underline{\sigma}$-$\underline{\varepsilon}$ relation for plane stress conditions in θ-φ plane is used

$$\begin{bmatrix} \sigma_{\varphi\varphi} \\ \sigma_{\varphi\theta} \\ \sigma_{\varphi r} \\ \sigma_{\theta\theta} \end{bmatrix} = \underline{D}\ \underline{\varepsilon} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & 0 & 0 & \nu \\ 0 & (1-\nu)/2 & 0 & 0 \\ 0 & 0 & (1-\nu)/2 & 0 \\ \nu & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{\varphi\varphi} \\ \gamma_{\varphi\theta} \\ \gamma_{\varphi r} \\ \varepsilon_{\theta\theta} \end{bmatrix} \tag{12}$$

where E is Young's modulus and ν the Poisson ratio of the material.
Using superposition, total strain $\underline{\varepsilon}$ is

$$\varepsilon = \underline{\varepsilon}_b + \underline{\varepsilon}_s = \sum_{k=1}^{2} \left[\underline{B}_b^k \underline{B}_s^k\right] \begin{bmatrix} \underline{U}_G^k \\ \underline{U}_s^k \end{bmatrix} \tag{13}$$

and the stiffness Matrix:

$$\underline{K} = \int_V \underline{B}^T\ \underline{D}\ \underline{B}\ dV \tag{14}$$

We have 14 (=6+8) degrees of freedom per node. Many integrations of eq.(14) have been performed analytically.

5. Example Test Problem

Pipe element ELSPOL was introduced in PIPES-Code developed during recent years at NTU-Athens. For sake of clarity and due to limited space a short example for straight pipe is given. Data for the problem is given in Fig.3 and has been received (with results) from BATHE and ALMEIDA [8]. Displacement $w_s(\theta)$ is shown in Fig.4 while $w_s(x)$ and $dw_s(x)/dx$ in Fig.5a,5b. ELSPOL gave excellent results with only four 2-node elements.

References

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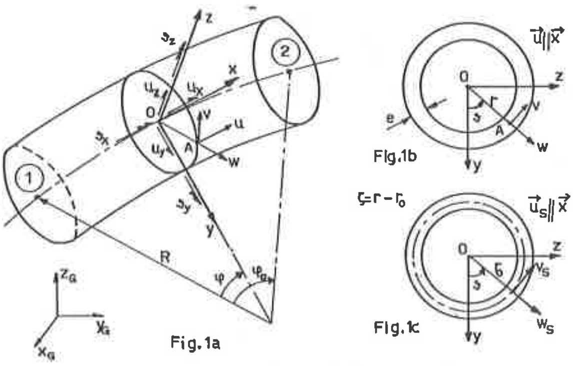
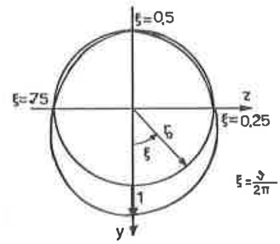
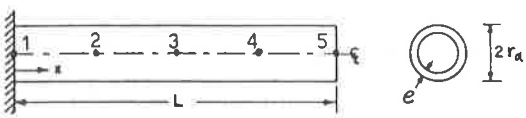


Fig.1 Elbow pipe element, coordinates systems and displacements



$$N_1(\xi)=\begin{cases} -128\xi^4+144\xi^3-44\xi^2+1 & ;0\leqslant\xi\leqslant 0.5 \\ -128(1-\xi)^4+144(1-\xi)^3-44(1-\xi)^2+1 & ;0.5\leqslant\xi\leqslant 1\end{cases}$$

Fig.2 Shell shape functions around middle circumference



SHELL BOUNDARY CONDITIONS

AT $x = 0$: $w_s = 0.,\ \frac{dw_s}{dx} = 0.$

AT $x = L$:

ϑ(DEG)	45	135	225	315
V_S(in.)	0.52	-0.52	0.52	-0.52

ANALYSIS PARAMETERS

$L = 4.8$ in $\qquad E = 2.8\times10^7$ psi

$r_a = r_0 + \frac{e}{2} = 8.0$ in.

$e = 0.37$ in.

Fig.3 Straight cantilever pipe test problem, E=Young's modulus

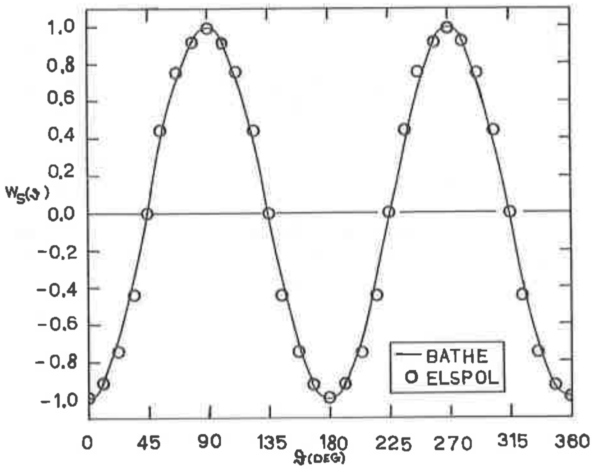
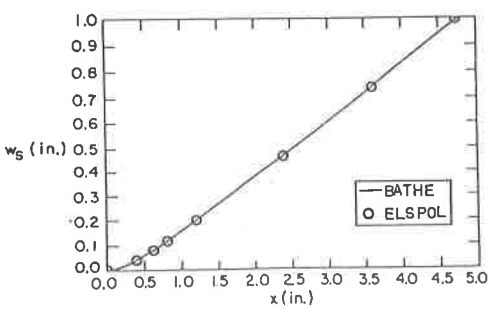


Fig.4 Shell displacement w_s at node 5



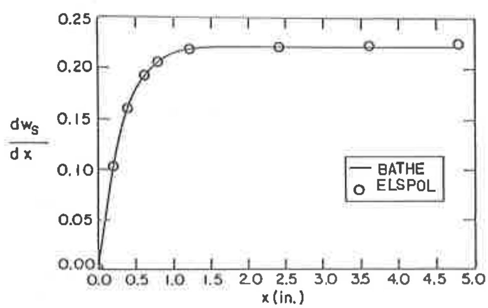


Fig.5a Predicted response of cantilever straight pipe using four 2-node finite elements

Fig.5b (cont.)