A General Equation of Motion for Thick Elastic Shells with Arbitrary Shape

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Abstract

In this paper, general equations of motion for thick elastic shells with arbitrary shape are derived by using the variational principle. In the equations the displacements, angles of rotation of cross sections and transverse normal strain on the middle surface are considered as independent variables. Besides the usual membrane stresses and stress couples, the equation also contains additional terms denoting the influences of transverse shear deformation, rotatory inertia and transverse extrusion.

Introduction

In dealing with the problems of thick shells, classical theory of thin shells, which ignores the effects of the transverse shear deformation, the rotatory inertia and the transverse extrusion on Kirchhoff's assumption, may be in error. Recently, structures with thick shells are extensively used in mechanical engineering, civil engineering, chemical industry, nuclear industry and offshore engineering. Therefore, many researchers have investigated the theory of thick shells. The static equilibrium equations for thick shells, but no boundary conditions, were given in [4]; Whereas equations of motion for thick cylindrical shell were give in [3]. In this paper, general equations of motion and natural boundary conditions for thick elastic shells with arbitrary shape, deduced from the three-dimensional theory of elasticity, are derived by application of Hamilton's principle. Besides the usual membrane stresses and stress couples, the present equation of motion also contains the effects of transverse shear deformation, rotatory inertia and transverse extrusion. The proposed equations can be reduced to equations of motion for thick plate and thick shell with special shape such as thick revolution shell, thick sphere shell and thick circular cylindrical shell, etc. In the present theory if the dynamic effect is neglected, general equations may be reduced to static equilibrium equations.

1. Displacement-Strain Relations

Orthogonal curvilinear coordinate axes on the middle surface of the shell are shown. The principal radii of curvature of the middle surface are denoted by $R_1=R_1(x,y)$, and $R_2=R_2(x,y)$, respectively. The radii of curvature of a thin lamina at a distance z from the middle surface are

$$R_1(x,y,z) = R_1 + z, \qquad R_2(x,y,z) = R_2 + z.$$

Let $A_1(x,y)$ and $A_2(x,y)$ denote the Lame coefficients of the middle surface, then the Lame coefficients of the lamina considered above are

$$\left.\begin{aligned} \tilde{A}_1 &= A_1\left(1+\frac{z}{R_1}\right), \\ \tilde{A}_2 &= A_2\left(1+\frac{z}{R_2}\right), \\ \tilde{A}_3 &= 1 \qquad , \end{aligned}\right\} \tag{1.1}$$

As well known, the following equation holds for the infinitesimal in a shell:

$$(d\tilde{S})^2 = (d\tilde{S}_1)^2 + (d\tilde{S}_2)^2 + (d\tilde{S}_3)^2 \tag{1.2}$$

where

$$\left.\begin{aligned} d\tilde{S}_1 &= \tilde{A}_1 dx = A_1\left(1+\frac{z}{R_1}\right)dx, \\ d\tilde{S}_2 &= \tilde{A}_2 dy = A_2\left(1+\frac{z}{R_2}\right)dy, \\ d\tilde{S}_3 &= \tilde{A}_3 dz = dz. \end{aligned}\right\} \tag{1.3}$$

Taking account of the effects of transverse shear, rotatory inertia and transverse extrusion, the components of the displacement of lamina in the co-- ordinate system will be approximated by u, v, w, which are taken as

$$\left.\begin{aligned} \tilde{u}(x,y,z,t) &= u(x,y,t) + z\psi_x(x,y,t) \ , \\ \tilde{v}(x,y,z,t) &= v(x,y,t) + z\psi_y(x,y,t) \ , \\ \tilde{w}(x,y,z,t) &= w(x,y,t) + z\psi_z(x,y,t) \ , \end{aligned}\right\} \tag{1.4}$$

u, v, and w are the displacement components of a particle on the middle surface, ψ_x and ψ_y are the angles of rotation of a cross section in the x-z plane and y-z plane, respectively. ψ_z is the transverse normal strain. The strain components of the lamina are:

$$\left.\begin{aligned} \tilde{e}_{11} &= \frac{1}{\tilde{A}_1}\frac{\partial \tilde{u}}{\partial x} + \frac{1}{\tilde{A}_1\tilde{A}_2}\frac{\partial \tilde{A}_1}{\partial y}\tilde{v} + \frac{1}{\tilde{A}_1\tilde{A}_3}\frac{\partial \tilde{A}_1}{\partial z}\tilde{w}, \\ \tilde{e}_{22} &= \frac{1}{\tilde{A}_2}\frac{\partial \tilde{v}}{\partial z} + \frac{1}{\tilde{A}_2\tilde{A}_3}\frac{\partial \tilde{A}_2}{\partial z}\tilde{w} + \frac{1}{\tilde{A}_2\tilde{A}_1}\frac{\partial \tilde{A}_2}{\partial x}\tilde{u}, \\ \tilde{e}_{33} &= \frac{1}{\tilde{A}_3}\frac{\partial \tilde{w}}{\partial z} + \frac{1}{\tilde{A}_3\tilde{A}_1}\frac{\partial \tilde{A}_3}{\partial x}\tilde{u} + \frac{1}{\tilde{A}_3\tilde{A}_2}\frac{\partial \tilde{A}_3}{\partial y}\tilde{v}, \\ \tilde{e}_{23} &= \frac{\tilde{A}_3}{\tilde{A}_2}\frac{\partial}{\partial y}\left(\frac{\tilde{w}}{\tilde{A}_3}\right) + \frac{\tilde{A}_2}{\tilde{A}_3}\frac{\partial}{\partial z}\left(\frac{\tilde{v}}{\tilde{A}_2}\right), \\ \tilde{e}_{31} &= \frac{\tilde{A}_1}{\tilde{A}_3}\frac{\partial}{\partial z}\left(\frac{\tilde{u}}{\tilde{A}_1}\right) + \frac{\tilde{A}_3}{\tilde{A}_1}\frac{\partial}{\partial x}\left(\frac{\tilde{w}}{\tilde{A}_3}\right), \\ \tilde{e}_{12} &= \frac{\tilde{A}_2}{\tilde{A}_1}\frac{\partial}{\partial x}\left(\frac{\tilde{v}}{\tilde{A}_2}\right) + \frac{\tilde{A}_1}{\tilde{A}_2}\frac{\partial}{\partial y}\left(\frac{\tilde{u}}{\tilde{A}_1}\right). \end{aligned}\right\} \tag{1.5}$$

2. Energy Considerations

The strain energy of the shell is

$$V=\frac{1}{2}\iiint(\tilde{\sigma}_{11}\tilde{e}_{11}+\tilde{\sigma}_{22}\tilde{e}_{22}+\tilde{\sigma}_{33}\tilde{e}_{33}+\tilde{\sigma}_{23}\tilde{e}_{23}+\tilde{\sigma}_{31}\tilde{e}_{31}+\tilde{\sigma}_{12}\tilde{e}_{12}) \cdot \tilde{A}_1\tilde{A}_2\tilde{A}_3\,dxdydz \tag{2.1}$$

The expression for the kinetic energy T of the shell will be

$$T=\frac{1}{2}\iiint\rho\left[\left(\frac{\partial\tilde{u}}{\partial t}\right)^2+\left(\frac{\partial\tilde{v}}{\partial t}\right)^2+\left(\frac{\partial\tilde{w}}{\partial t}\right)^2\right]\tilde{A}_1\tilde{A}_2\tilde{A}_3\,dxdydz \tag{2.2}$$

The potential energy of external force is

$$V_e=-\iint_S(f_x\tilde{u}+f_y\tilde{v}+f_z\tilde{w})ds \tag{2.3}$$

The potential energy of body force is

$$V_G=-\iiint_V(F_x\tilde{u}+F_y\tilde{v}+F_z\tilde{w})\tilde{A}_1\tilde{A}_2\tilde{A}_3\,dxdydz \tag{2.4}$$

3. Variation, Equation of Motion and Appropriate Natural Boundary Conditions

The mathematical expression of Hamilton's principle is

$$\delta\int_{t_1}^{t_2}L\,dt=0 \tag{3.1}$$

where

$$L=T-(V+V_e+V_G) \tag{3.2}$$

Substituting Eqs.(2-1), (2-2), (2-3) and (2-4) into Eq.(3-2), the variation (3-1) can be put into the following form:

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(L_1\delta u+L_2\delta v+L_3\delta w+L_4\delta\psi_x+L_5\delta\psi_y+L_6\delta\psi_z)dxdy\Big]dt$$
$$+\int_{x_1}^{x_2}\int_{y_1}^{y_2}(L_7)_{t_1}^{t_2}dxdy+\int_{t_1}^{t_2}\left[\int_{y_1}^{y_2}(L_8)_{x_1}^{x_2}+\int_{x_1}^{x_2}(L_9)_{y_1}^{y_2}dx\right]dt=0 \tag{3.3}$$

where

$$L_1=-J_0\frac{\partial^2u}{\partial t^2}-J_1\frac{\partial^2\psi_x}{\partial t^2}+\frac{\partial}{\partial x}(N_{11}A_2)+\frac{\partial}{\partial y}(N_{21}A_1)+N_{12}\frac{\partial A_1}{\partial y}-N_{22}\frac{\partial A_2}{\partial x}$$
$$+N_{13}\frac{A_1A_2}{R_1}+N_{31}^{*}A_1A_2+N_{11}^{G}A_1A_2 \quad,\ \cdots, \tag{3.4}$$

where

$$\left.\begin{aligned}J_0&=\rho I_0A_1A_2\\J_1&=\rho I_1A_1A_2\\J_2&=\rho I_2A_1A_2\end{aligned}\right\}.$$

Eq.(3-3) must hold for arbitary values of δu , δv , δw , $\delta\psi_x$, $\delta\psi_y$ and $\delta\psi_z$. We thus obtain the six equations of motion

$$\left.\begin{aligned}L_1=0,\ L_2=0,\ L_3=0\\L_4=0,\ L_5=0,\ L_6=0\end{aligned}\right\} \tag{3.5}$$

4. Stress-Displacement Relations

From the three-dimensional theory of elasticity, we obtain

$$\left.\begin{aligned}\tilde{\sigma}_{11}&=(\lambda+2\mu)\tilde{e}_{11}+\lambda(\tilde{e}_{22}+\tilde{e}_{33}),\ \tilde{\sigma}_{12}=\mu\tilde{e}_{12},\\ \tilde{\sigma}_{22}&=(\lambda+2\mu)\tilde{e}_{22}+\lambda(\tilde{e}_{33}+\tilde{e}_{11}),\ \tilde{\sigma}_{13}=\mu\tilde{e}_{13},\\ \tilde{\sigma}_{33}&=(\lambda+2\mu)\tilde{e}_{33}+\lambda(\tilde{e}_{11}+\tilde{e}_{22}),\ \tilde{\sigma}_{23}=\mu\tilde{e}_{23}.\end{aligned}\right\}\tag{4.1}$$

where the Lame's constants of elasticity λ and μ are expressed in terms of Young's modulus E and Poisson's ratio ν , by

$$\left.\begin{aligned}\lambda&=\frac{E\nu}{(1+\nu)(1-2\nu)},\quad \mu=\frac{E}{2(1+\nu)}\\ \text{and}\qquad \lambda+2\mu&=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\end{aligned}\right\}\tag{4.2}$$

In Eq.(4-1) employing the form of Eq.(1-5), then substituting it into(2-4), after some calculation, the following relations are obtained between the shell-Stress and shell-displacement components:

$$\left.\begin{aligned}N_{11}&=(\lambda+2\mu)\Big[\alpha_1\Big(\frac{1}{A_1}\frac{\partial u}{\partial x}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial y}v+\frac{w}{R_1}\Big)+\beta_1\Big(\frac{1}{A_1}\frac{\partial\psi_x}{\partial x}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial y}\psi_y\\ &\quad+\frac{\psi_z}{R_1}\Big)\Big]+\lambda\Big\{h\Big[\Big(\frac{1}{A_2}\frac{\partial v}{\partial y}\ \frac{1}{A_1A_2}\frac{\partial A_2}{\partial x}u+\frac{w}{R_2}\Big)+\psi_z\Big]\Big\},\ \cdots,\\ M_{11}&=(\lambda+2\mu)\Big[\beta_1\Big(\frac{1}{A_1}\frac{\partial u}{\partial x}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial y}v+\frac{w}{R_1}\Big)+\gamma_1\Big(\frac{1}{A_1}\frac{\partial\psi_x}{\partial x}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial y}\psi_y\\ &\quad+\frac{\psi_z}{R_1}\Big)\Big]+\lambda\Big\{\frac{h^3}{12}\Big[\Big(\frac{1}{A_1}\frac{\partial\psi_y}{\partial y}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial x}\psi_x+\frac{\psi_z}{R_2}\Big)+\frac{\psi_z}{R_2}\Big]\Big\},\ \cdots.\end{aligned}\right\}\tag{4.3}$$

5. Displacement Equation of Motion for Thick Shells

Substituting Eq.(4-3) into the shell-stress equation of motion (3-5), displacement equation of motion for thick shell in matrix notation is obtained

$$[B]\cdot\frac{\partial^2}{\partial t^2}\{P\}+[C]\{P\}+\{Q\}=\{0\}\tag{5.1}$$

Here [B] and [C] are 6x6 matrices, {P}, {Q} and {0} are 6x1 vectors. In physical interpretation, [B] is the mass matrix, [C] is the rigidity matrix, {P} is the displacement vector, {Q} is the load vector. {0} is a zero vector. Note that {C} is a differential operator matrix. The elements of the matrices [C], [B] and the vectors {P}, {Q} are given by

$$\begin{aligned}&P_1=u,\quad P_2=v,\quad P_3=w\\ &P_4=\psi_x,\quad P_5=\psi_y,\quad P_6=\psi_z\end{aligned}\tag{5.2}$$

$$\left.\begin{aligned}Q_1&=(N_{31}^{*}+N_{11}^{G})A_1A_2,\quad Q_4=(M_{31}^{*}+M_{11}^{G})A_1A_2\\ O_2&=(N_{32}^{*}+N_{22}^{G})A_1A_2,\quad Q_5=(M_{32}^{*}+M_{22}^{G})A_1A_2\\ Q_3&=(N_{33}^{*}+N_{33}^{G})A_1A_2,\quad Q_6=(M_{33}^{*}+M_{33}^{G})A_1A_2\end{aligned}\right\}\tag{5.3}$$

$$\left.\begin{aligned}&B_{11}=B_{22}=B_{33}=-J_0,\\ &B_{44}=B_{55}=B_{66}=-J_2,\\ &B_{14}=B_{25}=B_{36}=B_{41}=B_{52}=B_{63}=-J_1\end{aligned}\right\}\tag{5.4}$$

(other element B_{ij} of the matrix B vanishes)

$$
\left.
\begin{aligned}
C_{11} = {} & (\lambda + 2\mu)\frac{\alpha_1 A_2}{A_1}\frac{\partial^2}{\partial x^2} + \mu\frac{\alpha_2 A_1}{A_2}\frac{\partial^2}{\partial y^2} + (\lambda + 2\mu)\frac{\partial}{\partial x}\left(\frac{\alpha_1 A_2}{A_1}\right)\frac{\partial}{\partial x} \\
& + \mu\frac{\partial}{\partial y}\left(\frac{\alpha_2 A_1}{A_2}\right)\frac{\partial}{\partial y} + \lambda h\frac{\partial}{\partial x}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial x}\right) - \mu h\frac{\partial}{\partial y}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial y}\right) \\
& - \mu\frac{\alpha_1}{A_1 A_2}\frac{\partial A_1}{\partial y}\frac{\partial A_1}{\partial y} - (\lambda + 2\mu)\frac{\alpha_2}{A_1 A_2}\frac{\partial A_2}{\partial x}\frac{\partial A_2}{\partial x} - K_1^2\mu\frac{\alpha_1}{R_1}\frac{A_1 A_2}{R_1}, \\
\cdots, & \\
C_{66} = {} & K_1^2\mu\frac{\gamma_1 A_2}{A_1}\frac{\partial^2}{\partial x^2} + K_2^2\mu\frac{\gamma_2 A_1}{A_2}\frac{\partial^2}{\partial y^2} + K_1^2\mu\frac{\partial}{\partial x}\left(\frac{\gamma_1 A_2}{A_1}\right)\frac{\partial}{\partial x} \\
& + K_2^2\mu\frac{\partial}{\partial y}\left(\frac{\gamma_2 A_1}{A_2}\right)\frac{\partial}{\partial y} - (\lambda + 2\mu)\frac{\gamma_1}{R_1}\frac{A_1 A_2}{R_1} - (\lambda + 2\mu)\frac{\gamma^2}{R_2}\frac{A_1 A_2}{R_2} \\
& - \lambda\frac{h^3}{2}\frac{A_1 A_2}{R_1 R_2} - (\lambda + 2\mu) I_0 A_1 A_2 \text{ 。}
\end{aligned}
\right\} \quad (5.5)
$$

6. Discussion

Eq.(5-1) is the equation of motion for thick shell with arbitrary shape. If all the elements of matrix B vanish, it reduces to the static equilibrium equation for the thick shell. Otherwise, if the Lame's coefficients A_1, A_2 and the principal radii of curvature of the shell are specialized, it reduces to the dymanic equation of motion for thick shell with special shapes.

It remains to estimate the precisions of various theories of shells. In our opinion, classical theory can be expected to held for shells, whose maximum ratio h/R, i.e., the ratio of thickness to the radiu of curvature of the middle surface, can be considered as negligible in comparison with 1. For moderate thick shells, $(h/R)^2$, but not h/R, can be negligible. As to thick shells, even $(h/R)^2$ can not be considered as negligible in comparison with 1.

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