Three-Dimensional Rupture in Isotropic Elastic Material for the Simulation of Earthquake

H.J. Pradlwarter
Institut für Mechanik, Universität Innsbruck, Technikerstr. 13, A-6020 Innsbruck, Austria

H. Kameda
School of Civil Engineering, Kyoto University, Kyoto 606, Japan

Abstract

An exact and general solution is introduced for the displacement field in a homogeneous, unbounded isotropic medium caused by an arbitrarily moving dislocation field (all modes) like fault rupture causing earthquakes. It is valid for any position relative to the fault and may due to its simplicity replace those solutions usually divided into a far- and a near-field solution. Next we present a straightforward algorithm to determine the dynamic fault interaction for an arbitrarily distributed stress- and critical stress-intensity field. The efficiency of the method allows one to simulate the rupture propagation appearing in earthquakes and other kind of fracture even when the dislocation field is discretized by several hundred or thousand unknowns.

1. Introduction

A recent trend in seismology and earthquake engineering has been to estimate the radiation pattern of body waves on the rock surface by using ray theory and a dislocation model mostly based on the double couple- and far-field- solution [1]. The estimation of the displacement field near the rupturing fault is still object of intensive research work, which is imbied by lack of seismological records, unknown stress field and interaction on the fault. There is also a deficiency of methods to calculate the propagating crack with realistic assumptions as for the corresponding body waves near the fault. Only a few practical 2-D models for earthquakes have been proposed, for example those by S.Das and K.Aki [2]-[3] who model earthquakes as propagating plane shear-crack. The method proposed herein can be specialized to the 2-D case which leads to a very simple and efficient algorithm. This paper presents one outline of the method, whose details are found in [4].

2. Displacement- and Strain-Field Using "Basis-Dislocations"

The general solution is presented for the displacement field \( \ddot{u}(x,t) \) at instant \( t \) at a point \( x \) in a homogeneous, unbounded isotropic medium caused by a dislocation field \( \dot{\delta}(x,t), t \leq t \), arbitrarily distributed over the internal surface \( S(z) \). In the case of pure slip, the solution is given by Aki and Richards (Eq.14.37 in [1]) as the integral of the
double couple solution for a point dislocation. Only one additional function is necessary to expand this solution to the general case. Both expressions are too lengthy to be given here, so that see [1] and [4] regarding their details.

A dramatic simplification of the general solution can be accomplished by representing the dislocation $\Delta u(\vec{s}, t)$ as linear-combination (integral) of 'basis-dislocations' $D_u(\psi, \Delta \vec{w})$, which are easily derived from integrating $\Delta u(\vec{s}, t)$ by parts:

$$\Delta u(\vec{s}, t) = \int_{-\infty}^{\infty} \frac{1}{2} (t-\tau)^{\beta} \frac{\partial}{\partial \tau} \Delta u(\vec{s}, \tau) d\tau = r_0 \int_{0}^{\infty} D_u(\psi, \Delta \vec{w}) \cdot \Delta \vec{w}(\vec{s}, \psi) d\psi$$

(1)

where $r_0$ is an arbitrary fixed distance, $\psi$ is the normalized time difference $\psi \equiv (t-\tau)/r_0$ in which $\beta$ represents the shear-wave velocity, $D_u(\psi, \Delta \vec{w}) \equiv \psi^2 H(\psi)/2$ is the 'basis-dislocation' where $H(\psi)$ means the Heaviside's step function and $\Delta \vec{w}(\vec{s}, \psi) \equiv a^2 \Delta u(\vec{s}, \tau)/\partial \tau^3$, $r_0/\beta^3$ is the corresponding linear-factor which determines the progress of the dislocation $\Delta u(\vec{s}, t)$ over time. All terms on the right side of (1), except $r_0$, are dimensionless. We determined the displacement field due to a 'basis-dislocation' and obtained the following compact solution for an arbitrarily moving dislocation:

$$\tilde{u}(\vec{x}, \tau) = \frac{r_0}{4\pi} \int_{S} \frac{1}{r^3} \cdot \vec{f}_x(\vec{y}, \vec{n}, \vec{u}) \cdot \vec{W}_x(\vec{y}, \vec{n}, \vec{r}) \cdot dS(\vec{y}) : \chi = a, b \text{ and } c$$

(2)

Einstein's summation convention for repeated sub- or superscripts is implied in (2) and throughout this paper. In the above expression, $\eta \equiv r/r_0$ is the normalized distance, $\vec{r} = \vec{x} - \vec{S}$ is the spatial vector, $r = |\vec{r}|$ is the distance, $\vec{y} \equiv \vec{r}/r$ is the direction of $\vec{r}$ (see Fig.1a).

![Fig.1a: Vectors $\vec{x}, \vec{s}, \vec{r}, \vec{y}$ and $\vec{u}$ and distance $r$](image1)

![Fig.1b: Fault surface $S'$ and $S$ and its normal $\vec{u}$](image2)

$$\vec{n} = \frac{\Delta u(\vec{s}, \tau)}{\Delta u(\vec{s}, \tau)}$$

is the dislocation direction where $\Delta u(\vec{s}, \tau) = |\Delta u(\vec{s}, \tau)|$, $\vec{u}$ is the fault normal (see Fig.1b), $\vec{z} = \vec{s}/r_0$ is the normalized fault coordinate, $\kappa \equiv \alpha/\beta$ is the ratio between longitudinal- and shear-wave velocity which is a material constant $\kappa^2 = (1-2\nu)/(2-2\nu)$ depending on Poisson's ratio $\nu$, $\alpha^2 = (\lambda+2\mu)/\rho$ is the longitudinal- and $\beta^2 = \mu/\rho$ is the shear-wave velocity, $\lambda$ and $\mu$ are Lamé's constants and $\rho$ is the density. The functions $\vec{f}_x(\vec{y}, \vec{n}, \vec{u}), \chi = a, b$ or $c$ account for the geometry and $\vec{f}_c$ vanishes for the case of pure slip, where $\vec{n} \cdot \vec{u} = n_0 q_n = 0$:

$$\vec{f}_a = \eta_p q_p n_0 \vec{n}; \quad \vec{f}_b = \eta_p q_p n_0 \vec{n}; \quad \vec{f}_c = \eta_p q_p \vec{n}$$

(3)

$$\vec{W}_x(\vec{y}, \vec{n}, \vec{r}, \chi = a, b \text{ or } c \text{ depends on the dislocation history } \Delta u(\vec{s}, t), \tau \leq \tau;$$

$$\vec{W}_x(\vec{y}, \vec{n}, \vec{r}, \chi) = \int_{-\infty}^{\infty} \vec{f}_x(\vec{y}, \vec{n}, \vec{r}, \chi) \cdot \Delta \vec{w}(\vec{z}, \vec{y}) d\psi = \sum_{\chi} \int_{-\infty}^{\infty} \vec{f}_x(\vec{y}, \vec{n}, \vec{r}, \chi) \cdot \Delta \vec{w}(\vec{z}, \vec{y})\text{ for } \chi = a, b, c$$

(4)
where the last expression is a discretized approximation in which $\Delta \hat{\omega}(\xi, \eta)$ represent a lumped acceleration of $\Delta u(\xi, \tau)$ at instant $\tau = t - \gamma_0 \psi_{*}/\beta$. The three influence functions $I_a(\eta, \kappa, \psi; \Delta \hat{\omega})$, $\chi = a, b$, and $c$ result from the 'basis-dislocations' $Du(\eta, \kappa; \Delta \hat{\omega})$:

\[ I_a(\eta, \kappa, \psi; \Delta \hat{\omega}) = H_1 \cdot \left[ 3\psi^2(1-\kappa^2) - \psi^2(1-\kappa^4) \right] / 4 + H_2 \cdot \left[ 5\psi^2 - 6\psi^2 \kappa^2 - \eta^2 \kappa^4 \right] / 4 \]  
\( (5a) \)

\[ I_b(\eta, \kappa, \psi; \Delta \hat{\omega}) = H_2 \cdot \left[ 2\psi^2 \kappa^2 - \eta^2 (1-\kappa^4) \right] / 4 + H_2 \cdot \left[ -\psi^2 / \eta^2 + 2\psi^2 \kappa^2 - \eta^2 \kappa^4 \right] / 4 \]  
\( (5b) \)

\[ I_c(\eta, \kappa, \psi; \Delta \hat{\omega}) = H_1 \cdot \left[ -2\psi^2 \kappa^2 + \eta^2 (1-2\kappa^2 + 3\kappa^4) \right] / 4 + H_2 \cdot \left[ -\psi^2 / \eta^2 + 2\psi^2 (1-\kappa^2) + \eta^2 (-2\kappa^2 + 3\kappa^4) \right] / 4 \]  
\( (5c) \)

where $H_1 \equiv H(\psi - \eta)$ and $H_2 \equiv H(\psi - \eta \cdot \kappa) - H(\psi - \eta)$.

It should be pointed out that eq. (2) is exact, i.e., comprises all terms which are often split into a near- and a far-field solution. In fact, only all terms together give a short compact solution and leads to no singularity for any point $\bar{\mathbf{x}}$ near the fault surface or on it. With a view to all investigations using the far-field solution it may be noted that the exact solution in eq. (2) is, in practical terms, as simple and applicable for any location.

The strain components $e_{ij} = (u_{i,j} + u_{j,i})/2$, where the comma between subscripts indicates a spatial derivative, is readily derived from eq. (2):

\[ e_{ij}(\bar{\mathbf{x}}, \bar{\mathbf{l}}) = \frac{1}{B_0^2} \int_{\mathcal{S}} \left( \frac{1}{\eta^2} \cdot p_{ij}(\gamma, \bar{\mathbf{n}}, \bar{\mathbf{u}}) \cdot \omega_{k} + \frac{1}{\eta^2} \cdot q_{ij}(\gamma, \bar{\mathbf{n}}, \bar{\mathbf{u}}) \cdot \omega_{k} \right) \, dS(\bar{\mathbf{z}}) \]  
\( (6) \)

We have omitted the parameters in $\omega_{k}(\gamma, \bar{\mathbf{n}}, \bar{\mathbf{u}})$ and the apostrophe ' indicates a partial derivative $\partial / \partial \eta$. The six functions $p_{ij}$ and $q_{ij}$ for $\chi = a, b, c$ follow in which $\hat{\delta}$ represents the Kronecker-symbol:

\[ p_{ij} = p_{ij}^{\delta} \gamma \cdot \left[ 2p_{ij}^{\delta} \gamma_{ij}^{\delta} \right] + 2p_{ij}^{\delta} \gamma_{ij}^{\delta} \]  
\[ p_{ij} = p_{ij}^{\delta} \gamma_{ij}^{\delta} \]  
\[ q_{ij} = q_{ij}^{\delta} \gamma \cdot \left[ 2q_{ij}^{\delta} \gamma_{ij}^{\delta} \right] + 2q_{ij}^{\delta} \gamma_{ij}^{\delta} \]  
\( (7) \)

If the dislocation field $\Delta u(\xi, \tau), \tau \leq \tau$ is continuous over time, the strain using eq. (6) will be finite at all points on the fault surface where the dislocation and its spatial derivatives are continuous. In section 4 we present a simple, and in practical terms, exact method to calculate the dominant contribution to the strain resulting from the dislocation in the neighborhood of $\bar{\mathbf{x}}$ where $\eta_{\text{min}}$ is small or zero.

3. Fault-Interaction Between Dislocation- and Stress-Field

Eq. (6) is used to determine the interaction on the fault. The method developed herein is applicable for any case. But to show the main idea and not to complicate the notation, we confine ourselves to the case of an even fault surface. As no closed-form solution can exist in general, we resort to a numerical technique. We discretize the field in space and time, using the right side of eq. (4) with $t = m \cdot \Delta t$, $\eta = n \cdot \Delta \eta$, $\psi_{*} = (n-m)\psi_{0}$, where

\[ - 323 - \\ M1K 1/7 \]
$\psi_0 = \Delta \cdot \beta / \gamma_0$ and divide the fault into quadratic subdomains and use the shape- or weighting-function $h(\tilde{z}, \tilde{z})$ illustrated in Fig. 2. There are three different kinds of stresses or equivalent forces by using the subsequent relations.

$$ F_{ik} = \int_0^1 \int_0^1 \sigma_{ij}(\tilde{z}, \tilde{z}) h(\tilde{z}, \tilde{z}) \cdot d\tilde{z} \cdot d\tilde{z} ; \quad h(\tilde{z}, \tilde{z}) = \frac{3}{16} \cdot (1 - z_1 - z_1) ; \quad \Delta \tilde{z} = (1 - z_1 - z_1)$$

acting on the two moving fault surfaces, namely $F_{ik}$ due to the initial stress, $\dot{F}_{ik}$ due to friction and $\ddot{F}_{ik}$ due to the dislocation. The stress caused by the dislocation field is related to the strain-field through $\lambda_{ij} = \lambda (\epsilon_{i1} + c_{22} \epsilon_{22}) \delta_{ij} + 2 \gamma_{ij}$. The equivalent force can be expressed as:

$$ d_{ik} = \mu_0 \sum_{n=0}^{\infty} \sum_{L=1}^{\infty} \sum_{z=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p_{ik} \cdot I_{ij}(\eta^{ij}(\tilde{z}, \tilde{z}, \psi_{mn}) d\tilde{z} - Q_{ik} \cdot I_{ij}(\eta^{ij}(\tilde{z}, \tilde{z}, \psi_{mn}) d\tilde{z} \quad (8)$$

where the notation $F_{ik}$ and $\dot{F}_{ik}$ is specified in Fig. 2 and where $\eta^{ij} = |\tilde{z} - \tilde{z}|$, $\Delta \tilde{z} = |\psi(\tilde{z}, \tilde{z})|$. The geometry constants $P_{ik}$ and $Q_{ik}$ will be given in the next section. For the nth time step, all the linear factors $\Delta \tilde{z}$ in eq. (8), except the last where $m = n - 1$, are already determined. The condition $F_{ik} = 0$ and $\dot{F}_{ik} = \ddot{F}_{ik}$ and eq. (8) are used to obtain $\Delta \tilde{z}$ for $m = n - 1$. This leads to an asymmetric equation system of order $2L$ where $L$ is the number of nodal-points. But actually, no equation system has to be solved, a simple Gauss-Seidel algorithm is sufficient due to the strong dominance of the Pivot-coefficients. According to eq. (8), the number of computations would grow at a rate of $L^2 N^2$ (N number of time steps), an unwieldy number when $L = 500-1000$ and $N = 20-40$. It can be reduced dramatically without losing accuracy to less than $L \cdot N \cdot 300$ multiplications by making use of the relatively small influence of dislocations far from the points being considered.

4. Geometry Constants $P_{ik}$ and $Q_{ik}$

The geometry constants $P_{ik}$ and $Q_{ik}$ may be interpreted as proportional factors to the weighted stress field integrated over the domain $\Sigma(\tilde{z})$ where $W_k$ respectively $W_x$ in eq. (6) equals the weighting function $h(\tilde{z}, \tilde{z})$:

$$ P_{ik} = \frac{1}{4\pi} \int_\Sigma h(\tilde{z}, \tilde{z}) \cdot d\Sigma(\tilde{z}) = \int_\Sigma \frac{1}{4\pi} (\frac{\nu}{1 - 2\nu} \pi_{ik} \cdot \eta_{0}^{ij} + \pi_{ik}^{ij} h(\tilde{z}, \tilde{z}) \cdot d\Sigma(\tilde{z})$$

$$ Q_{ik} = \frac{1}{4\pi} \int_\Sigma h(\tilde{z}, \tilde{z}) \cdot d\Sigma(\tilde{z}) = \int_\Sigma \frac{1}{4\pi} (\frac{\nu}{1 - 2\nu} \pi_{ik}^{ij} \cdot \eta_{0}^{ij} + \pi_{ik}^{ij} h(\tilde{z}, \tilde{z}) \cdot d\Sigma(\tilde{z})$$

---

324 --
\( p_{\beta i} = p_{\gamma j} (\tau_{\gamma i} = \delta_{ij}, \tau_{\gamma j} = \delta_{ij}) \), \( \chi = a, b \) or \( c \) and \( q_{\beta i} \) with the same parameters are defined in eq. (7). \( \mu_{\beta \gamma} \) represent the sum \( p_{\beta i} \cdot \mu_{\beta \gamma} + p_{\gamma j} \cdot \mu_{\beta \gamma} \) and similarly does \( \mu_{i \gamma} \). Both expressions in eqs. (9a-b) are easy to evaluate for large \( \eta = |z^2 - z^2| \) but extremely difficult for \( \eta \leq 2 \) due to infinite stress at points where the weighting function has edges or corners. Using the following relations in eqs. (10a-b) we solved all the above integrals in a very efficient, and in practical terms, exact manner. In comparison, we needed an exorbitant amount of computer time just to verify \( p_{\beta i} \) for three digits, in spite of applying a sophisticated integration technique and reducing theoretically the integration order by one.

\[
P(k, l) = \frac{1}{2\pi} \sum_{m=2}^{2} \sum_{n=2}^{2} \frac{a(1,1,1,1)}{Q(2k+2l+2n)}
\]

\[
Q(k, l) = \frac{1}{8\pi} \sum_{m=2}^{2} \sum_{n=2}^{2} \frac{a(1,1,1,1)}{Q(2k+2l+2n)}
\]

where \( a(0,0) = 36, a(1,0) = a(0,1) = 24, a(1,1) = 16, a(2,0) = a(0,2) = 6, a(1,2) = a(1,2) = 4 \) and \( a(2,2) = 1 \).

\( P(k, l) \) in eq. (10a) and \( Q(k, l) \) in eq. (10b) can be replaced by \( P_{\beta i} \) and \( Q_{\beta i} \) respectively where \( k = z^2 - z^2 \) and \( l = z^2 - z^2 \). The relation in eqs. (10a-b) allow us to calculate \( P_{\beta i} \) and \( Q_{\beta i} \) successively from values for nodal points \( z^2 \) around twice as far and leads for \( P_{\beta i} \) with \( \eta = |z^2 - z^2| \leq 2 \) to an equation system with nine unknowns if we take the symmetry or asymmetry into account. \( Q_{\beta i} \) does not exist for \( \eta \leq 1 \), which makes sense if we note that \( \psi_{z} \) in eq. (6) is zero for \( \eta = 0 \).

5. Example for Slip with Rupture Condition i.e. Stress-Intensity Factor

We choose as example problem the slip in a quadratic fault (see Fig.3) due to uniform initial stress \( \sigma_{01} \). All internal nodal points are released suddenly at an instant \( t = 0 \) and the slip starts. Until the time \( t_{s}(s) = d(s)/a \), where \( d(s) \) is the shortest distance to the slip boundary, the dislocation is identical with the one-dimensional case for which \( \Delta u(s,t) = \sigma_{01} \beta / \mu \) grows linearly. The final (static) slip is reached after \( t = 2a / \beta \) preceeded by weak oscillations in the time interval \( [a / \beta, 2a / \beta] \). The static slip at the mid-point is \( 1.574\sigma_{01} a / \mu \) which is slightly smaller than the theoretical value \( 1.6\sigma_{01} a / \mu \) for the 2-D in-plane slip (\( \nu = 0.2 \) is assumed).

![Fig.3: Progress over time of the slip and the stresses at the slip-boundary](image)

- 325 -
Especially simple is the estimation of the dynamic stress-intensity factor,

\[ \kappa(s,t) = \frac{2}{\pi} \sqrt{r_0} \cdot a(s,t) \]  

which is together with the given critical stress-intensity factor the criterion to release a nodal point. According to Sneddon is the static stress-intensity factor \( \kappa = 2/\pi \cdot \sigma_{II} d^{0.5} \approx 0.637 \sigma_{II} d^{0.5} \) for a circular slip area while we obtain for our quadratic slip area \( \kappa_2 = 0.703 \sigma_{II} d^{0.5} \) (in-plane) and \( \kappa_3 = 0.632 \sigma_{II} d^{0.5} \) (anti-plane).

6. Conclusion

The theoretical and practicable numerical means for a realistic simulation of earthquakes are available. The dynamic interaction on the fault can be determined for an arbitrarily distributed initial stress field, critical stress-intensity field and any relation for friction. Although the discussion in this paper has been focused on earthquake problems, the general solution developed herein can be applied to many other subjects of engineering interest, including crack propagation and arrest in finite elastic bodies, reflection and refraction on a arbitrarily shaped half-space, impacts on elastic bodies like plates and shells, soil-foundation interaction, etc. Pursuit in such directions is underway.

7. Acknowledgements

This paper is based on the work conducted by the first author during his stay at Kyoto University under the program of the International Course in Civil Engineering. His participation is based on an agreement between the Faculty of Engineering, Kyoto University and University of Innsbruck. It is gratefully acknowledged that the stay of the first author in Kyoto was fully supported by the Ministry of Education, Science and Culture (MOKUSHO), Japanese Government. The authors would like to extend their deep appreciation to Professor G.I. Schueller for his effort to establish the agreement between the two universities.

References

[4] Pradlwarter, H.J., and H.Kameda., "Elastic Waves Near the Fault due to a Dislocation Field and Rupture Propagation", 1985, Research Report No.65-ST--*+), School of Civil Engineering, Kyoto University, Kyoto 606, Japan. *: number will be announced at the conference