A C° Continuous Linear Beam/Bilinear Plate Flexure Element

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ABSTRACT

A C° continuous linear and bilinear flexure elements, based on a first order shear deformable theory, are designed for beams and plates respectively. An antiparameter is introduced in the shear energy part of the total energy expression. Based on numerical experimentation this antiparameter is assigned a bounded number to avoid 'locking' of elements in such formulations.

INTRODUCTION

The early displacement based finite element formulations for flexure problems relying on 'Kirchhoff' hypothesis required slope continuity between adjacent elements i.e. C^1 continuity in shape functions . This is because the energy expression in a Kirchhoff theory contains second derivatives of the transverse displacement component. Compatible, incompatible and complicated higher-order C^1 continuous elements have been derived in the past[1].

In recent years C° continuous elements based on shear deformable theories and which use independent interpolation of slopes and displacements have been developed. This is mainly due to the ease in the development of computer programmes and also the formulation [1-8].

However, the linear and bilinear elements of this family become very stiff when the thickness is reduced i.e. when system conforms to Kirchhoff mode. This phenomenon, called 'locking' arises due to linear/bilinear element discretization. The trouble arises due to the existence of spurious discrete shear energy which turns out to be of the order $(h/t)^2$ of flexural energy, h and t being effective element size and thickness respectively and as $t \rightarrow 0$, $h/t \rightarrow \infty$. This situation poses real problem in the numerical analysis. Remedial measures suggested to-date to overcome this difficulty are,

1. Use of discrete Kirchhoff procedure in which element matrix equation is stabilized by tying together the two independent degrees of freedom w, θ at discrete points such that $\theta_i = \partial w / \partial x_i$. The method turns out to be complicated in implementation.

2. Use of selective/reduced integration procedure in which the shear energy term is under-integrated [2,3]. Many viesw this technique as mere tricks rather than methods although heuristic justification

of these procedures has been provided [4]. The approach introduces unwanted spurious zero energy modes in plates over and above the actual rigid body modes. This poses many problems, e.g. these elements give oscillatory results in the case of corner supported plates.

Belytschko, et al.[5,6] have proposed use of a stabilization matrix with a free parameter to get rid of the spurious modes. Although this method is effective, the efficiency of the original element is lost and the choice of free parameter requires one's judgement. Hughes and Tezduyar [7] have proposed a bilinear plate element free from spurious modes but they use complicated shape functions.

We present here a simple approach. The total energy is split into bending and shear energies in the usual manner. But an antiparameter is introduced in the shear energy expression. Full exact integration is used in the formulation. It is shown that this antiparameter controls locking characteristi cs. An estimate, based on numerical experimentation, is proposed for this free parameter for general use.

BASIC IDEA

What is presented below is an extension of a half-baked idea reported in [8]. A two-noded linear beam element as shown in Fig.1 is considered. Following the usual notations we have,

$$\delta^e = (w_1, \theta_1, w_2, \theta_2)^T \quad \text{.... (1)}$$

$$N_1 = (1 - x/h) \; ; \quad N_2 = x/h \quad \text{.... (2)}$$

$$u = N \delta^e \quad \text{.... (3)}$$

$$\underset{\sim}{\varepsilon} = (\chi, \phi)^T = (d\theta / dx \, , \, dw/dx + \theta)^T \quad \text{.... (4)}$$

We introduce here an antiparameter α such that

$$\phi = \alpha (dw/dx + \theta) \quad \text{.... (5)}$$

The expression for strain energy U is written as,

$$U = 1/2 \int EI \, (d\theta/dx)^2 \, dx + 1/2 \int k \, GA_s \alpha^2 \, (dw/dx + \theta)^2 \, dx \quad \text{.... (6)}$$

For a beam of rectangular cross-section,

$$k = 5/6 \;, \quad A_s = b\,t \;, \quad G = E / 2(1 + \nu) \;, \alpha = p (t / L) \quad \text{.... (7)}$$

The standard procedure is used for the derivation of the element properties and an estimate of the free parameter 'p' is obtained to ensure lower-bound monotonic convergence to true solution by subjecting the formulation to extensive numerical experimentation.

The above idea is readily extended to bilinear plate element of the type shown in Fig.2.

NUMERICAL EVALUATIONS AND CONCLUSIONS

The new elements so developed were tested on a variety of beam and plate problems with various load and boundary conditions. The results of the evaluations are shown in Figs. 3 - 13. The pathalogical case of a corner supported plate was also tried and the successful results are presented in Table - 1 (such a result is not possible with even a selectively reduced integrated element).

We see from the figures drawn that the finite element displacement solution converges monotonically to the exact solution for a particular value of 'p' . We propose a value of 'p' between 6 and 12 for beams and 9 and 15 for plates. The eigenvalue analysis for a single plate element with 'p' in this range produced only three zero eigenvalues corresponding to the three rigid-body modes.

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TABLE I

Central displacement for a corner just supported square plate under UDL
(a / t = 10^4 , E = 10.92×10^{12} , = 0.3)

Mesh	9.0	9.5 p	10.0
1x1	0.00305	0.00277	0.00253
2x2	0.0117	0.0110	0.0104
3x3	0.0176	0.0167	0.0161
4x4	0.0212	0.0205	0.0198
6x6	0.0248	0.0242	0.0237
8x8	0.0264	0.0259	0.0256

Exact: 0.026

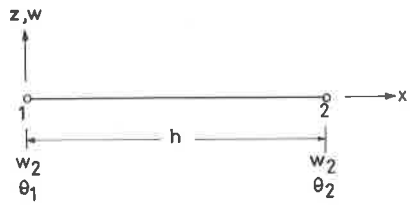

FIG. 1: 2-NODED LINEAR ELEMENT

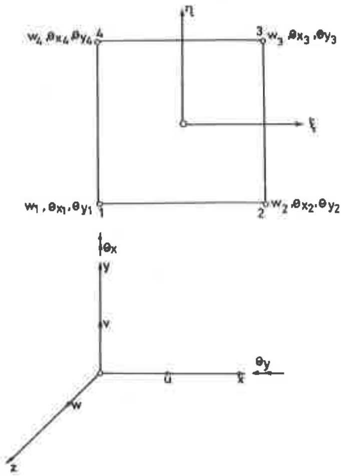

FIG. 2: 4-NODED QUADRILATERAL ELEMENT

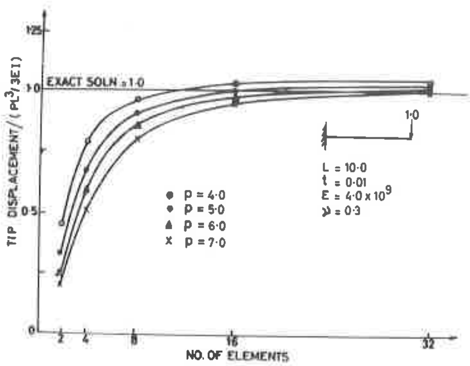

FIG.3: CANTILEVER BEAM UNDER TIP LOAD

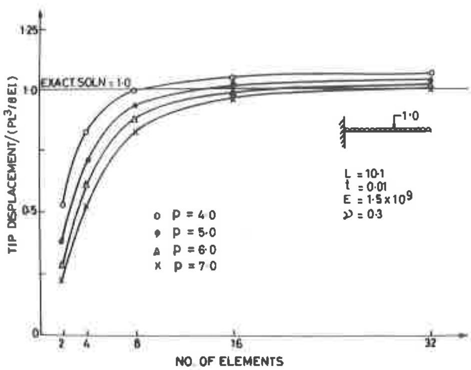

FIG. 4: CANTILEVER BEAM UNDER UDL

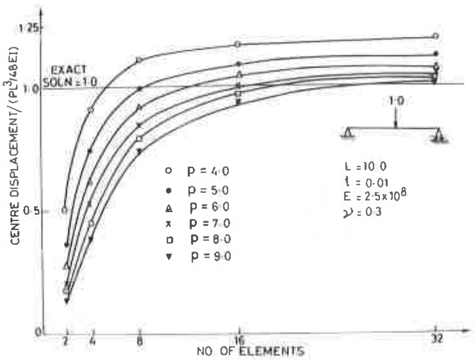

FIG. 5: SIMPLY SUPPORTED BEAM UNDER CENTRAL CONCENTRATED LOAD

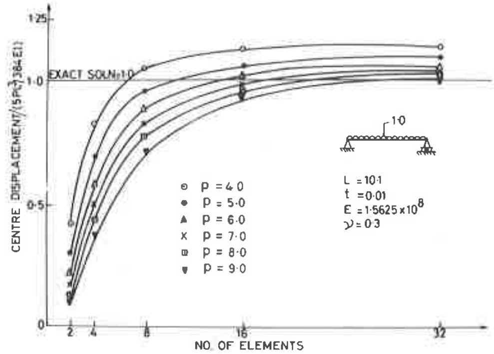

FIG. 6: SIMPLY SUPPORTED BEAM UNDER UDL

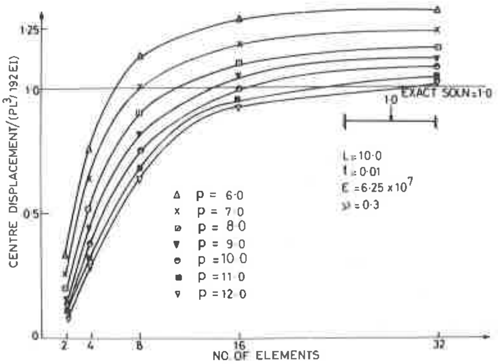

FIG. 7: CLAMPED BEAM UNDER CENTRAL CONCENTRATED LOAD

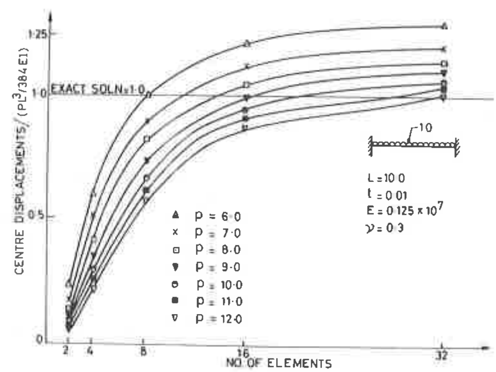

FIG. 8: CLAMPED BEAM UNDER UDL

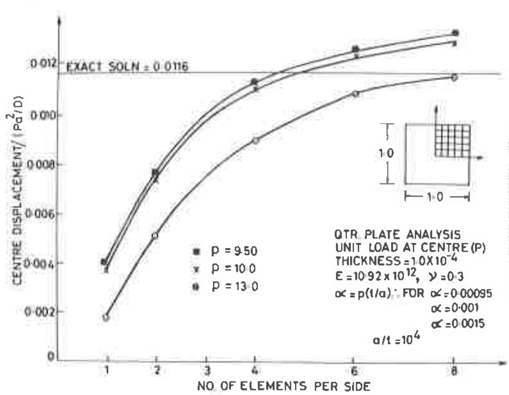

FIG. 9: SIMPLY SUPPORTED SQUARE PLATE UNDER CENTRAL CONCENTRATED LOAD

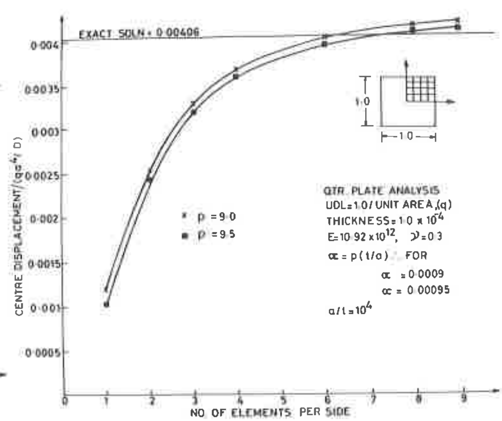

FIG.10 SIMPLY SUPPORTED SQUARE PLATE UNDER UDL

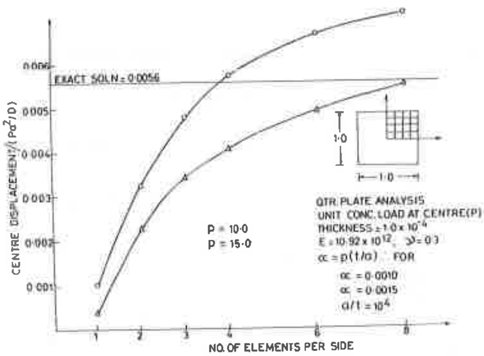

FIG 11 CLAMPED SQUARE PLATE UNDER CENTRAL CONCENTRATED LOAD

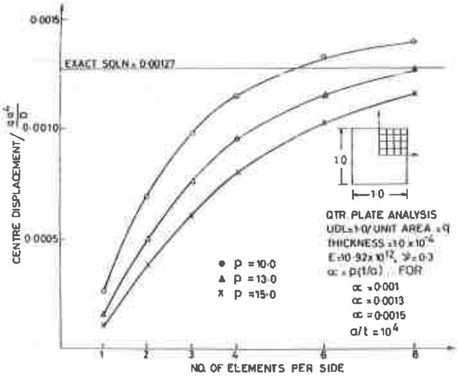

FIG 12: CLAMPED SQUARE PLATE UNDER UDL

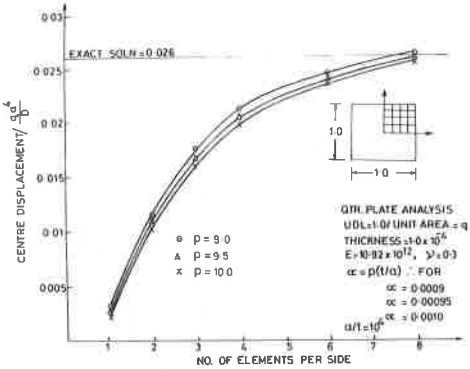

FIG. 13: CORNER SUPPORTED SQUARE PLATE UNDER UDL