A Constitutive Model of Cyclic Plasticity in Multi axial Non-Proportional Loading

E. Tanaka, S. Murakami, M. Ooka
Nagoya University, Dept. of Mechanical Engineering, Chikusa-ku, Nagoya 464, Japan

Abstract

A novel constitutive model of cyclic plasticity under general states of loading is developed on the basis of the previous experiments of the present authors. In view of the information available from the experiments, we first introduce a proportional and a non-proportional non-hardening region in plastic strain space to describe the characteristic features of cyclic hardening behavior; proportional or non-proportional cycles are assumed to induce no isotropic hardening in these non-hardening regions. Then, by incorporating the hardening parameters defined by these regions, the stress-plastic strain relation is formulated by modifying the two surface model in advanced flow theories of plasticity. The proposed model describes the cyclic behavior not only for proportional and circular cycles of constant strain amplitude, but also that for a series of these cycles of different strain amplitudes.

1. Introduction

A number of constitutive models of cyclic plasticity have been developed by various methods thus far. Some of them [1-5] were formulated by modifying the classical flow theory of plasticity based on the combined isotropic-kinematic hardening rule, and succeeded in describing the cyclic behavior of proportional loadings with sufficient accuracy [5]. In this progress of constitutive modelling of cyclic plasticity, in particular, the notion of two-surface plasticity [1-2] and that of non-hardening strain region [4] played important roles.

No models mentioned above, however, could describe the marked cyclic hardening of non-proportional loadings, because they all were formulated on the basis of uniaxial cyclic experiments, and the microstructural mechanisms of cyclic hardening under non-proportional loadings differ essentially from those under uniaxial ones.

One of the most rational methods to formulate the model of non-proportional cyclic plasticity will be to establish appropriate internal variables and the corresponding evolution equations which can describe the predominant microstructural changes caused by non-proportional cyclic loadings, and then to incorporate them into the advanced flow theories of plasticity mentioned above. In order to facilitate this approach, the present authors recently performed a series of cyclic tests [6-8], and elucidated the essential features of cyclic hardening under proportional and non-proportional loadings.

The present paper is concerned with the formulation of a constitutive model applicable to general multi-axial cyclic loadings on the basis of the information available from these tests.
2. **Characteristic Features of Cyclic Hardening**

In References [6-8], the present authors performed a series of plastic strain controlled cyclic tests on type 316 stainless steel tubes at room temperature under combined axial force and torque. Several representative strain paths including proportional, cruciform, starlate in eight directions, square and circular path were chosen in a plastic strain space \( (\varepsilon^P, \gamma^P/\sqrt{3}) \) of von Mises type, where \( \varepsilon^P \) and \( \gamma^P \) denote axial and torsional plastic strain; the tests were carried out for different histories of these paths as well as of various magnitudes of strain amplitude.

The information available from these tests is summarized as follows:

1. There exists a cyclic hardening mechanism for non-proportional loadings besides that for proportional loadings, and the cyclic hardening due to the former mechanism is more salient when the direction of the plastic strain rate vector deviates more significantly from that of the plastic strain.

2. There exists a certain region in the plastic strain space inside which arbitrary proportional loadings induce no isotropic hardening. At the saturated state of cyclic hardening under proportional cycles with fixed strain limits, the size of this region is equal to the plastic strain amplitude of the cycles. We approximate this region by a sphere and call it a proportional non-hardening region, hereafter (see Fig. 1).

3. There also exists a region inside which arbitrary proportional and non-proportional loadings induce no isotropic hardening. At the saturated state of cyclic hardening of circular cycles, the size of this region is equal to the equivalent plastic strain amplitude of the cycles. We again approximate this region by another sphere and call it a non-proportional non-hardening region, hereafter. This region naturally is contained inside the proportional non-hardening region mentioned above (see Fig. 1).

4. The cyclic hardening can be described by a single scalar isotropic hardening variable. This variable and the sizes of these two non-hardening regions are related uniquely to one another.

3. **Formulation of a Constitutive Model**

3.1 **Non-Hardening Regions**

Let us first express the non-proportional and the proportional non-hardening region introduced in the preceding section by the following equations (see Fig. 1):

\[
\begin{align*}
q_N &= (2/3)(\varepsilon^P_{i+} - a_{i+}) - 2/3 \rho_N \leq 0, \quad q_P &= (2/3)(\varepsilon^P_{i+} - a_{i+}) - 2/3 \rho_P \leq 0
\end{align*}
\]

where \( \varepsilon^P_{i+} \), \( a_{i+} \), and \( \rho \) denote a plastic strain, the center of the non-proportional and the proportional non-hardening region, respectively, and \( q_N \) and \( q_P \) are the sizes of the regions.

We now formulate the evolution equation for \( a_{i+} \). The non-proportional non-hardening region moves and/or expands only when the plastic strain point \( \varepsilon^P_{i+} \) is located on the surface \( q_N = 0 \) and moves outwards. Then, we introduce the following function to describe the motion of this region:

\[
\Gamma = <(2/3)\varepsilon^P_{i+} / \varepsilon^P_{EQ}>, \quad \Gamma_N = 0 \quad \text{when} \quad q_N < 0
\]

where \( \varepsilon^P_{EQ} \) denotes a equivalent plastic strain rate, and \( \varepsilon^P_{EQ} = [(2/3)\varepsilon^P_{i+}]^{1/2} \), \( a_{i+} = (\varepsilon^P_{i+} - a_{i+})/\rho_N \)

In eq. (3), furthermore, the symbol < > is the Macaulay bracket defining the operation \( <L> = - 72 - 

L 2/4
L.H(L), where H(L) is the Heaviside unit function, with H(0) = 0.

As mentioned previously, the non-proportional non-hardening region is located inside the proportional non-hardening region (see Fig. 1), and the two regions never intersect. Hence, we assume that \( \alpha_{ij} \) moves in conformity with MKOZ's kinematic rule [9]. Let us denote by \( \varepsilon_{ij}^P \)
the point on the surface \( \varepsilon_{ij} = 0 \) having the same external normal at \( \varepsilon_{ij}^P \) on the surface \( \varepsilon_{ij} = 0 \):

\[
\varepsilon_{ij}^P = \beta_{ij} + (p_{ij}/p_{ij}^0) (\varepsilon_{ij} - \alpha_{ij})
\]

(6)

Thus, if we introduce the normalized tensor \( n_{ij}^m \) of \( \varepsilon_{ij}^P - \varepsilon_{ij} \):

\[
n_{ij}^m = (\varepsilon_{ij}^P - \varepsilon_{ij})/\delta, \quad \delta = [(2/3)(\varepsilon_{ij}^P - \varepsilon_{ij})(\varepsilon_{ij}^P - \varepsilon_{ij})]^{1/2}
\]

(7),(8)

and use the consistency condition of \( \varepsilon_{ij} = 0 \), we obtain the evolution equation for \( \alpha_{ij} \):

\[
\dot{\alpha}_{ij} = (\Gamma_{ij}^P g_{ij}) - (n_{ij}^m/(2/3)n_{ij}^m n_{ij}^m)
\]

(9)

It is noted that if the point \( e_{ij}^P \) coincides with the point \( e_{ij}^P \), then eq. (9) becomes indeterminate. This difficulty can be overcome by formulating the evolution equation for \( \beta_{ij} \) in such a way that the two points \( e_{ij}^P \) and \( e_{ij}^P \) never coincide. For this purpose, we first assume that the point \( e_{ij}^P \) moves only when the normal component \( \delta_N \) of the tensor \( \varepsilon_{ij}^P - \varepsilon_{ij} \) on the surface \( \varepsilon_{ij} = 0 \) is equal to a certain small constant \( \delta_0 \). If the point \( e_{ij}^P \) moves toward the outside of the surface \( \varepsilon_{ij} = 0 \). If we further assume that the point \( e_{ij}^P \) moves in the direction of \( n_{ij}^m \) at the rate equal to the normal component of \( \varepsilon_{ij}^P \) on the surface \( \varepsilon_{ij} = 0 \), then \( e_{ij}^P \) can be expressed in the form

\[
\varepsilon_{ij}^P = \Gamma_{ij}^P g_{ij}, \quad \delta_N = (2/3)n_{ij}^m n_{ij}^m
\]

(10),(11)

\[
\Gamma_{ij} = (2/3)n_{ij}^m e_{ij}^P \quad \text{when} \quad \delta_N = \delta_0, \quad \Gamma_{ij} = 0 \quad \text{when} \quad \delta_N > \delta_0
\]

(12)

Finally, the assumption that \( \beta_{ij} \) moves in the direction of \( n_{ij}^m \), and the use of the consistency condition of the surface \( g_{ij}^P \), \( \beta_{ij} \), \( p_{ij}^0 \) lead to

\[
\dot{\beta}_{ij} = (\Gamma_{ij}^P g_{ij}) - (p_{ij}^0 n_{ij}^m)
\]

(13)

We are now in a position to formulate the evolution equations for the radii \( p_N \) and \( p_P \) of the two non-hardening regions. As mentioned previously, when the plastic strain point \( e_{ij}^P \) is located inside the two regions (i.e., \( q_N < 0 \) and \( \delta_N > \delta_0 \), \( p_N \) and \( p_P \) do not change. Hence, we first discuss the case of \( q_N = 0 \) and \( \delta_N > \delta_0 \) and assume that only non-proportional loadings induce cyclic hardening in this case. According to the results in the preceding section, the cyclic hardening can be induced only when the plastic strain point \( e_{ij}^P \) moves toward the outside of the surface \( \varepsilon_{ij} = 0 \), and is more significant with increasing degree of non-proportionality between the plastic strain and the plastic strain rate. Thus, we assume that \( p_N \) is proportional to the component of \( e_{ij}^P \) projected to the tangential plane at \( e_{ij}^P \):

\[
p_N = c_N e_{ij}^P g_{ij}^{*P}
\]

(14)

\[
\Gamma_{ij} = D(1 - q_N) = R(2/3)e_{ij}^P n_{ij}^m \quad \text{when} \quad q_N = 0, \quad \Gamma_{ij} = 0 \quad \text{when} \quad q_N < 0
\]

(15)

where \( c_N \) is a material constant, and \( D \) is a function describing the saturation of cyclic hardening under non-proportional loadings; this function will be defined later.

As regards the evolution equation for \( p_P \), let us consider only the mechanism of cyclic hardening induced by proportional loadings; this mechanism can operate in the case of \( q_N = 0 \) and \( \delta_N = \delta_0 \). As proposed by OHNO [4], the cyclic hardening induced by this loading develops when the plastic point \( e_{ij}^P \) moves toward the outside of the proportional non-hardening region,
and the magnitude of hardening is proportional to the normal component of $\dot{\varepsilon}_{ij}$ on this surface:

$$\dot{\rho}_P = c_P \dot{\varepsilon}_{ij}^N \varepsilon_{eq}^P$$

where $c_P$ is a material constant within the range from zero to unity.

Now, as mentioned previously, there exists the unique relation between the sizes of the two non-hardening regions. Hence, between $\rho_N$ and $\rho_P$ holds the relation

$$\rho_N = r(\rho_P) \text{ or } \rho_P = s(\rho_N), \quad (d\rho_P/ds(\rho_N))(ds(\rho_N)/d\rho_N) = 1$$

By combining this relation with eqs. (14) and (16), we can express $\dot{\rho}_N$ and $\dot{\rho}_P$ in the form

$$\dot{\rho}_N = [c_{N} \Gamma_T + (d\rho_P/ds(\rho_N))(c_P \Gamma_T + c_{P} \Gamma_T^P)] \varepsilon_{eq}^P, \quad \dot{\rho}_P = [(ds/d\rho_N)(c_{N} \Gamma_T + c_{P} \Gamma_T^P)] \varepsilon_{eq}^P$$

Finally, we need to determine the function $D$ in eq. (15). In order to describe the cyclic hardening-saturation process under the circular cycles, we chose the relation

$$D = \exp\{-b_2(1/b_1 \chi - 1)\}$$

where $b_1$ and $b_2$ are material constants, and $\chi$ denotes an internal variable decreasing with progression of the cyclic hardening. Since the center $\alpha_{ij}$ moves slower with progression of the cyclic hardening, the evolution equation for $\chi$ was determined in the form

$$\dot{\chi} = [(2/3)\alpha_{ij} \alpha_{ij}^P]^{1/2} - b_1 \varepsilon_{eq}^P$$

3.2 Stress-Plastic Strain Relation

Now, let us modify the two surface plasticity model of KRIEG [1] by incorporating the non-hardening regions developed in Section 2. This modification can be accomplished by the same procedure as OHNO's [5]. We first express the loading surface $\vec{F} = 0$ and the bounding surface $\vec{F} = 0$ in the form

$$f = (3/2)(s_{ij} - R_{ij}) (s_{ij} - R_{ij}) - \kappa^2, \quad \bar{f} = (3/2)(s_{ij} - \bar{R}_{ij}) (s_{ij} - \bar{R}_{ij}) - \bar{\kappa}^2$$

where $s_{ij}$ and $R_{ij}$ are a deviatoric stress and the center of loading surface, and $\kappa$ denotes the size. Moreover, $R_{ij}$ and $\bar{\kappa}$ are the center and the size of the bounding surface, respectively. Figure 2 shows the configuration of these two surfaces. Then, a deviatoric stress $\vec{S}_{ij}$ on the surface $\vec{F} = 0$ is defined when the stress point $s_{ij}$ is located on the surface $f = 0$, and the external normal at $\vec{S}_{ij}$ is the same as $s_{ij}$ on the surface $\bar{f} = 0$. That is,

$$\vec{S}_{ij} = \bar{R}_{ij} + (s_{ij} - R_{ij})(\bar{\kappa}/\kappa)$$

In general, an elastic region may expand as plastic deformation proceeds. However, we can consider this effect by introducing a bounding surface. Hence, we assume that $\kappa$ is a constant value $\kappa_0$. According to the aforementioned observations, $\bar{\kappa}$ is uniquely connected with the sizes $\rho_N$ and $\rho_P$ of the two non-hardening regions, and is described by a single isotropic hardening variable $q$. Then, we can choose the function $q = q(\rho_N)$ or $q(\rho_P)$ arbitrarily, because we can consider this effect when we characterize the function

$$\bar{f} = \bar{f}(q)$$

Thus, we assume

$$\dot{q} = \dot{\rho}_P/c_P = [(ds/d\rho_N)/c_P] \dot{\rho}_N = [(c_{N}/c_P)(ds/d\rho_N)^{\Gamma_T} + \Gamma_T^P \varepsilon_{eq}^P$$

In order to specify the relation between plastic strain rate and stress rate, we must further establish the evolution equation for $R_{ij}$. In a plastic state, a stress point is located on the loading surface, and moves outwards. Then, the loading surface moves inside the
bounding surface, but the two surfaces never intersect. Hence, we assume a rule for \( R_{ij} \)

\[
R_{ij} = \frac{A_{ij}}{\bar{R}_{ij} - s_{ij}}
\]  

(27)

where \( A \) is a material function; in the present study, we adopted the form

\[
A = a_1 + a_2 (1 - (3/2) \bar{s}_{ij} \bar{m}_{ij})
\]  

(28)

In the above equation, \( a_1 \) and \( a_2 \) are material constants, and \( \bar{s}_{ij} \) and \( \bar{m}_{ij} \) denote the external normal on \( f = 0 \) and the normalized tensor of \( \bar{R}_{ij} - s_{ij} \):

\[
\bar{s}_{ij} = (s_{ij} - R_{ij})/\bar{r}_{ij}, \quad \bar{m}_{ij} = (s_{ij} - R_{ij})/\Delta, \quad \Delta = [(3/2) (s_{ij} - s_{ij})(s_{ij} - s_{ij})]^{1/2}
\]  

(29), (30), (31)

If we assume the associated flow rule and use the consistency condition of the loading surface together with eqs. (27) and (24), we obtain

\[
e_{ij}^P = (3/2) (L/\bar{r}_{ij}) \bar{m}_{ij}, \quad H_0 = (3/2) \bar{m}_{ij} \bar{R}_{ij} + (\bar{k} - \bar{k}_0),
\]  

(32), (33)

\[
R_{ij} = \frac{A_{ij}}{\bar{R}_{ij} - s_{ij} + (\bar{k} - \bar{k}_0) \bar{m}_{ij}},
\]  

(34)

\[
L = (3/2) e_{ij}^P \bar{m}_{ij} \]  

when \( f = 0 \), \( L = 0 \) when \( f < 0 \)

(35)

Let us now specify the evolution equation for the center \( \bar{R}_{ij} \) of the bounding surface. For this purpose, we assume the following form for \( \bar{R}_{ij} \) according to OHNO [4]:

\[
\frac{\dot{R}_{ij}}{R_{ij}} = \frac{1}{(2/3)} \frac{\dot{e}_{ij}^R}{\bar{e}_{ij}^R} - k \frac{\dot{R}_{ij}}{R_{ij}} e_{ij}^P
\]  

(36)

where the second term on the right hand side of the above equation is a term introduced to describe the cyclic relaxation, and \( k \) is a material constant. We further assume that \( \bar{X} \) is divided into the sum of the constant part \( k \) and the variable part \( \bar{X} \) which is specified by the relation

\[
(2/3) (\Delta e_{ij}^R) \bar{R}_{ij}^P = (\Delta t/\mathbf{q}) (e_{ij}^P - \mathbf{q})
\]  

(37)

From eq. (37), \( X \) is obtained, and substitution of this into eq. (36) yields the relation

\[
\dot{R}_{ij} = (2/3) k \bar{e}_{ij}^E + (2/3) (\Delta c/\mathbf{q}) (1 - (c_p/c_p) (ds/\mathbf{d}) T_{ij}^* - T_{ij}) e^P - k \bar{R}_{ij} e^P
\]  

(38)

Finally, we specify the relation between the bounding stress rate and the plastic strain rate. If we introduce the external normal \( \bar{n}_{ij} \) at \( s_{ij} \) on \( \bar{f} = 0 \), i.e.,

\[
\bar{n}_{ij} = (\bar{s}_{ij} - \bar{R}_{ij})/\bar{k}
\]  

(39)

then, \( \bar{e}_{ij}^P \) is in the direction of \( \bar{m}_{ij} \). By use of this relation and the consistency condition of the surface \( \bar{f} = 0 \) together with eqs. (25), (26) and (38), we obtain

\[
\dot{e}_{ij}^P = (3 \bar{L}/2 \bar{E}) \bar{m}_{ij}^b, \quad e^P = k + (\Delta c/\mathbf{q}) (1 - (3/2) k \bar{m}_{ij}^b \bar{R}_{ij}^P)
\]  

(40), (41)

\[
\bar{L} = (3/2) \bar{e}_{ij}^P \bar{m}_{ij} \]  

when \( f = 0 \), \( \bar{L} = 0 \) when \( f < 0 \)

(42)

3.3 Material Functions and Material Constants

The proposed constitutive model involves two material functions \( \bar{K} = \bar{K}(q) \), \( \bar{c}_p = s(c_p) \), and ten material constants \( \bar{c}_0 \), \( \bar{k} \), \( \bar{c}_p \), \( \bar{c}_p \), \( \bar{c}_p \), \( \bar{a}_1 \), \( \bar{b}_1 \), \( \bar{b}_2 \) and \( \bar{c}_0 \). Since this model coincides with OHNO's model [5] in the case of simple cyclic loadings, the material function \( \bar{K}(q) \) is determined from the stress-plastic strain curve under monotonic simple loading and the saturated stress amplitude versus plastic strain amplitude relation under proportional cyclic loadings, similar to OHNO's model [5]. In the present study, the function
\[ \tilde{k}(q) = \tilde{k}_0 \left( \frac{\lambda}{m \tilde{k}_0} \right)^m \]  

(43)

was adopted, where \( \tilde{k}_0 \), \( \lambda \) and \( m \) are material constants.

The function \( s(\rho_N) \), on the other hand, can be determined on the basis of the relation \( \sigma^s(\rho_N) \) between the saturated stress amplitude \( \sigma^s \) and the equivalent plastic strain amplitude \( \Delta \varepsilon^p/2 \) (for \( \Delta \varepsilon^p/2 = \rho_N \) in this case) under circular cycles. The result is

\[ s(\rho_N) = \left( \frac{\sigma^s(\rho_N)}{\tilde{k}_0} \right)^{1/m} \left( m \tilde{k}_0 c_p/\lambda \right) \]  

(44)

The other material constants are also determined from the experimental results mentioned above.

4. Comparison of Theoretical Predictions with Experimental Results

Let us now show the results simulated by the proposed model. Figure 3 shows the stress-plastic strain diagram for the torsional cycles of the plastic strain amplitude of 0.2 %. We see that the theoretical predictions are very similar to the experimental results. The predictions and the experimental results of the stress path for the circular cycles of the equivalent plastic strain amplitude of 0.2 % are shown in Fig. 4. The simulation of the present models is again very satisfactory. Figure 5, on the other hand, shows the relations between the stress amplitude and the accumulated plastic strain for the torsional and the circular cycles; both amplitudes are changed from 0.1 % to 0.2 %, and from 0.2 % to 0.4 %. The theory simulates the experimental results with relatively high accuracy.

References

Fig. 1 Schematic representation of the proportional and non-proportional non-hardening regions

Fig. 2 Schematic representation of the loading and bounding surfaces

Fig. 3 Stress-plastic strain diagram for torsional cycles

Fig. 4 Stress path for circular cycles

Fig. 5 Cyclic hardening curves for torsional and circular cycles