Response of Linear Oscillators to Uncertain Random Excitations

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Abstract
Statistical and probabilistic mean failure rates are determined for oscillators with linear and nonlinear nonhysteretic restoring forces subjected to ground acceleration processes which depend on uncertain parameters. The statistical mean failure rates account for the uncertainty in the ground acceleration while the probabilistic mean failure rates disregard this uncertainty. It is shown that unconservative designs can result when the analysis does not account for the statistical uncertainty in the seismic excitation.

1. Introduction
A common hypothesis in seismic analysis is that the models of structural mechanical properties and ground acceleration are perfectly known, Bolotin [1] and Lin [2]. The statistical uncertainty in these models is generally disregarded, although it can be significant. For example, the coefficients of variation of estimates of the predominant period, the strong-motion duration, and the standard deviation of the ground acceleration process can exceed 50%, as shown by Vanmarcke and Lai [3] and Lai [4].

The objective of this paper is the characterization of the seismic response of nonhysteretic oscillators with linear and nonlinear restoring forces. It is assumed that the structural mechanical properties are deterministic and perfectly known. However, the seismic ground acceleration is modeled probabilistically and depends on statistically uncertain parameters. The paper also evaluates differences between statistical and probabilistic descriptors of structural response. The statistical descriptors account systematically for the statistical uncertainty in the parameters of the ground acceleration process and are developed in this study. On the other hand, the probabilistic descriptors disregard the statistical uncertainty. They are common to current methods of seismic analysis.

2. Seismic Excitation
The ground acceleration is represented by the random process

\[ X(t) = g(t) Z Y(t) \]  

(1)

in which \( g(t) \) is a specified modulation function, \( Z \) denotes a positive random variable which controls the scale of the excitation, and \( Y(t) \) is a zero-mean stationary Gaussian process.
It is assumed that \( g(t) \) is a unit step function and \( Y(t) \) is white with a one-sided spectrum of intensity one. More general models can be considered for \( g(t) \) and \( Y(t) \) without significant difficulties. The scale variable \( Z \) follows an inverted gamma-2 probability law with the density

\[
f_Z(z) = \frac{2}{\Gamma(\frac{n}{2})} \frac{n\delta^2}{2} \frac{1}{z^{n+1}} \exp\left(-\frac{n\delta^2}{2z^2}\right)
\]

(2)

in which \( \Gamma(\cdot) \) is the gamma function. The mean, mode, and variance of \( Z \) are \( \mu_Z = \frac{\Gamma(\frac{n}{2}; \frac{n}{2})}{\Gamma(\frac{n}{2})} \sqrt{\frac{n}{2}} \delta \), \( \mu_Z = \delta \sqrt{\frac{n}{n+1}} \), and \( \sigma_Z^2 = n\delta^2/(n-2) - \mu_Z^2 \). Note that \( f_Z \) approaches a delta function centered at \( z = \delta \) as \( n \) increases indefinitely. In this case, there is no uncertainty in the scale of the ground acceleration process and \( Z = \delta \). From statistical analyses of ground acceleration records by Vanmarcke and Lai [3], it results that the parameter \( n \) in Eq. 2 is approximately equal to 5.

The density in Eq. 2 is consistent with current seismic hazard studies which postulate densities of a similar shape for most seismic parameters. It was employed by Bayesian statisticians to quantify the uncertainty in the standard deviation of Gaussian variables, Zellner [5]. This density is applied in the next section to the development of statistical descriptors of the peak response. In contrast, the probabilistic response descriptors are based on the assumption that the scale variable is not uncertain and takes on a particular value \( z_0 = \alpha \delta \), in which

\[
\alpha = \sqrt{\frac{n}{n+1}} + k \sqrt{\frac{n}{n-2} \frac{\Gamma(\frac{n}{2}; \frac{n}{2})}{\Gamma(\frac{n}{2})}}
\]

(3)

and \( k \) is positive, negative, or zero. Note that \( z_0 \) coincides with the mode of \( Z \) for \( k = 0 \).

3. Structural Response

Consider the response \( R(t) \) of a simple oscillator to the ground seismic acceleration \( X(t) \) in Eq. 1. The response satisfies the differential equation

\[
R(t) + \zeta \omega_0 \dot{R}(t) + g(R(t)) = -X(t)
\]

(4)

and depends on the restoring force \( g(R(t)) \), the frequency parameter \( \omega_0 \), and the damping ratio \( \zeta \). Two restoring forces are considered, the linear function

\[
g(R(t)) = \omega_0^2 R(t)
\]

(5)

and the nonlinear function
\[ g(R(t)) = \frac{u_0^2}{\lambda} \text{sqn}(R(t)) \left[ 1 - e^{-\lambda |R(t)|} \right] \quad (6) \]

in which \( \text{sqn}(r) = -1; 0; \) or \( 1 \) for \( r < 0; r = 0; \) or \( r > 0. \) The latter function is approximately equal to \( u_0^2 R(t) \) for \( \lambda |R(t)| \ll 1 \) but can differ significantly from the linear function in Eq. 5 for relatively large values of \( \lambda |R(t)|. \) It provides a model for structures with softening stiffness characteristics.

From Eqs. 4 to 6, the response is a zero-mean process with time-dependent characteristics during the initial phase of the motion and becomes stationary for large values of \( t. \) The process follows a Gaussian distribution when the restoring function is linear (Eq. 5). However, non-Gaussian responses result for the nonlinear restoring function in Eq. 6.

The probability that the peak response exceeds a specified (strength) level \( r \) can be approximated and bounded from the mean rate at which \( |R(t)| \) exceeds \( r. \) This mean failure rate can be obtained simply from the integral, Lin [2],

\[ v(r/z) = 2 \int_0^r f(r, \tilde{r}/z) \, d\tilde{r} \quad (7) \]

in which \( f(r, \tilde{r}/z) \) is the joint density of \( R(t) \) and \( \tilde{R}(t) = dR(t)/dt, \) conditional on \( Z = z. \) When \( R(t) \) is a stationary Gaussian process, it can be obtained from

\[ v(r/z) = \frac{\hat{\sigma}}{\sqrt{2\pi} \sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad (8) \]

where \( \sigma \) and \( \hat{\sigma} \) are the standard deviations of the response and its derivative for \( Z = 1. \)

From Eqs. 2 and 7, the corresponding unconditional mean failure rate \( v(r) \) is, Grigoriu [6],

\[ v(r) = \int_0^\infty v(r/z) f_Z(z) \, dz \quad (5) \]

**Linear Oscillators.** The steady-state response of the oscillator in Eqs. 4 and 5 is a stationary Gaussian process for any value of \( Z. \) The probabilistic mean failure rate function is, from Eq. 8,

\[ v(r/z_0) = \frac{\hat{\sigma}}{\sqrt{2\pi} \sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad (10) \]

in which \( Z = z_0 = \alpha \delta \) is a postulated value of the scale parameter and \( \xi = r/\sigma_0 \) denotes a standardized threshold, in which \( \sigma_0 = \sqrt{\pi\sigma^2/(4\pi \omega_0^2)} \) is the standard deviation of the steady-state response of the linear oscillator for \( Z = 0. \) From Eq. 9, the statistical mean failure rate has the expression

\[ v(r) = \frac{\hat{\sigma}}{\sqrt{2\pi} \sigma} \left(1 + \frac{\xi^2}{\pi}\right)^{-n/2} \quad (11) \]
Figure 1 shows the variation of the ratio \( v(r) / v(r_0) \) with \( n \) for selected values of the standardized threshold \( \xi = r / r_0 \), when \( z_0 \) coincides with the mode of \( Z \) (i.e., \( k = 0 \)). Note that \( v(r) \) is significantly larger than \( v(r/z_0) \), particularly for values of \( n \) relevant to seismic analysis, e.g., \( n = 5 \). Similar results have been found for other values of \( k \).

Nonlinear Oscillators. It can be shown that the joint density of the steady-state response vector \( \{ R(t), \hat{R}(t) \} \) in Eqs. 4 and 6 is, Lin [2],

\[
f(r, \hat{r}/z) = c \exp \left[ -\frac{1}{\sigma^2} \left[ \frac{2w_0^2}{\lambda} \left( |r| + \frac{1 - e^{-\lambda|r|}}{\lambda} \right) + \hat{r}^2 \right] \right] \tag{12}
\]

for any value \( z \) of \( Z \). In this equation, \( c \) is a normalization constant. Note that \( R(t) \) and \( \hat{R}(t) \) are independent at any instant \( t \) and the random variable \( \hat{R}(t) \) is normally distributed. However, the process \( \hat{R}(t) \) is not Gaussian because its integral, \( R(t) \), does not follow a Gaussian distribution.

The probabilistic mean failure rate of the response can be obtained from Eqs. 7 and 12 for \( z = \xi = 0 \) as while the statistical mean failure rate follows from Eq. 9. Figure 2 shows ratios \( v(r)/v(r/z_0) \) for \( k = 0 \) and selected standardized thresholds \( \xi \). As in Fig. 1, these ratios are large for small values of \( n \) and approach one as \( n \) increases indefinitely. Note that unconservative designs can result when the analysis does not account for the statistical uncertainty in the scale of the ground acceleration process.

4. Conclusions
Mean failure rates were determined for nonhysteretic oscillators with linear and nonlinear restoring forces subjected to statistically uncertain seismic excitations. It was found that the mean failure rate can be underestimated significantly if the analysis does not account for the statistical uncertainty in the ground acceleration process.

5. References

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Figure 1. Ratio of Statistical to Probabilistic Mean Failure Rates for Linear Oscillators with Restoring Force in Eq. 5.

Figure 2. Ratio of Statistical to Probabilistic Mean Failure Rate for Nonlinear Oscillators with Restoring Force in Eq. 6.