Simplified Analysis of a Cantilever Beam Subjected to Elastic Follow-up and Cyclic Loadings

U.L. Nguyen, D.J. Lee
National Nuclear Corporation Ltd., Booths Hall, Chelford Road, Knutsford, Cheshire WA16 8QZ, U.K.

Abstract

An approximate method for an elastic-plastic creep problem of a cantilever beam subjected to monotonic and cyclic transverse displacement loads with holdtimes is presented. The constitutive equations in the form of Ramberg-Osgood and Norton’s law are assumed. An expression relating creep strain accumulation due to follow-up for loadings in excess of yield is derived. Under cyclic loadings, the simplified analysis predicts ratchetting causing progressive distortion in the curvature of the cantilever. The severity of the ratchetting increases with the cyclic displacement amplitude. The approximate results also give a good agreement with the finite element solution.

1. Introduction

In designing nuclear plant operating at elevated temperatures some consideration is required in assessing the effects of elastic follow-up. Frequently, structures are subjected to thermal loadings and are operating under alternating plasticity and creep dwell. The degree of follow-up due to elastic energy dissipation in the body through creep stress redistribution is of particular interest since it has significant bearing on the cyclic structural response. Owing to the imbalance of the stress-strain cycles, the effects of the follow-up in addition to enhancing the strain can lead to ratchetting or incremental distortion. A typical follow-up situation occurs at the built-in end of a cylinder having an axial temperature gradient [Ref 1]. The effect of follow-up is similar to having a plastic hinge at the built-in end.

By studying the response of a cantilever, Takeda et al [Ref 2] applied the results to a particular flat plate-shell problem under multi-axial stress states. It was found that the strain enhancement due to follow-up is close to that predicted by the simple cantilever theory. The purpose of this paper is to determine the response of the cantilever under the influence of creep and plasticity and under cyclic conditions. Ramberg-Osgood and Norton’s laws are assumed in the constitutive equations. An approximate method is used to evaluate the creep strain enhancement and the response of the cantilever subjected cyclic transverse displacement load with dwell times. The results are verified by comparison with finite element solutions.

2. Approximate method

Consider a cantilever beam as shown in Fig 1 of a unit width subjected to a displacement load \( \delta \). It is assumed that the axial stress \( \sigma \) and strain \( \varepsilon \) are related by the Ramberg-Osgood and Norton’s law:

\[
\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p + \dot{\varepsilon}_c
\]

where

\[
\dot{\varepsilon}_e = \frac{\dot{\varepsilon}}{\varepsilon}, \quad \dot{\varepsilon}_p = n \left( \frac{\sigma}{E_p} \right)^{m-1} \frac{\dot{\sigma}}{E_p}, \quad \dot{\varepsilon}_c = \left( \frac{\sigma}{E_c} \right)^n
\]

The \( E \)'s, \( m \) and \( n \) are material constants. The plastic strain rate, \( \dot{\varepsilon}_p \), is assumed to satisfy the associated flow rule.
The kinematic condition requires that

\[ \dot{\epsilon} = \int_0^l k(x, t) \times dx \]  

where \( k \) is the rate of change in curvature and is related to \( \dot{\epsilon} \) by

\[ \dot{\epsilon} = k(x, t) \cdot y \]  

\( \dot{k} \) can be decomposed as follows

\[ \dot{k} = \dot{k}_e + \dot{k}_p + \dot{k}_c \]  

where

\[ \dot{k}_e = \dot{k}_e y, \quad \dot{k}_p = \dot{k}_p y, \quad \dot{k}_c = \dot{k}_c y \]  

In general, \( \dot{k}_e \)'s are functions of \( y \), but under limiting circumstances, they are approximately independent of \( y \). By considering the limiting cases in terms of purely-elastic, grossly-plastic and steady-state creep conditions and using equilibrium conditions, it can be shown that \( \dot{k}_e \)'s can be related to the end load \( P \) as follows [Ref 3]

\[ \dot{k}_1 = \frac{P}{EI_1}, \quad \dot{k}_p = m P^{n-1} \left( \frac{x}{E P^m} \right)^n, \quad \dot{k}_c = \left( \frac{P}{P^c} \right)^n \]  

where

\[ I_k = \frac{2k}{1 + 2k} \left( \frac{h}{2} \right)^{1 + 2k} \]  

Substituting [7] and [5] into [3], an approximate non-linear differential equation relating \( \dot{\epsilon} \) and \( P \) is obtained:

\[ \dot{\epsilon} = b \dot{\epsilon} + \frac{m P^{n-1}}{EI_1} \left( \frac{P E}{P^c} \right)^n + \frac{P^{n-1}}{E P^c} \]  

where

\[ I_k = \frac{1 + \dot{k} + 2}{k + 2} \]  

3. Elastic follow-up under monotonic loading

Figure 2 shows a typical load path encountered during reactor plant operations. The displacement load history is analogous to a temperature rising to a steady-state condition. The loading path OA in Fig 2 is sufficiently rapid to dismiss creep dissipation. In contrast during the 'dwell' period following A of Fig 2, creep deformation is dominant and no further plastic action occurs. The corresponding stress-strain history is illustrated in Fig 3. Consequently, equation [8] can be broken down as follows:

for quasi-static instantaneous load (path OA in Fig 2)

\[ D = \frac{h}{L} \left( \frac{h}{2} \right)^{m+2} \left( \frac{x}{E P^m} \right)^m \]  

where \( D \) and \( x \) are non-dimensionalised displacement and load variables:

\[ D = \frac{\delta}{L} \left( \frac{h}{2} \right), \quad x = \frac{P}{E} \left( \frac{h}{2} \right)^{-2} \]  

and \( a = E/E_p \)
The maximum strain \( \varepsilon_{m} \), occurring at \( y = \frac{h}{2} \) and \( x = L \), is given by (\( y = \frac{h}{2} \))

\[
\varepsilon_{m} = \frac{3}{2} \beta + \frac{(2m + 1)^{m}}{2m} \frac{a^{n}}{\chi} \chi^{n}
\]

(10)

for creep dwell (path AB in Fig 2)

\[
\dot{\varepsilon}_{m} = \frac{3}{2} \beta + \frac{(2n + 1)^{n}}{2n} \frac{b^{n}}{\chi} \chi^{n} = 0
\]

(11)

where

\[
b = \frac{E/K}{E}
\]

From equation [11] and [12], the follow-up strain \( \Delta \varepsilon_{c} \) as defined in Fig 3 for \( t \rightarrow \infty \) is given by

\[
\Delta \varepsilon_{c} = \left( \frac{n - 1}{2} \right) \chi_{0}
\]

(13)

where \( \chi_{0} \) satisfies equation [9].

The total strain is given by equation [10] and [13]:

\[
\varepsilon_{t} = \varepsilon_{m} + \Delta \varepsilon_{c} = \left( \frac{n + 2}{2} \right) \chi_{0} + \frac{(2m + 1)^{m}}{2m} \frac{a^{n}}{\chi_{0}} \chi^{n}
\]

(14)

In terms of strain enhancement factors, these can be defined as follows:

\[
K_{e} = \frac{\varepsilon_{t}}{\varepsilon_{e}}, \quad K_{c} = \frac{\varepsilon_{t}}{\varepsilon_{c}}
\]

(15)

where \( \varepsilon_{e} \) corresponds to elastically calculated solution, i.e. \( \varepsilon_{e} = 3\Delta \). \( K_{e} \) and \( K_{c} \) can be shown to be related by

\[
K_{c} = \frac{(n - 1)(n + 2) + 3(m - n)K_{e}}{3(n - 1)}
\]

(16)

\( K_{e} \) is generally known as the strain enhancement owing to plastic deformation and is approximately dependent on the load \( \chi_{0} \) as follows

\[
K_{e} = 1 + \frac{n + 2}{m + 2} \frac{Y}{1 + Y} = 2 \frac{a^{n}}{m + 2} \frac{(2m + 1)^{m} \chi_{0}^{m - 1}}{1 + Y}
\]

Thus \( K_{e} \) varies from 1 to \( m + 2 \) with increasing load \( \chi_{0} \). \( K_{c} \) defined the total strain enhancement factor inclusive of plastic and creep deformation. According to the approximate method \( K_{e} \) and \( K_{c} \) are related linearly.

According to [13], \( \Delta \varepsilon_{c} \) can be expressed in terms of the initial stress \( \sigma_{0} \) (see Fig 3) as follows

\[
\Delta \varepsilon_{c} = \Delta K_{c} \frac{\sigma_{0}}{E}
\]

(17)

where \( \sigma_{0}/E \) evidently corresponds to the pure relaxation creep strain and \( \Delta K_{c} \) is bounded by

\[
\frac{n - 1}{3} \leq \Delta K_{c} \leq \frac{n - 1}{2m + 1}
\]

(18)

The equalities apply to conditions of pure-elastic and gross-plastic deformation respectively. For very large \( m \) corresponding to perfect plasticity then \( \max \Delta K_{c} = \frac{n - 1}{2} \).
4. Cyclic loadings

A typical stress-strain history due to a cyclic displacement load, $\delta$, with a hold time introduced at peak displacements is illustrated in Fig 4. It is assumed that the hold time is sufficiently large to approximate the stress state to be zero at points I, L, and J, K of Fig 4. Virgin material behaviour is assumed under cyclic plasticity.

Using expression [8], it can be shown that the non-dimensionalised displacement, $D$, and end loads $X_J$, $X_K$ at J and K respectively are approximately related by

$$
-D = \frac{1}{4} x_J - \left(\frac{2m+1}{2m}\right)^m a^m \left(\frac{-x_J}{m+2}\right)^m \left(x_J < 0\right) \tag{19}
$$

and

$$
D = \frac{1}{4} (x_K - x_J) + \left(\frac{2m+1}{2m}\right)^m a^m \left(\frac{x_K}{m+2}\right)^m \left(x_K > 0\right) \tag{20}
$$

Similarly, the strain quantities, $\Delta e$'s, at $x = L$ and $y = h/2$, defined in Fig 4 can be shown to be related to $x$'s as follows

$$
\Delta e_B = -\frac{3}{2} x_J + \left(\frac{2m+1}{2m}\right)^m a^m \left(-x_J\right)^m \tag{21}
$$

$$
\Delta e_F = \frac{3}{2} (x_K - x_J) + \left(\frac{2m+1}{2m}\right)^m a^m \left(x_K\right)^m \tag{22}
$$

and

$$
\Delta e_C = \left(\frac{n-1}{2}\right) x_K \tag{23}
$$

The ratchet strain $\Delta e_R$ is determined by

$$
\Delta e_R = \Delta e_F + \Delta e_C - \Delta e_B \tag{24}
$$

Substituting [21] to [23] into [24], and using [19] and [20], it can be shown that

$$
\Delta e_R = \left(\frac{n-1}{2}\right) x_K \cdot \left(x_K > 0\right) \tag{25}
$$

Therefore since in general $m > n$ for most metals, ratchetting will occur in the direction opposite to the initial direction of straining.

5. Comparison with finite element solution

(1) Monotonic-loading

A cantilever of dimension $L = 300$ and $h = 12$ was considered. The material constants for the constitutive equation [1] were assumed as follows

$$
E = 2 \times 10^5, \quad E_p = 250, \quad K_C = 562, \quad n = 25 \quad \text{and} \quad n = 8
$$

A typical stress-strain response produced by these constants under a displacement load $\delta = 12$ is shown in Fig 3. The solution was obtained by using 2 node beam elements from the finite element program ABAQUS (Ref 4).

The results are plotted in Fig 5 and 6 in terms of the strain enhancement factors $K_{e}$, $K_C$ and $\Delta K_{e}$. The approximate solution from the simplified analysis are also presented. A linear correlation between $K_{e}$ and $K_C$ is observed in Fig 5 and is consistent with the approximate solution (c.f. equation [16]). The follow-up enhancement factor $\Delta K_{e}$ obtained from finite element results does appear to be bounded from above by equation [16].
Cyclic loading

Material isotropic hardening is assumed in the finite element analysis. Furthermore it is assumed that the stress-strain cycles have reached a steady state situation as shown in Fig 4. The residual stress at the end of long dwell time is assumed zero. Thus it is only necessary to consider the load path corresponding to the stress-strain history defined by I to K in Fig 4. This is performed by applying one complete displacement cycle \( \delta \) with a long creep dwell introduced at the end of the cycle. Figure 7 shows a typical stress-strain cycle from the finite element analysis for \( \delta = 10 \text{ mm} \). Figure 8 and 9 present a comparison of the results of the finite element analysis and the simplified analysis in terms of the follow-up strain \( \Delta \varepsilon_R \) (c.f. equation [23]) and the ratchet strain \( \Delta \varepsilon_R \) (c.f. equation [25]) with respect to a load factor \( \lambda \). The initial ratchet strain \( \Delta \varepsilon_0 \) is normally 5 which corresponds to the maximum plastic strain of 0.01% in the cantilever.

From Fig 8 and 9, there appears to be a reasonable agreement between the two analyses although the approximate ratchet strain \( \Delta \varepsilon_R \) tends to overestimate the finite element solution. It is interesting to note that in Fig 9, the ratchet strain \( \Delta \varepsilon_R \) below a load factor of 1.4 is insignificant.

b. Acknowledgements

The authors wish to acknowledge National Nuclear Corporation Limited for their kind permission to publish the work.

7. References


1. Cantilever beam

2. Typical monotonic displacement load.

3. Stress-strain curve for \( \delta = 12 \text{ mm} \).

5. Strain enhancement factors from finite element analysis.

6. Elastic follow-up factor from finite element analysis.

7. Stress–strain history under cyclic displacement load (δ = 10 mm).

8. Follow-up strain $\Delta \varepsilon_c$ resulting from cyclic loadings.

9. Ratchet strain $\Delta \varepsilon_R$ resulting from cyclic loadings.