A Yield Condition for Anisotropic Materials

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ABSTRACT

On the basis of the notion of local reference configuration the total compatible deformation has been divided into incompatible elastic and plastic parts. A yield function whose arguments are Kirchhoff stress defined with regard to local reference configuration as well as plastic local deformation gradient has been defined. Isotropy group has been stated and applied to this yield function and the list of relevant invariants has been explicitly shown for isotropic and transversely isotropic materials. Some special cases have been given and Bauschinger effect has been explained by means of the residual stress. Then this yield function has been reformulated in order to take into account permanent deformation history effects. Some special cases of the new yield function have also been considered and consequences to the set of constitutive relations have been briefly mentioned.

1. Introduction

In order to formulate a constitutive theory of anisotropic elastic-plastic solids it is necessary to have the corresponding yield condition. In papers dealing with a yield condition for anisotropic materials the following assumptions have been introduced: (a) a factor of transverse isotropy is taken into account in |1, 2|, (b) the vector of transverse isotropy direction is used in |3| and (c) an explicit yield function for materials with orthorombic symmetry is formulated in |4|. Nevertheless, in |1,2,3,4| no attention is paid to a possible connexion between isotropy group and yield function. On the other hand, in |5| a yield condition is expressed by Kirchhoff stress related to the global reference configuration without special reference to isotropy group.

The aim of this paper is to formulate a yield condition for anisotropic materials by making use of isotropy group and the concept of local varying natural state reference configuration.

2. A Geometry of Plastic Deformation

Consider an elastoplastically deformed body, $\mathcal{B}$, represented by continuous configurations $\chi_t(\mathcal{B})$ changing in euclidean three-dimensional space, E_3, with time $t \in R$. To each of these configurations there corresponds a continuous (global) natural state reference configuration, $\kappa(\mathcal{B}) \subset E_3$, serving for identification of particles, X, usually by means of a vector field $\underset{\sim}{X}$. Define an infinitesimal open ball in κ by d - neighbourhood of point X:

$$\mathcal{N}(X) = \{Z : |\underset{\sim}{Z} - \underset{\sim}{X}| < d\} \quad , \qquad d \in R^+ \quad , \qquad \mathcal{N}(X) \subset \kappa(\mathcal{B}) \ . \tag{2.1}$$

Besides κ and χ_t it is useful to introduce another two configurations: $\nu_t(\mathcal{N})$ and ${}_o\chi_t(\mathcal{B})$. First of them is a local natural state reference configuration obtainable by an imagined cutting of $\mathcal{N}(X)$ out of $\chi_t(\mathcal{B})$ and a complete relaxation of $\mathcal{N}(X)$. In this way the local material isomorphism between $\nu_t(\mathcal{N})$ and $\kappa(\mathcal{B})$ would be reached. The second, ${}_o\chi_t(\mathcal{B}) \subset E_3$, is partially relaxed since it possesses residual stresses, ${}_{res}T$, while external stresses, ${}_{ext}T$, caused by surface tractions and body forces are relaxed since these are removed.

For any two points in $\mathcal{N}(X)$ their relative positions (of Figure 1) are denoted by (notations for configurations and their associated normed vector spaces are tacitly identified)

$$d\underset{\sim}{X} \in \kappa(\mathcal{B}) \ , \quad d\underset{\sim}{\xi} \in \nu_t(\mathcal{N}) \ , \quad d_o\underset{\sim}{x} \in {}_o\chi_t(\mathcal{B}) \quad , \quad d\underset{\sim}{x} \in \chi_t(\mathcal{B}) \ . \tag{2.2}$$

Accordingly, introducing material base vectors $\underset{\sim}{G}_K$ in κ, structural $\underset{\sim}{h}_\alpha$ in ν_t as well as spatial base vectors $\underset{\sim}{g}_k$ in χ_t and ${}_o\chi_t$ the following mapping functions are defined |6|:

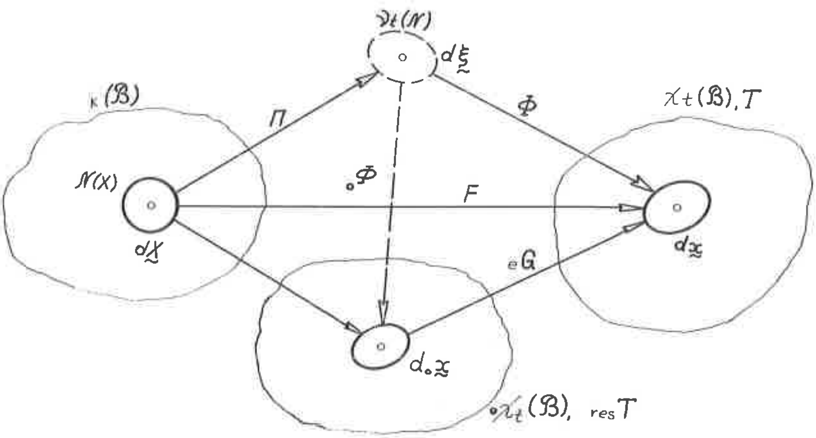

Figure 1. Basic configurations and mappings

$$\Pi : \kappa \to \nu_t, \quad d\underset{\sim}{\xi} = \Pi d\underset{\sim}{X}, \quad (d\xi)^\kappa = \Pi^\kappa_K dX^K, \tag{2.3}$$

$$\Phi : \nu_t \to \chi_t, \quad d\underset{\sim}{x} = \Phi d\underset{\sim}{\xi}, \quad dx^k = \Phi^k_\kappa (d\xi)^\kappa \tag{2.4}$$

$$F : \kappa \to \chi_t, \quad d\underset{\sim}{x} = F d\underset{\sim}{X}, \quad dx^k = F^k_K dX^K, \tag{2.5}$$

$$^eG : {}_o\chi_t \to \chi_t, \quad d\underset{\sim}{x} = {}^eG d_o\underset{\sim}{x}, \quad dx^k = {}^eG^k_{.\ell} d_o x^\ell, \tag{2.6}$$

$$^pG : \kappa \to {}_o\chi_t, \quad d_o\underset{\sim}{x} = {}^pG d\underset{\sim}{X}, \quad d_o x^k = {}^pG^k_{.K} dX^K, \tag{2.7}$$

where the afore mentioned mapping functions are reffered to as:

Π - local plastic deformation gradient (plastic distortion),

Φ - local elastic deformation gradient (elastic distortion),

F - total deformation gradient,

^pG - global plastic deformation gradient,

^eG - global elastic deformation gradient,

whereas Φ, Π are incompatible while F, ^eG, ^pG, are compatible |6|.

On the other hand, considering all the above second order tensors as linear mappings and taking into account the rule of composite functions we get

$$F = \Phi\Pi, \quad F = {}^eG\,{}^pG, \tag{2.8}$$

$$^pG = {}^o\Phi\Pi, \quad {}^eG = \Phi\,{}^o\Phi^{-1} \tag{2.9}$$

where $^o\Phi$ is elastic distortion of ${}_o\chi_t$ - configuration.

3. Constitutive equations

Let us introduce, after |5|, the sets

$$\bar\Lambda = \{\bar\Lambda_e, \bar\Lambda_p\}, \quad \bar\Lambda_e = \{\Phi, \theta, \mathrm{grad}_\chi \theta\}, \quad \bar\Lambda_p = \{\Pi, g\} \tag{3.1}$$

$$\bar\Gamma = \{\psi, \eta, T_\chi, \underset{\sim}{q}_\chi\}, \tag{3.2}$$

where:

θ - the absolute temperature,

g - a scalar hardening parameter,

ψ - the free energy density,

η - the entropy density,

T_χ - Cauchy symmetric stress tensor in $\chi_t(\mathcal{B})$,

$\underset{\sim}{q}_\chi$ - the heat flux vecotor in $\chi_t(\mathcal{B})$,

$\bar\Lambda$ - the set of variables of internal state of body,

$\bar\Lambda_e$ - the instant deformation-temperature configuration of $\mathcal{N}(X)$,

$\bar\Lambda_p$ - the set of permanent deformation history functions and

$\bar\Gamma$ - the instant local reaction of $\mathcal{N}(X)$.

Constitutive functions may be given by

$$\bar{\mathfrak{R}} : \bar\Lambda \to \bar\Gamma, \tag{3.3}$$

or, after material frame indifference principle

$$\mathfrak{R} : \Lambda \to \Gamma \tag{3.4}$$

Here

$$\Lambda_e = \{E_e, \theta, \mathrm{grad}_\nu \theta\}, \quad \Lambda_p = \bar\Lambda_p, \quad \Gamma = \{\psi, \eta, T_\nu, \underset{\sim}{q}_\nu\} \tag{3.5}$$

were

$$E_e = \tfrac{1}{2}(\Phi^T\Phi - 1_\nu), \quad \mathrm{grad}_\nu\theta = \Phi^T \mathrm{grad}_\chi \theta, \quad \underset{\sim}{q}_\nu = \frac{\rho_\chi}{\rho_\nu}\Phi^{-1}\underset{\sim}{q}_\chi \tag{3.6}$$

are respectively elastic strain tensor, temperature gradient and heat flux related to $\nu_t(\mathcal{N})$ while

$$T_\nu = \frac{\rho_\nu}{\rho_\chi}\Phi^{-1}T_{\chi}\Phi^{-T} \tag{3.7}$$

is Kirchhoff stress tensor related to $\nu_t(\mathcal{N})$.

In order to have a full set of constitutive equations we need evolution equations which have to fulfil two following conditions |5|

- time scale invariance,
- any change in the set of permanent history is possible only if plastic distortion changes.

Therefore, evolution equations read (superposed dot denotes material derivative)

$$\dot{\Lambda}_p = \mathcal{D}(\Lambda)\dot{\Lambda}_e \tag{3.8}$$

or, explicitly,

$$\dot{\Pi} = \mathcal{D}_1|\dot{E}_e| + D_2\dot{\theta} + D_3\overline{\mathrm{grad}_\nu\theta}^{\cdot}, \tag{3.9}$$

$$\dot{g} = t_r\{D_4\dot{\Pi}\}, \tag{3.10}$$

where: $\mathcal{D}_1$ - a fourth order tensor, D_3 - a third order tensor as well as D_2, D_4- second order tensors, all of them being functions of Λ.

If Drucker's normality law holds then (3.8) may be adequately modified. In such a situation as a hardening parameter elementary plastic power is usually chosen.

By making use of the second law of thermodynamics, (3.4) and (2.8), we obtain the following inequality

$$-(\eta+\partial_\theta\hat{\Psi})\dot{\theta} + \frac{1}{\rho_\nu}\mathrm{tr}\{(T_\nu-\rho_\nu\partial_{Ee}\hat{\Psi})\dot{E}_e\} - \partial_{\mathrm{grad}_\nu\theta}\hat{\Psi}\overline{\mathrm{grad}_\nu\theta}^{\cdot} +$$

$$+\frac{1}{\rho_\nu}\mathrm{tr}\{(\mathcal{L}|T_\nu|-\rho_\nu\partial_\Pi\hat{\Psi})\dot{\Pi}\} - \frac{1}{\theta\rho_\nu}q_\nu.\mathrm{grad}_\nu\theta \geq 0, \tag{3.11}$$

($\partial_x y$ denote partial derivatives), where

$$\mathcal{L} = \mathcal{L}(E_e,\Pi) \tag{3.12}$$

is a fourth order tensor. Now, in each elastic region (for $\dot{\Pi} = 0$) by means of the analysis of admissible thermodynamic processes |9| the following restrictions on (2.4) hold:

$$\partial_{\mathrm{grad}_\nu\theta}\hat{\Psi} = 0, \qquad \eta = -\partial_\theta\hat{\Psi}, \quad T_\nu = \rho_\nu\partial_{Ee}\hat{\Psi}. \tag{3.13}$$

Let us postulate

$$(T_\nu)_{Ee=o} = 0 \tag{3.14}$$

on the basis of local isomorphism between $\nu_t(\mathcal{N})$ and $\kappa(\mathcal{B})$ and define

$$(T_\nu)_{\Phi=o_\Phi} = {}_{res}T_\nu. \tag{3.15}$$

In this way 1. it is shown that $\nu_t(\mathcal{N})$ is totaly relaxed and 2. residual stress in ${}_o\chi_t(\mathcal{B})$ is naturally introduced and defined.

Besides the use of these definitions this approach differs from |5| essentially in the exploitation of local varying reference configuration. The local isomorphism between $\nu_t(\mathcal{N})$ and $\kappa(\mathcal{B})$ eliminates a need for consideration of rotations of $\nu_t(\mathcal{N})$ with regard to $\kappa(\mathcal{B})$, and in fact introduces distant paralelism noneuclidean structure in a global $\nu_t(\mathcal{B})$ (cf.|6|). Hence, an elaboration of these rotations such as in |10| is unnecessary and misleading.

4. A Yield Condition Without History Effects

Let us introduce a scalar function

$$f = \hat{f}(T_\nu, \Pi) \tag{4.1}$$

and denote by f* the following restricted time derivative

$$f^* = (\dot{f})_{\dot{\Pi}=0} = \mathrm{tr}\{\partial_{T_\nu} \hat{f} \dot{T}_\nu\}, \tag{4.2}$$

Then, as a elastic region in stress space we define the interior of the surface f = 0 for fixed Π, i.e.

$$\mathcal{R}_E = \{T_\nu : f(T_\nu, \Pi) < 0 \ , \quad \dot{\Pi} = 0\} \tag{4.4}$$

The boundary of this region is the yield surface. On the surface the following three cases are possible

$$f = 0 \quad \begin{cases} f^* < 0 \ , \\ f^* = 0 \ , \\ f^* > 0 \ , \end{cases} \tag{4.4}$$

called, respectively, unloading, neutral loading and loading. In the case of loading, plastic distortion as well as the local reference configuration $\nu_t(\mathcal{N})$ change with time.

The isotropy group, G, is defined as the subset of the set of rotation tensors R (the corresponding group operation is tacitly taken into account)

$$\check{G}_K = R G_K \ , \qquad (R \in G) \tag{4.5}$$

leaving (3.4) invariant. Obviously, this holds when $\kappa(\mathcal{B})$ is considered as a reference configuration and (4.5) defines all material coordinate frames in $\kappa(\mathcal{B})$ which give the same form of (3.4).

As far as $\nu_t(\mathcal{N})$ is concerned isotropy group gives all rotation tensors R_ν

$$\check{h}_\varkappa = R_\nu h_\kappa \ , \qquad (R_\nu \in G) \tag{4.6}$$

leaving (3.4) and (3.8) invariant and (4.6) defining all structural anholonomic coordinate frames in $\nu_t(\mathcal{N})$ which give the same form of (3.4) and (3.8).

It is very important to note that (4.1) and (4.4) depend on material of the considered body $\mathcal{B}$ and must obey isotropy group. Therefore

$$\hat{f}(T_\nu, \Pi) = \hat{f}(R_\nu T_\nu R_\nu^T, R_\nu \Pi R) \ , \qquad (R, R_\nu \in G) \tag{4.7}$$

Due to the local isomorphism between $\kappa(\mathcal{B})$ and $\nu_t(\mathcal{N})$ it is allowed to identify

$$G_K = \delta^\lambda_K h_\lambda \tag{4.8}$$

leading to

$$R^K_{\ .L} = \delta^K_\alpha \delta^\beta_L R_\nu{}^{.\alpha}{}_\beta \ , \tag{4.9}$$

If instedad of (4.1) we postulate a more specific form

$$F = \hat{F}(T_\nu, B_p) \tag{4.10}$$

where

$$B_p = \Pi\Pi^T \tag{4.11}$$

is the left Cauchy-Green plastic deformation tensor and $\hat{F}$ has features (4.2)-(4.4), then invariance originated from isotropy group reads

$$\hat{F}(T_\nu, B_p) = \hat{F}(R_\nu T_\nu R_\nu^T,\ R_\nu B_p R_\nu^T) \tag{4.12}$$

Suppose now that G = SO(3)-the group of all orthogonal rotation tensors (with det R_ν = 1). Then the most general form of (4.10) appears as

$$\hat{F}(T_\nu, B_p) = \hat{F}(I_1, I_2, I_3, J_1, J_2, J_3, K_1, K_2, K_3, K_4) \tag{4.13}$$

with invariants |11|

$$I_1 = trT_\nu,\ I_2 = t_rT_\nu^2,\ I_3 = t_rT_\nu^3,\ J_1 = t_rB_p,\ J_2 = t_rB_p^2,\ J_3 = t_rB_p^3,$$
$$K_1 = tr\{T_\nu B_p\},\ K_2 = t_r\{T_\nu^2 B_p\},\ K_3 = t_r\{T_\nu B_p^2\},\ K_4 = t_r\{T_\nu^2 B_p^2\} \tag{4.14}$$

Without loss of information J_3 may be missed because det Π = 1.

In the special case when

$$\tilde{F}(\ldots) = \frac{I_2}{k^2} - 1 \quad , \quad k = \text{const.} \tag{4.15}$$

Huber-Mises yield condition of ideal plasticity takes place.

If the second important case of transverse isotropy takes place the list of invariants appearing in yield function becomes much larger. Let the transverse isotropy direction be given by the unit vector $\underset{\sim}{A}_\nu$ and denote $\underset{\sim}{A}_\nu \times \underset{\sim}{A}_\nu$ by A_ν. Then

$$\hat{F}(T_\nu, B_p) = \tilde{F}(I_1, I_2, I_3, J_1, J_2, J_3, K_1, K_2, K_3, K_4, L_1, L_2, M_1, M_2, N_1, N_2, N_3, N_4) \tag{4.16}$$

where the new invariants amount to

$$L_1 = t_r\{T_\nu A_\nu\},\ L_2 = t_r\{T_\nu^2 A_\nu\},\ M_1 = t_r\{B_p A_\nu\},\ M_2 = t_r\{B_p^2 A_\nu\},$$
$$N_1 = t_r\{T_\nu B_p A_\nu\}, N_2 = t_r\{T_\nu^2 B_p A_\nu\},\ N_3 = t_r\{T_\nu B_p^2 A_\nu\},\ N_4 = t_r\{T_\nu^2 B_p^2 A_\nu\} \tag{4.17}$$

In the special case when |3|

$$\tilde{F}(\ldots) = \frac{I_2}{k_1^2} + \frac{L_2}{k_2^2} - 1,\quad k_1 = \text{const.},\quad k_2 = \text{const.} \tag{4.18}$$

we get Spencer's generalization of Mises yield condition for ideal plasticity of transversely isotropic (and possibly inextensible) materials.

Finally, it is worth noting that Bauschinger effect

$$\hat{f}({}_{res}T_\nu + {}_{ext}T_\nu, \Pi) \neq \hat{f}({}_{res}T_\nu - {}_{ext}T_\nu, \Pi) \tag{4.19}$$

naturally follows from the deformation geometry depicted on Fig. 1 anddefinition (3.15).

5. A Yield Condition With History Effects

Let us take history effects into account by the yield function

$$f = \hat{f}(T_\nu, \Lambda_p) \equiv \hat{f}(T_\nu, \Pi, g) \tag{5.1}$$

with material derivative

$$\dot{f} = f^* + tr\{(\partial_\Pi f + D_4)\dot{\Pi}\} \tag{5.2}$$

where g satisfies (3.10) and f* is defined by (4.2). The permanent deformation history is involved into the scalar hardening parameter

$$g = \int_{-\infty}^{t} t_r\{D_4[\dot{\Lambda}(\tau)[\dot{\Pi}(\tau)\}d\tau. \tag{5.3}$$

If the evalution equation (3.9) has the following special form (motivated by Drucker's normality law when $B(\Lambda) = \partial T_\nu \hat{f}$)

$$\dot{\Pi} = \dot{\mu} B(\Lambda) \ , \tag{5.4}$$

then from $\dot{f} = 0$ in the case of loading we have

$$\dot{\Pi} = -Bf^*/t_r\{(\partial_\Pi \hat{f}+D_4)B\} = -Bt_r\{\partial T_\nu \hat{f}\dot{T}_\nu\}/t_r\{\partial_\Pi \hat{f}+D_4)B\} \ . \tag{5.5}$$

Suppose also that

$$g = \hat{g}(w_p) \tag{5.6}$$

where w_p is the work done on plastic deformation. Then it is easy to show that

$$D_4 = g'\Pi^{-1}T_\nu C_e \ , \quad (C_e = \Phi^T\Phi) \ . \tag{5.7}$$

In the special case of hardening of isotropic materials with Mises yield function

$$\hat{f}(\ldots) = \frac{I_2}{g(w_p)} - 1 \tag{5.8}$$

we would have

$$f^* = \frac{2}{g} t_r\{T_\nu \dot{T}_\nu\}, \quad \partial_\Pi \hat{f} = 0 \ , \quad \dot{\mu} = -f^*/t_r\{D_4 B\} \tag{5.9}$$

while in the second special case of hardening of transversely isotropic materials with Spencer's yield function

$$\hat{f}(\ldots) = \frac{I_2}{k_1^2(w_p)} + \frac{L_2}{k_2^2(w_p)} - 1 \ , \tag{5.10}$$

the corresponding expressions would read

$$f^* = 2tr\{(\frac{1_\nu}{k_1^2} + \frac{A_\nu}{k_2^2})T_\nu \dot{T}_\nu\}, \quad \partial_\Pi \hat{f} = 0 \ , \ \hat{g}' = -2(I_2\frac{k_1'}{k_1^3} + L_2\frac{k_2'}{k_2^3}) \tag{5.11}$$

At the end of this section it is worth noting that the decomposition

$$T_\nu = {}_{res}T_\nu + {}_{ext}T_\nu \tag{5.12}$$

allows for explicit kinematic hardening provided that an evolution equation for ${}_{res}T_\nu$ is introduced on the basis of (3.9), (3.12) and (3.15). The procedure would be similar as in |12|.

6. Conclusion

The concept of local varying reference configuration is very useful. It makes possible a consequent formulation of constitutive equations of elastoplasticity as well as a clear interpretation of residual stresses and Bauschinger effect. When applied to some stability problems |13| it permits a very clear decomposition into small elastic and finite plastic strains leading to a convenient partial linearization of the corresponding hypoelastic constitutive equation.

The main conclusion is that each yield condition is a constitutive inequality and the exploitation of this fact in the paper on the basis of isotropy group has given a whole list of new invariants.

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