An Asymptotic Solution Procedure for Power Law Creep and Non-Linear Deformation Problems

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Abstract

A non-dimensional parameter, having a critical value of unity, has been introduced for use when material deformations such as creep rates are defined as a function of the equivalent shear stress. For power law creep, for example, its value depends on the creep index n and it is then given by (n-1)/n. The variation of the stresses in a creeping structure is substantially linear with this parameter, so long as its critical value is not too closely approached. Use has been made of the above concepts to arrive at solutions to creep problems for conditions which are sufficiently far removed from those critical situations which may physically be related, in limit load terms, to the onset of structural collapse.

1. Introduction

The stress distribution in a structure depends on the creep law relating the strain rate to the effective stress and a parameter (M^2) has been introduced in which to express this. For a non-linearly viscous material M^2 is (n-1)/n; more generally, M^2 measures the extent to which the material behaviour is non-linear ($0 < M < 1$)and an expression for determining M^2 is given in the Appendix. Conditions at a point may be envisaged in terms of the local value of M, or the latter at a point of reference. For power law creep these two values are the same.

2. Asymptotic Stress Solutions

So long as conditions are sufficiently removed from those which make M = 1, the state of stress may be envisaged as a series in terms of powers of M^2. For power law creep the normalised value of the stress $T = \sigma/\sigma_N$ may thus be expressed if h is written for (n-1)/n:

$$T(h) = T^o + h\,T^I + h^2\,T^{II} + \dots$$

There is evidence to suggest that the effects of the higher derivatives of T with respect to h are normally small and a useful approximation may be obtained as:

$$T = T^{o} + h\,T^{I} + O(h^2)$$

A corresponding expression for the normalised strain rates $\dot{S} = \dot{\varepsilon}/\dot{\varepsilon}_N$ is:

$$\dot{S}(h) = \dot{S}^{o} + h\,\dot{S}^{I} + h^2\dot{S}^{II} + \ldots \qquad (2)$$

When envisaged as a Maclaurin series:

$$\dot{S}(h) = \dot{S}(o) + h\,\dot{S}'(o) + \frac{1}{2!}\,h^2\,\dot{S}''(o) + \ldots$$

It therefore follows that $\dot{S}^{o}$ and $\dot{S}^{I}$ etc may be derived from:

$$\dot{S}(h) = T^{1/(1-h)}, \qquad \dot{S}'(h) = d\left(T^{1/(1-h)}\right)/dh, \qquad \ldots$$

and it is thus found that:

$$\dot{S}^{o} = T^{o}, \qquad \dot{S}^{I} = T^{I} + T^{o}\log T^{o}, \qquad \ldots$$

which are equivalent to expressions derived by Calladine (1).

3. Calladine's Method

For boundary conditions which correspond, Calladine pointed out that the creep stress distribution is identical to that for an analogous non-linear elastic material; $\dot{S}$ is then construed in terms of the von Mises effective strain S. Since T^{o} is the solution for h = o, it is the stress which would arise in a linearly elastic structure. Noting that the gradient of T with respect to h when the latter tends to zero is the same as dT/dn when n = 1, use may be made of Calladine's observations concerning the latter. These lead to recognising $S^{I}(T^{I})$ as a linear elastic material relationship, with an initial strain at each point as defined by $S^{(i)} = T^{o}\log T^{o}$. The implied lack of fit provides a source of self straining. The self-equilibrating stresses so arising define T^{I} (and hence the stress re-distribution to the order of h) as the solution of a linear-elastic self-straining problem. Calladine considered an analysis of that sort by no means trivial but commensurately difficult thermo-elastic problems are now routinely solved by computer-aided methods; the lack of fit which $S^{(i)}$ defines is one of shape rather than size however. For uniaxial problems the relevant values of S and T are their moduli; taking account of correspondences between signs it is then found that:

$$S^{o} = T^{o}, \qquad S^{I} = T^{I} + S^{(i)}, \qquad S^{(i)} = T^{o}\log\left|T^{o}\right|$$

4. An Illustrative Example

Fig 1 shows a rectangular beam subjected to a moment M; at a distance y above the neutral axis σ^o is My/I. Noting that T^o is thus proportional to y, ky say, the equivalent thermal strain is proportional to y log ky. The requirement for plane sections to remain plane, together with the condition that the couple arising from the self-equilibrating 'thermal' stresses must be zero, enables the latter to be found:

$$\sigma^I = -\sigma^o\ (\log|y/Y| + 1/3) \qquad \ldots\ \sigma^o = My/I.$$

see fig 1. The maximum stress so obtained is such that:

$$\sigma_{max}/\sigma^o = 1 - h/3 = (2n + 1)/3n$$

It is the same expression as results from integrating the power law relationship see, for example, Boyle and Spence(2). This problem is one for which the maximum stress does, in fact, vary linearly with h, and hence with $m = n^{-1}$, see fig 2. More generally, so long as h is sufficiently far from the value unity (at which $n \to \infty$, which is a singular case) the errors are generally found to be acceptable and on the safe side, exceptionally the streses being underestimated by ~5% for $3<n<5$. The corresponding strain rates may be estimated from the specified power law relationship. Because of the non-linear nature of the latter, the variation of $\dot{\varepsilon}$ with h is non-linear - except for values of h which tend to zero, so that n ~1.

5. Structural Deflections

Similar considerations apply to the rates of displacement at each point in a structure as apply to the strain rates. It is only for values of h which tend to zero that accurate solutions are obtainable from:

$$q \approx q^o + hq^I$$

Use may be made of skeletal or, more generally, reference stress concepts to arrive at deflections q which are valid for a wider range of h or n values. For the illustrative problem considered there is a point at which the stress remains independent of the value of h (or n) according to the asymptotic stress solution. To the order of h, it is such that $\sigma^I = 0$, see fig 1; the corresponding value of y is given by:

$$\log|y/Y| = -1/3$$

This skeletal stress causes a creep rate which may be used to infer the corresponding rate of curvature. Use may be made of reference stresses for the same purpose see, for example, Anderson et al (3). For simple problems the skeletal stress is the reference stress. The latter may be defined by T_R in:

$$q = f\ S\ (T_R) \tag{5}$$

for the deflection q in an equivalent non-linear elastic problem. The dependence of T_R on h may be minimised through requiring, for $T_R(h)$, that $T_R{}^I = T_R'$ (o) be zero. Equating terms in:

$$q^o + hq^I + \ldots = \quad f(S^o + hS^I + \ldots)_R$$

with the above definition of the reference stress:

$$q^o = fS_R^o = f\ T_R{}^o \qquad \ldots \quad T_R{}^o = T_R$$

$$q^I = fS^I_R \ = \ f(T^I_R \ + \ T^o_R \log T^o_R\ ) \ \ldots \quad T^I_R = 0$$

to the degree of accuracy implied. The reference stress so defined is such that:

$$\log T_R = q^I/q^o$$

in which both q^o and q^I are presumed known; this may also be used to deduce the factor:

$$f = q^o/T_R \tag{7}$$

For the beam problem considered, for example, it is found from $q^o = \kappa^o$, $q^I = \kappa^I$ that:

$$\sigma_R = (MY/I)\ \exp\ (-1/3)$$

which is the same as the stress at the skeletal point. More generally, T_R and f may be derived from q^O and q^I to deduce a solution for q from expression(5). It is thus found for a rectangular beam problem that if $\varepsilon = B\sigma|\sigma|^{n-1}$, then:

$$\kappa = F(n)(Y^{n-1}/I^n)\ BM|M|^{n-1}$$

This is of the same form as the exact solution, see for example Anderson et al (3). The deflections are however over-estimated (by between 9 and 24% of the range $3 < n < 5$), since the reference stress solution predicts:

$$F(n) = (\exp - 1/3)^{n-1}$$

instead of: $F(n) = (1 + 1/2n)^n\ (2/3)^n$

6. Conclusions

Identified needs include requirements to refine and simplify current methods of analysis. An important area, for nuclear power plant operating at elevated temperatures, is concerned with creep and related non-linear deformations. The latter have been envisaged as a function of a parameter $M^2 = (n-1)/n$ for power law creep. This has been used as a basis for solutions to a degree of accuracy which, in most cases, is higher than the variability commonly found among the relevant materials data. The objective has been towards reducing dependence on over-restrictive 'elastic design routes' on the one hand without giving rise to difficult inelastic analyses as an alternative.

7. References

1. CALLADINE, C R - "A rapid method for estimating the greatest stress in a structure subject to creep". Proc Instn Mech Engrs, Vol 178, Pt 3L, 198-206 1963-1964.

2. BOYLE, J T, SPENCE, J - "Stress analysis for creep". Butterworths, (London 1983).

3. ANDERSON R G, GARDNER, L R T, HODGKINS W R, "Deformations of uniformly loaded beams obeying complex creep laws". J Mech Eng Sci. 5 238-244 (1963).

8. Appendix - Definition of the parameter M

The rate of energy dissipation per unit volume W, for a shear strain rate $\dot{\gamma}$ and a non-linear viscosity μ (which may be expressed either in terms of $\dot{\gamma}$ or the shear stress $\tau = \mu\dot{\gamma}$) is such that: $dW = \mu\dot{\gamma}d\dot{\gamma}$

This is analogous in form to an expression used in the analysis of compressible flows - the so-called Bernoulli equation for which $-dp = \rho u du$. The density ρ in the latter case may be expressed, either as a function of the velocity u, or of the mass flow per unit area in $m = \rho u$. For such problems it is meaningful to introduce a Mach number = u/a, in which a is the velocity of sound, namely $\sqrt{(dp/d\rho)}$. Equivalent parameters for non-linear creep are:

$$a^2 = -\frac{dW}{d\mu} = -\mu\dot{\gamma}\,\frac{d\dot{\gamma}}{d\mu}$$

and $M = \dot{\gamma}/a$, which may thus be expressed, since $\tau = \mu\dot{\gamma}$, as:

$$M^2 = 1 - \frac{\dot{\gamma}}{\tau}\,\frac{d\tau}{d\dot{\gamma}}$$

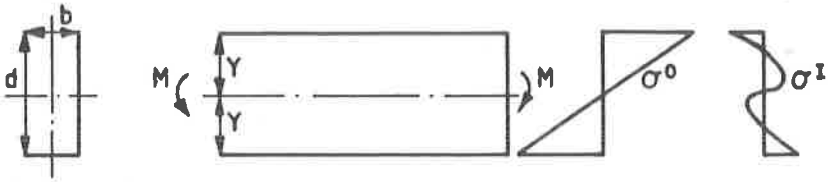

Fig 1 - BENDING OF A NON-LINEAR BEAM

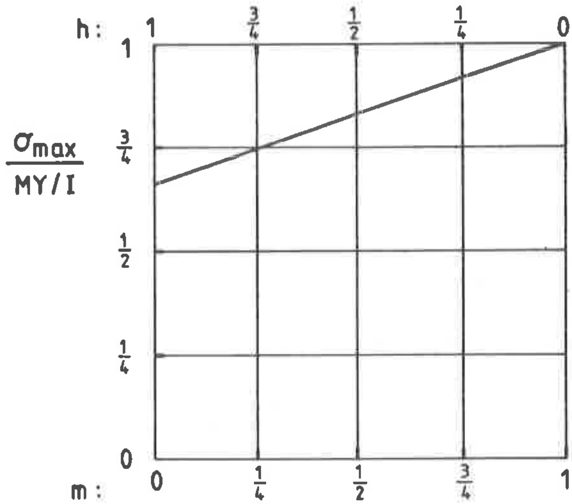

Fig 2 - MAXIMUM BENDING STRESS FOR A RECTANGULAR SECTION