Seismic Response of a Gyroscopic System by the Response Spectrum Method

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Abstract

When applied to a gyroscopic system, the response spectrum method is different than when it is applied to a classical dynamic system in several aspects. First, the equation of motion of a gyroscopic system is not self-adjoint and cannot be uncoupled by the classical normal modes. Second, the spectral responses of a gyroscopic system are not determined at the natural frequencies and the given damping values because of coupling of the natural frequencies with the rotating speed. Third, the responses of a gyroscopic system depend not only on the 'displacement' terms but also on the 'velocity' terms, making response combination much more complicated.

1. Introduction

Some rotating equipment in a nuclear power plant are required to remain operational or be capable of free-wheeling during a seismic event. When qualification by tests is not feasible, a seismic analysis of the rotor-bearing system will instead have to be performed to show that the bearings will not fail, the clearance between the rotor and the stator will not close, and the shaft will not be overstressed.

A rotating dynamic system is also known as a gyroscopic system because its equation of motion contains gyroscopic terms which are recognized as skew-symmetric coefficient matrices. Seismic response of gyroscopic systems has been studied by Assis and Duff [1] and Srinivazan and Soni [2], both using the time-history method. This paper determines the seismic response of a gyroscopic system by using the response spectrum method. Despite its long-standing popularity in earthquake engineering, the response spectrum method has not yet been extended to the seismic analysis of gyroscopic systems. As a start, this paper considers a low-speed rotor, one that rotates at a speed below its first critical speed.

2. Equation of Motion

Figure 1 shows a rotating dynamic system which is excited in two transverse directions simultaneously by support accelerations \( \ddot{u} \) and \( \ddot{v} \). Figure 2
Figure 1: A Rotating System Subjected to Biaxial Excitation.

Figure 2: Free-Body Diagram of An Element of the System.
shows a free-body diagram of an infinitesimal element dz. Let $\mathbf{i}$, $\mathbf{j}$ be the unit vectors of a fixed frame and $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ be the unit vectors of a moving frame, attached to the rotating system as shown in Figure 2. The absolute acceleration of dz is equal to

$$\ddot{\mathbf{x}} = \ddot{\mathbf{r}} + \omega \times (\omega \times \mathbf{r}) + \dot{\omega} \times \mathbf{r} + 2\omega \times \dot{\mathbf{r}}$$

(1)
or in orthogonal components:

$$\ddot{\mathbf{x}} = (\ddot{u} + \ddot{v} - 2\omega \dot{v} - \omega^2 \mathbf{r})_{\mathbf{i}} + (\ddot{v} + \ddot{\omega} + 2\omega \dot{\omega} - \omega^2 \mathbf{r})_{\mathbf{j}}$$

(2)

The above expression can be used to obtain the inertia force $\mu A(z)\ddot{\mathbf{x}} dz$, $\mu$ being the mass density and $A(z)$ the cross-sectional area. The viscous damping force is given by $cA(\dot{\mathbf{r}} + \omega \times \mathbf{r}) dz$, where $c$ is a damping value proportional to $\mu$, or $cA[(\dot{\mathbf{r}} - (\omega \mathbf{r}))_{\mathbf{i}} + (\dot{\mathbf{r}} + \omega \mathbf{r})_{\mathbf{j}}] dz$ in terms of orthogonal components. The equation of motion is derived as follows. It suffices to consider the $\mathbf{i}$ direction, as the result thus obtained can easily apply to the $\mathbf{j}$ direction.

Let $dM_\mathbf{i}$ be the bending moment and $dM_\mathbf{g}$ be the gyroscopic moment of the element dz. Then,

$$dM_\mathbf{g} = \mu I_d \omega^2 \frac{\partial \mathbf{r}}{\partial z}$$

(3a)

and

$$M'' + \mu \omega^2 (I_d \omega')' = V' = 0$$

(3b)

where $I_d$ is the diametric moment of inertia per unit length, $V$ is the shear force and $'$ denotes partial derivative with respect to $z$. With $M = -EI\mathbf{x}''$ and $V = \mu A(\ddot{u} + \ddot{v} - 2\omega \dot{v} - \omega^2 \mathbf{r}) + cA(\ddot{v} + \omega \mathbf{r}) = 0$, the equation for motion in the $\mathbf{i}$ direction is

$$(EI\mathbf{x}'')'' - \mu \omega^2 (I_d \omega')' + \mu A(\ddot{u} + \ddot{v} - 2\omega \dot{v} - \omega^2 \mathbf{r}) + cA(\ddot{v} + \omega \mathbf{r}) = 0.$$  (4)

Similarly, the equation for motion in the $\mathbf{j}$ direction is

$$(EI\mathbf{y}'')'' - \mu \omega^2 (I_d \omega')' + \mu A(\ddot{v} + \ddot{\omega} + 2\omega \dot{\omega} - \omega^2 \mathbf{r}) + cA(\ddot{\omega} + \omega \mathbf{r}) = 0.$$  (5)

The above two equations are to be discretized by substituting $X(z,t) = x(t)\xi(z)$ and $Y(z,t) = y(t)\xi(z)$, where $\xi(z)$ is a predetermined coordinate function which satisfies all the prescribed boundary conditions, in the equations, then by performing an inner product on the results so that

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 2\eta \Omega^2 & -2\omega \\ -2\omega & 2\eta \Omega^2 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} -2\eta \omega \Omega & \Omega^2 - \omega^2 \\ \Omega^2 - \omega^2 & -2\eta \omega \Omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\Gamma \begin{bmatrix} \ddot{u} \\ \ddot{v} \end{bmatrix}.$$  (6)

where

$$\Omega^2 = \int \left[ EI(\xi'')^2 + \mu \omega^2 (\xi')^2 \right] dz / \int \mu A (\xi')^2 dz,$$

$$\gamma = \frac{\xi}{2\mu \Omega},$$

$$\Gamma = \int A \xi dz / \int A \xi^2 dz.$$

In practice, the static deflected shape may be chosen as the coordinate function $\xi(z)$. Note that $\Omega$ is greater than the natural frequency because of the gyroscopic moment. Equation (6) is not self-adjoint. It contains gyroscopic terms

$$\begin{bmatrix} 0 & -2\omega \\ -2\omega & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & -2\eta \omega \Omega \\ 2\eta \omega \Omega & 0 \end{bmatrix}.$$
If $\ddot{u}$ and $\ddot{v}$ are given as explicit time-histories, equation (6) can be
solved by using the direct time-integration method. Here $\ddot{u}$ and $\ddot{v}$ are defined
by their floor response spectra and equation (6) is to be solved by the
spectral modal superposition method.

3. Modal Analysis

A detailed treatment of the eigenvalues and the eigenvectors, together
with their completeness and orthogonality, of a gyroscopic system can be
found in Meirovitch's works [3, 4]. The results obtained in this paper are
for subsequent modal spectral analysis of equation (6).

Consider the undamped homogeneous equation, that is, $\gamma = 0$,

$$\begin{bmatrix}
\ddot{x} \\
\ddot{y}
\end{bmatrix} + \begin{bmatrix}
0 & -2\omega \\
2\omega & 0
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} + (\Omega^2 - \omega^2) \begin{bmatrix}
x \\
y
\end{bmatrix} = 0 \quad .
$$

(7)

Let $\begin{bmatrix}
x \\
y
\end{bmatrix} = \exp(i\lambda t)p$, $p$ being a constant vector, so that equation (7)
becomes

$$\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}p = 0 \quad .
$$

(8)

Non-trivial solution of equation (8) exists if and only if

$$\det\begin{bmatrix}
\Omega^2 - \omega^2 - \lambda^2 & -2i\omega \lambda \\
2i\omega \lambda & \Omega^2 - \omega^2 - \lambda^2
\end{bmatrix} = 0 \quad ,
$$

(9)

from which it is obtained that $\lambda = \Omega \pm \omega$, each with a multiplicity of two.
Associated with the eigenvalues $\Omega + \omega$ and $\Omega - \omega$ are the one-dimensional
vector spaces spanned by the eigenvectors $(-1, 1)'$ and $(1, 1)'$ respectively.
Because these two vectors are linearly independent, they form a complete
system for equation (7). It can easily be shown that they are orthogonal
with respect to the matrices

$$\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} .
$$

Note that the system becomes unstable when $\omega = \Omega$. Here, it is assumed that
$\Omega > \omega$.

4. Seismic Response

The completeness of the eigenvectors allows the response $(x, y)'$ to be
expanded in modal components as

$$\begin{bmatrix}
x \\
y
\end{bmatrix} = f_1(t) \begin{bmatrix}
-1 \\
1
\end{bmatrix} + f_2(t) \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
f_1(t) \\
f_2(t)
\end{bmatrix} ,
$$

(10)

where $f_1(t)$ and $f_2(t)$ are projections to be determined. Substitution of
equation (10) into equation (6), with use made of the orthogonality condition,
yields the following two uncoupled equations.
\[ \ddot{f}_1 + 2(\eta \Omega + i\omega)\dot{f}_1 + (\Omega^2 - \omega^2 + 2i\eta \omega \Omega) f_1 = \frac{\gamma (-i\dot{u} - \ddot{v})}{2}, \quad (11) \]
\[ \ddot{f}_2 + 2(\eta \Omega - i\omega)\dot{f}_2 + (\Omega^2 - \omega^2 - 2i\eta \omega \Omega) f_2 = \frac{\gamma (i\dot{u} - \ddot{v})}{2}. \quad (12) \]

To solve equations (11) and (12), let \( f_1(t) \) and \( f_2(t) \) be denoted by \( f_1 = e_1(t)e^{-\eta \Omega t - i\omega t} \) and \( f_2 = e_2(t)e^{\eta \Omega t + i\omega t} \) so that equations (11) and (12) are simplified into
\[ \ddot{e}_1 + \Omega^2 e_1 = \frac{1}{2} \gamma (\Omega + i\omega) t, \quad (13) \]
\[ \ddot{e}_2 + \Omega^2 e_2 = \frac{1}{2} \gamma (\Omega - i\omega) t. \quad (14) \]

In equations (13) and (14), the damped frequency \( (1 - \eta^2)^{\frac{1}{2}} \) has

\text{been assumed to be equal to } \Omega \text{ because damping is small. The unit impulse response functions for both equations (13) and (14) are } h(t) = \sin(\Omega t / \Omega). \text{ Hence, the solutions in a convolution form for equations (13) and (14) for an at-rest initial condition are}
\[ e_1 = \frac{1}{2} \gamma h^*[(\Omega + i\omega) t], \quad (15) \]
\[ e_2 = \frac{1}{2} \gamma h^*[(\Omega - i\omega) t] = \bar{e}_1. \quad (16) \]

With use made of the above results, \( f_1 \) and \( f_2 \) are found to be
\[ f_1 = \frac{1}{2} \gamma \int \sin(\Omega t / \Omega e^{-\eta \Omega + i\omega}(t - t')^0) e^{-i\dot{u}(t - t')} dt', \quad (17) \]
\[ f_2 = \bar{f}_1. \quad (18) \]

From equation (10), it follows that
\[ x = i(-f_1 + f_2) = -c_2 H_2 \dot{u} - c_1 H_1 \dot{u} - \frac{c_2}{\Omega_2} \frac{d}{dt}(H_2 \dot{u}) + \frac{c_1}{\Omega_1} \frac{d}{dt}(H_1 \dot{u}), \quad (19) \]
\[ y = f_1 + f_2 = -c_1 H_1 \ddot{\dot{u}} - c_2 H_2 \ddot{u} - \frac{c_1}{\Omega_1} \frac{d}{dt}(H_1 \dot{u}) + \frac{c_2}{\Omega_2} \frac{d}{dt}(H_2 \dot{u}), \quad (20) \]

where
\[ H_n(t) = \frac{1}{\Omega_n} e^{-b_n \Omega_n t} \sin(\Omega_n t), \quad n = 1, 2 \]
\[ \Omega_1 = \Omega - \omega, \quad \Omega_2 = \Omega + \omega, \]
\[ c_1 = \gamma \Omega_1 / 2 \Omega, \quad c_2 = \gamma \Omega_2 / 2 \Omega, \]
\[ b_1 = \gamma \Omega_1 / \Omega_1, \quad b_2 = \gamma \Omega_2 / \Omega_2. \]

The expressions in equations (19) and (20) are obtained based on the assumption that for small damping the cosine term can be regarded as the derivative of the sine term. Since sine terms represent displacements, cosine terms may be taken as representing velocities. This being the case, the peak of the sine term and the peak of the corresponding cosine term would be approximately 90° out of phase. It is observed that
\[ |x|_{\text{max}} = |y|_{\text{max}} = \sqrt{[Mc_nS_n(b_n, \Omega_n)]^2 + [Mc_nS_v(b_n, \Omega_n)]^2}, \quad (21) \]
but \( |x|_{\text{max}} \) and \( |y|_{\text{max}} \) would be 90° out of phase with each other, according to the expressions in equations (19) and (20). In equation (21), \( S(b_n, \Omega_n) \) is the spectral displacement corresponding to a "pseudo damping factor" \( b_n \) and a frequency \( \Omega_n \) that has been shifted due to coupling of the natural frequency with the rotating speed, the symbol \( M \) denotes response combination.
which depends on the closeness of the frequencies in a manner similar to the
conventional modal combination, and u and v have been assumed to be indepen-
dent so that their effects are combined vectorially. Finally, the maximum
radial displacement is estimated as
\[
\| \phi \|_{\text{max}} = \sqrt{\max (x^2 + y^2)} = |x|_{\text{max}} = |y|_{\text{max}}.
\]
(20)
Compared with a classical dynamic system, response combination for a gyro-
scopic system is much more complicated. Little has been reported in this
area, so further study will be required in order that guidelines can be
established. This, however, is not in the scope of this paper and is important
even enough to warrant a separate study.

5. Conclusion

It is shown that the response spectrum method can be extended to the
seismic analysis of a gyroscopic system. There are some differences between
the analysis of a gyroscopic system and the analysis of a classical dynamic
system that are worthy of note. First of all, even when the gyroscopic system
is undamped, its equation of motion still cannot be uncoupled by the classical
normal modes. Secondly, there is coupling between the natural frequencies and
the rotating speed, suggesting that the gyroscopic effects should not always
be neglected. Thirdly, response combination is much more complicated for a
gyroscopic system and more studies in this area are needed before guidelines
on response combination can be laid down.

6. References

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