A Semi-Analytical Method to Predict the Ultimate Strength of Reinforced Concrete Members Subjected to In-Plane Stresses

H. Yoshikawa
Technical Research Institute, Hazama-Gumi, Ltd., Yono, Saitama 338, Japan
H. Umehara
Nagoya Institute of Technology, Showa-ku, Nagoya 466, Japan
T.-A. Tanabe
Nagoya University, Chigusa-ku, Nagoya 464, Japan

Abstract

Analytical approaches as well as experimental studies on the mechanical behavior of reinforced concrete panel-elements where in-plane stresses are dominant, have been carried out intensively so far. Various theories have been developed, each of them differing in the point where they put heavier stress and in the method of deriving formulations. In the paper, major theories on the in-plane shear load capacity of a regularly reinforced concrete panel are comprehensively compared. Then, the predictability of these theories is studied, comparing them with experimental results of some 100 specimens in terms of dimensionless descriptions. Finally, a semi-analytical method to estimate the ultimate shear strength and the failure mode of an orthogonally reinforced concrete panel is proposed, based on modified limit analysis methods for both the steel-yielding type and the concrete-crushing type.

1. EVALUATION OF PRINCIPAL THEORIES ON THE IN-PLANE SHEAR STRENGTH

It is quite a complicated problem if one tries to predict the entire behavior of a reinforced concrete panel including deformational characteristics. On the other hand, as far as the estimation of ultimate load capacity (maximum strength) is concerned, the analytical procedure becomes much simpler. Some theories such as limit analysis and other macroscopic modeling are considered to be good examples of this. The authors indicated in the previous study [1] that most of the formulae from various analytical models can be derived by the equilibrium condition alone, and that the difference between theories can be identified to be due to a different assumed stress state.

The stress state of orthogonally reinforcing bars, the cracked concrete and the applied internal forces are represented by stress matrices in their own local coordinate systems, which are shown in Table 1. R_x and R_y are called "equivalent" stresses of steel, defined by reinforcement ratios, P_x, P_y and the axial stresses in reinforcing bars σ_{sx}, σ_{sy}, i.e., $R_x = P_x\sigma_{sx}$, $R_y = P_y\sigma_{sy}$. In the case of cracked concrete, σ_n is the normal stress perpendicular to the direction of the crack which is usually neglected, and σ_c ($\leq f_c$) is the compressive stress parallel to the direction of the crack and τ_c is the shear stress along the crack.

When deriving a formula for the ultimate strength, it is required to determine these stress matrices according to the failure type and assumptions. For example, $R = Pf_y$ when reinforcement yielding type or $\sigma_c = f_c$ when concrete fails. (f_c is the compressive strength of concrete and f_y is the yield strength of reinforcement). The equilibrium condition of a reinforced concrete element in a properly chosen common coordinate system and the transformation law for the coordinate system are expressed as follows,

$$[\sigma_s]_\theta + [\sigma_c]_\theta = [F]_\theta \quad (1)$$

$$[\sigma]_\theta = [T]^t[\bar{\sigma}][T], \quad [T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (2)$$

in which $[\bar{\ }]$ and $[\ ]_\theta$ express stress matrices in local coordinates and in transformed coordinate systems, respectively, and [T] is a transformation matrix. Final results are summarized in Table 1. Most of the formulae proposed by the various researchers can be derived by means of the above equations (1) and (2) using stress matrices in Table 1, which may be different from their original methods.

For example, one can get Nielsen's formulae [2] making use of the following balanced equation choosing the x-y direction as a common coordinate system,

$$[\sigma_s]_{\theta=0} + [\sigma_c]_{\theta=-\beta} = [F_2]_{\theta=0} \quad (3)$$

Nielsen assumes that the concrete is a uniaxial compressive member which carries a load only in the cracking direction. Therefore, $\sigma_n = 0$ and $\tau_c = 0$, and the above equation can be written as,

$$\begin{bmatrix} R_x & 0 \\ 0 & R_y \end{bmatrix} + \begin{bmatrix} -\sigma_c\sin^2\beta & -\frac{1}{2}\sigma_c\sin 2\beta \\ -\frac{1}{2}\sigma_c\sin 2\beta & -\sigma_c\cos^2\beta \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \quad (4)$$

When the reinforcement bars in both directions have reached a tensile yield point, the shear stress is given as follows from Eq. (4) considering that $R_x = P_x f_y$, $R_y = P_y f_y$ and $\sigma_c < f_c$.

$$\tau_{xy} = \sqrt{(p_x f_y - \sigma_x)(p_y f_y - \sigma_y)}, \quad (\sigma_c < f_c) \quad (5)$$

When one of either of the reinforcements yield, and the compressive failure of concrete occurs, the shear stress τ_{xy} is provided as follows,

$$\tau_{xy} = \sqrt{(p_y f_y - \sigma_y)\{f_c - (p_y f_y - \sigma_y)\}} \quad (6)$$

in which the stress state is considered in such a way that $R_x < P_x f_y$, $R_y = P_y f_y$ and $\sigma_c = f_c$. Finally, a continuation of the above development successfully leads to the five formulae for the shear strength which constructs one closed failure criterion for a reinforced concrete panel in a plane stress state. Fig. 1 depicts the so-obtained Nielsen's criterion as well as the failure condition given by Ono and Tanaka [5] and the modified Baumman's theory [7], both of which can also be formulated in the same way.

It is found from the Figure that in the case of tensile failure or compressive failure of reinforcement (which correspond to domain I and V in Nielsen's criterion), these three theories provide almost the same results, while the theories reveal different strengths in-between areas where the concrete fails prior to the steel yielding.

The authors have investigated other major theories by Marti [4], Bažant and Tsubaki [6] and Collins [3] as well, based on the above-mentioned unified formulation.

It may be said from the investigation [1] that each of the major formulae for the ultimate strength of a reinforced concrete panel can be clearly derived solely from the equilibrium equation (1) in one common coordinate system and that the differences among the theories mainly depend upon the assumed stress matrix for the concrete. The interrelationship between the theories is briefly summarized as follows.

As previously stated, Nielsen's formulae can directly be formulated from Eqs (1) and (2), whereas Bažant and Tsubaki [6] added to Eq. [1] the following Coulomb's frictional law based on the slip-free criterion,

$$|\tau_c| < -k\,\sigma_n \qquad (7)$$

in which k denotes a frictional coefficient on cracked surfaces. Marti [4] introduced the modified Coulomb yield function as a fracture behavior of concrete in plastic analysis. This means that

$$\sigma_n = f_t \qquad (8)$$

at the tensile failure instead of the assumption made by many researchers that $\sigma_n = 0$. (f_t is the tensile strength of concrete.) Ono and Tanaka [5] developed the three failure criteria in the principal stress state resulting in the upper and lower bounds for load carrying capacity of a shear wall. Their criteria are shown in Fig. 1.

More detailed descriptions can be found in [1].

2. COMPARISONS WITH TEST RESULTS

Values predicted by the theories stated so far are compared to the experimental data, which are presently available from over seventy specimens reported by eight research groups [1]. These tests were carried out by either torsional loading on a hollow cylindrical specimen or in-plane direct loading on a flat panel where reinforcement ratios and membrane stresses induced by mechanically applied forces, internal pressure or prestressing forces are chosen as changing parameters. Results obtained from horizontal loading tests using shear walls or hollow cylinder specimens are excluded here because stresses observed in those experimental works are not uniformly distributed over a specimen, which makes it difficult to directly convert force into stress. Failure of reinforced concrete panels is supposed to be divided into three modes as follows: both reinforcing bars in two directions yield (mode I); reinforcing bars in one direction yield and concrete crushed (mode II); and concrete crushed with no reinforcement yielded (mode III).

To allow a general comparison of the ultimate in-plane shear strength, the non-dimensional stress values normalized relative to the compressive strength of concrete f_c are employed, such that,

$$\eta = \frac{\tau}{f_c}, \quad \xi = \frac{\sigma}{f_c}, \quad \psi = \frac{pf_y}{f_c} \qquad (9)$$

by which all the mathematical expressions of various models can be represented. (Hereafter τ indicates the shear stress at the ultimate stage, so subscripts xy are omitted, and R indicates the equivalent stress of reinforcing steel when yielded; $R = Pf_y$)

Fig. 2 shows that measurements roughly agree with the values calculated by Nielsen and the conventional formula $\tau = Pf_y$, and it may be said that the measurement slope in mode I is much closer to that of Bažant's slip-free criterion, and an intercept of measured data with the vertical axis is not zero, which corresponds to ζ used in Marti's theory. These may be suggesting that the adoption of the Mohr-Coulomb failure criterion is justified in the lower degree of reinforcement and that the tensile stress in the reinforcement bars is increased due to slippage along cracked surfaces.

The regression curve, $\eta = 0.85\psi + 0.024$, is obtained from the experimental data in mode I of Fig. 2, which means the frictional coefficient k = 1.6 in Bažant's theory and the ratio of tensile-compressive strength of concrete $\zeta = 0.024$ in Marti's theory.

Fig. 3 describes a change of the shear strength with respect to the normal stress. As can be seen in Fig. 3, the theoretical predictions give a reasonable explanation of a change of in-plane shear strength through failure modes I, II and III. (Note that the constants, ν, ζ and κ are chosen as the best fitting values so that the theories can agree with the experimental data.)

In Fig. 4, only the experimental results in mode III are plotted with respect to the compressive strength of concrete, comparing them to the values calculated by the failure criterion $\tau = 0.5f_c$ and by the Japanese drafted CCV code [13]. It is found that all the theoretical predictions overestimate the experimental data.

3. A SEMI-ANALYTICAL METHOD TO ESTIMATE THE IN-PLANE SHEAR STRENGTH AND THE FAILURE MODE

Even though we accept the reasonable agreement of the theoretical and experimental values in the reinforcement yielding region, it should be recognized that there still exists room for improvement of the differences and of simplification of the methods. Moreover, in the case of the concrete crushing failures, there is the significant discrepancy between the predicted and experimental values, and theoretical approaches are not as available.

Hence, the authors propose the following semi-analytical procedure for the estimation of in-plane shear strength of reinforced concrete panels as well as for their failure mode. This method will be obtained by simplifying limit analyses formulated by Nielsen [2], Marti [4] and Bažant [6] and making a correlation with experimental observations referred to in the paper [1]. Proposed formulae are prepared for both reinforcement-yielding type (failure modes I and II) and concrete-crushing type (failure mode III). (Hereafter, equations in parentheses are expressed by metric units and not as dimensionless expressions)

Reinforcement-yielding type: If one takes,

$$\eta = a\psi + b\ , \qquad (\ \tau = aR + bf_c\ ) \tag{10}$$

as a general form of a shear strength formula for a reinforced concrete panel in a pure shear stress state, then this formula can represent all three analytically obtained expressions by Nielsen, Marti and Bažant by changing the constants a and b. The above equation corresponds to Eq. (5) by Nielsen when a = 1 and b = 0, and is identical with Marti's theory when $a = 1$ $b = \zeta$, and with Bažant's theory when $a = k/\sqrt{1+k^2}$ and b = 0 (See Fig. 2). We then introduce a new term, the "equivalent" degree of reinforcement, ψ^*, in the more extended form as,

$$\psi^* = \sqrt{(\psi_x - \xi_x)(\psi_y - \xi_y)}\ , \qquad (\ R^* = \sqrt{(p_x f_y - \sigma_x)(p_y f_y - \sigma_y)}\ ) \tag{11}$$

It is considered that the constants in Eq. (10) may be determined by adjusting them to the experimental data, supposing Eq. (10) is still applicable even when ψ is substituted with this equivalent degree of reinforcement ψ^*. Toward that purpose, measurements plotted in Fig. 2 are added by other data from specimens subjected to normal stresses in either one or two directions and are shown in Fig. 5(a) with respect to the equivalent degree of reinforcement ψ^*. By the least squares method a linear regression line is then obtained whose constants are a = 0.76 and b = 0.026, which may be in accordance with the investigation for Fig. 2. Now, a formula of the shear strength for the reinforcement-yielding type is represented as,

$$\eta = 0.76\,\psi^* + 0.026, \quad (\ \tau = 0.76\,R^* + 0.026\,f_c\ ) \tag{12}$$

Fig. 5(a) exhibits that equation (12) so obtained leads to a better agreement with experimental results arranged by the term of the equivalent degree of reinforcement ψ^* rather than the chosen three theories discussed in this chapter.

Concrete-crushing type: According to the authors' study [1], it is observed from both the experimental and calculated results that the shear strength governed by the concrete crushing (mode III) is influenced by the compressive strength of concrete f_c and the degree of reinforcement. Hence, we introduce a new dimensionless parameter, $(f_o/f_c)^{0.5}\psi^*$, which is assumed to be linearly related to the in-plane shear strength of this failure mode. Consequently, two straight lines are obtained as shown in Fig. 5(b). The average strength (solid line) is thus expressed as,

$$\eta = 0.29\left(\frac{f_0}{f_c}\right)^{0.5}\psi^* + 0.145, \quad \left(\ \tau = 0.29\left(\frac{f_0}{f_c}\right)^{0.5}R^* + 0.145\,f_c\ \right) \tag{13}$$

in which f_o is a reference strength for concrete. The value of f_o is taken to be 250 kgf/cm^2, or 24.5 MPa or so on, whose unit must be the same as that of f_c.

Determination of failure mode: Making use of following newly defined dimensionless terms,

$$\bar{\psi}_x = \psi_x - \xi_x\ , \quad \bar{\psi}_y = \psi_y - \xi_y \tag{14}$$

failure modes of an orthogonally reinforced concrete panel can be classified as follows.

$$\begin{aligned} &\text{mode I} && \bar{\psi}_x + \bar{\psi}_y \leqq \nu \\ &\text{mode II} && \bar{\psi}_x + \bar{\psi}_y > \nu\ , \text{ and } \bar{\psi}_x,\ \bar{\psi}_y < \tfrac{1}{2}\nu \\ &\text{mode III} && \bar{\psi}_x,\ \bar{\psi}_y \geqq \tfrac{1}{2}\nu \end{aligned} \tag{15}$$

4. THE USAGE OF THE PROPOSED METHOD

As for the application for the design use, one only needs to calculate the equivalent degree of reinforcement ψ^* and the parameter $(f_o/f_c)^{0.5}\psi^*$ from given conditions and then two values of η (or τ) using Eq. (12) and Eq. (13). The smaller value of η (or τ) is the shear strength. Fig. 6 compares the newly proposed formulae and the recently obtained test results from about thirty specimens. These experiments have been carried out by Vecchio and Collins [8], Muguruma and Watanabe et al. [10] and, Yamazaki and Yasaka et al. [9], whose results were not known before the authors' proposal (so these were not used when the constants of Eq. (12) and (13) were calibrated.) It can be seen from Fig. 6 that the authors' formulae are in good agreement with the measured values.

In Fig. 7 actual test data are compared to the proposed Eqs. (15), where the effectiveness factor of the compressive strength of concrete is assumed to be 0.6. Although some measurements in the neighborhood of boundaries of each mode do not coincide, this method can predict well the failure mode of actual test observations.

REFERENCES

[1] Yoshikawa, H., Umehara, H. and Tanabe, T.: Comprehensive Evaluation of Major Theories on the Ultimate Strength of Reinforced Concrete Panels Subjected to In-Plane Shear Forces and the Proposed Semi-Analytical Method for Estimation of Ultimate Strength, Proceedings of 2nd Colloquium on Shear Analysis of RC Structures, JCI, Oct., 1983 pp. 69∿78, (in Japanese), Concrete Library of JSCE No. 4, Dec., 1984 pp. 261∿281, (in English).

[2] Nielsen, M.P.: On the Strength of Reinforced Concrete Discs, Acta Polytechnica Scandinavica, Civil Engineering and Building Construction Series No. 70, Copenhagen, 1971.

[3] Collins, M.P.: Towards a Rational Theory for RC Members in Shear, Journal of the Structural Division, ASCE, ST 4 Vol. 104, April 1978, pp. 649∿666.

[4] Marti, P.: Plastic Analysis of Reinforced Concrete Shear Walls, Introductory Report of IABSE Colloquium, Copenhagen, 1979, Plasticity in Reinforced Concrete, 1979, pp. 51∿69.

[5] Ono, K. and Tanaka, H.: Limit Design of Reinforced Concrete Walls, Transactions of the Architectural Institute of Japan, No. 49, Sep., 1954, pp. 42∿48.

[6] Bažant, Z.P. and Tsubaki, T.: Concrete Reinforcing Nets: Optimum Slip-free Limit Design, Journal of the Structural Division, ASCE, Vol. 105, No. ST2, Feb. 1979, pp. 327∿346.

[7] Baumann, T.: Zur Frage der Netzbewehrung von Flächentragwerken, Der Bauingenieur, Vol. 47, Heft 10, 1972, pp. 367∿377.

[8] Vecchio, F. and Collins, M.P.: The Response of Reinforced Concrete to In-Plane Shear and Normal Stresses, University of Toronto, Department of Civil Engineering, No. 82-03, March, 1982.

[9] Yamazaki and Yasaka, A, et al: An Experiment on Reinforced Concrete Cylinders Subjected to both Thermal and Torsional Loads (Part I), Summaries of Technical Papers of Annual Meeting of Architectural Institute of Japan, Oct., 1984, pp. 2327∿2328.

[10] Muguruma, H. and Watanabe, F., et al: An Experimental Study on the Shear Resisting Behaviors of R/C Shear Wall, Proceedings of JCI 6th Conference, 1984, pp. 705∿708.

[11] Higai, I.: Plastic Theory on Shear Collapse in Reinforced Concrete Beams, Proceedings of the 36th Annual Conference of the Japan Society of Civil Engineers, V-150, 1981, pp. 298∿299.

[12] Exner, H.: On the Effectiveness Factor in Plastic Analysis of Concrete, IABSE Colloquium, 1980, pp. 35∿42.

[13] Ohsaki, Y., Ibe, Y. and Aoyagi, Y.: Drafted Japanese Design Criteria for Concrete Containment, SMiRT-6, J 1/2, 1981.

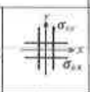
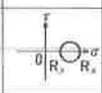
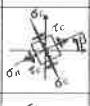
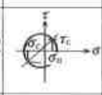
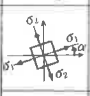
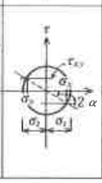
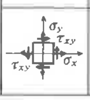
model		Mohr's circle	stress matrices	
			in local coordinate system	in transformed coordinate system
reinforcement			$[\bar{\sigma}_s]=\begin{bmatrix}R_x & 0\\ 0 & R_y\end{bmatrix}$	$[\sigma_s]_\theta=\begin{bmatrix}R_x\cos^2\theta+R_y\sin^2\theta & (R_x-R_y)\sin\theta\cdot\cos\theta\\ (R_x-R_y)\sin\theta\cdot\cos\theta & R_x\sin^2\theta+R_y\cos^2\theta\end{bmatrix}$
cracked concrete			$[\bar{\sigma}_c]=\begin{bmatrix}\sigma_n & \tau_c\\ \tau_c & -\sigma_c\end{bmatrix}$	$[\sigma_c]_\theta=\begin{bmatrix}-\tau_c\sin 2\theta-\sigma_c\sin^2\theta & \tau_c\cos 2\theta+\frac{1}{2}\sigma_c\sin 2\theta\\ \tau_c\cos 2\theta+\frac{1}{2}\sigma_c\sin 2\theta & \tau_c\sin 2\theta-\sigma_c\cos^2\theta\end{bmatrix}$ $(\sigma_n=0)$
internal applied forces			$[\bar{F}_1]=\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}$	$[F_1]_\theta=\begin{bmatrix}\sigma_1\cos^2\theta+\sigma_2\sin^2\theta & (\sigma_1-\sigma_2)\sin\theta\cos\theta\\ (\sigma_1-\sigma_2)\sin\theta\cos\theta & \sigma_1\sin^2\theta+\sigma_2\cos^2\theta\end{bmatrix}$
			$[\bar{F}_2]=\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}$	$[F_2]_\theta=\begin{bmatrix}\sigma_x\cos^2\theta+\sigma_y\sin^2\theta-2\tau_{xy}\sin\theta\cos\theta & (\sigma_x-\sigma_y)\sin\theta\cos\theta+\tau_{xy}(\cos^2\theta-\sin^2\theta)\\ (\sigma_x-\sigma_y)\sin\theta\cos\theta+\tau_{xy}(\cos^2\theta-\sin^2\theta) & \sigma_x\sin^2\theta+\sigma_y\cos^2\theta+2\tau_{xy}\sin\theta\cos\theta\end{bmatrix}$

Table 1 Stress Matrices for Materials and Applied Forces

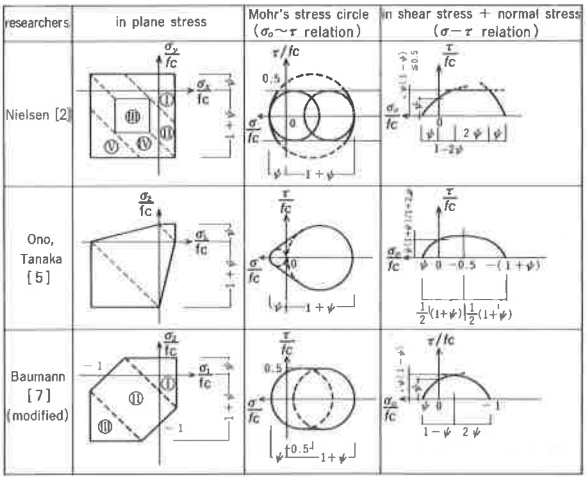


Fig. 1 Failure Criteria of a Reinforced Concrete Panel with Orthogonal and Isotropic Reinforcement

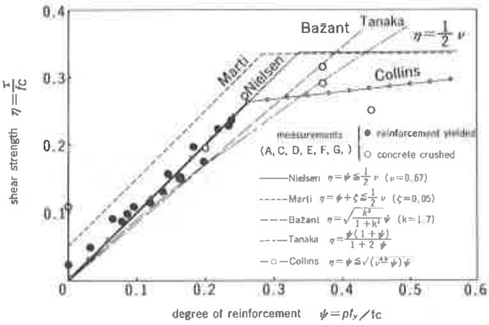


Fig. 2 Comparison with Test Results (ψ–η Relation in Pure Shear Stress)

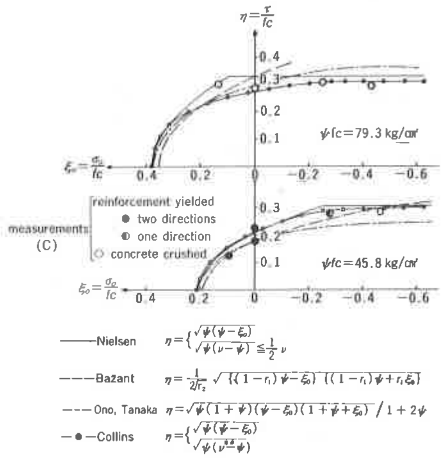


Fig. 3 Comparison with Test Results (ξ_0 – η Relation in Shear Stress and One Directional Normal Stress)

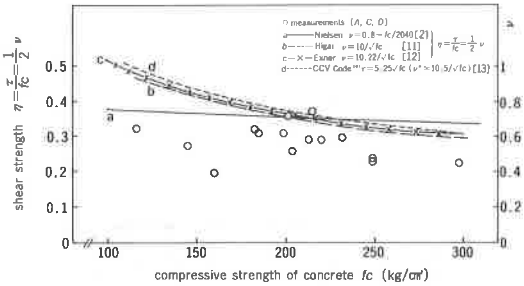


Fig. 4 In-Plane Shear Strength for Concrete-Crushed Failure (f_c – η Relation)

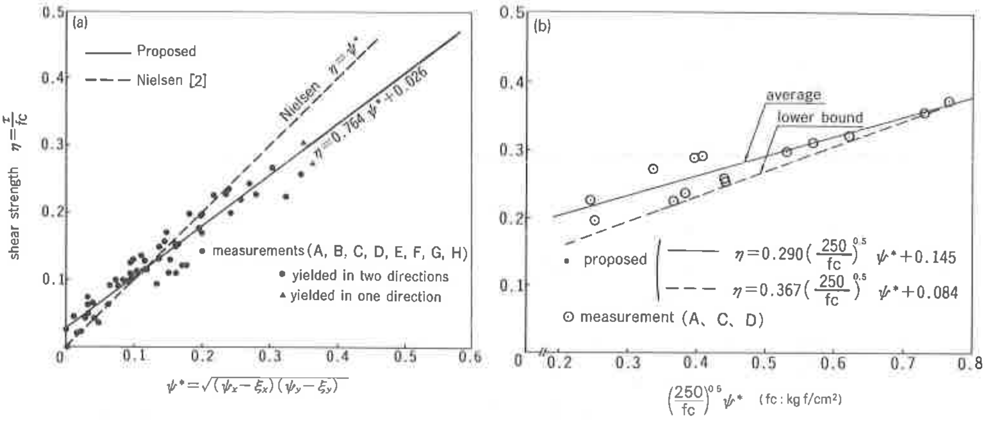


Fig. 5 Test Results and Obtained Regression Lines for
a) Reinforcement-Yielding Type and b) Concrete-Crushing Type

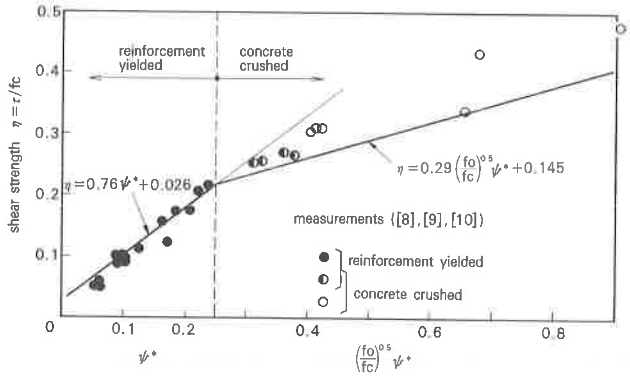


Fig. 6 Proposed Failure Criteria for In-Plane Shear Strength and Recently Available Test Data

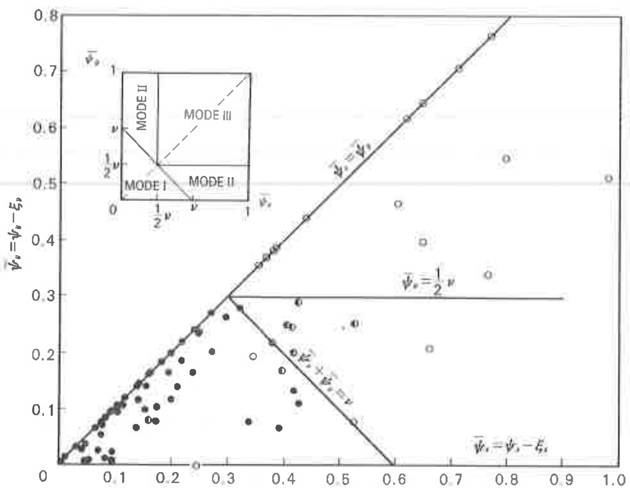


Fig. 7 Classification of Failure Modes and Test Results