Taylor-Galerkin Method for Time-Dependent Transport Problems

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Abstract

In this paper the recently proposed Taylor-Galerkin method is briefly illustrated and applied to derive time-accurate finite-element schemes for pure advection and mixed advection-diffusion problems governed by the non-linear Burgers' equation.
By contrast with the Petrov-Galerkin formulations of upwind-type, the Taylor-Galerkin approach provides a rational basis to improve the order of accuracy of the time-discretized equations without introducing special weighting functions, nor free parameters.
To produce an accurate time discretization, the method employs Taylor series expansions in the time-step including second-order and, when possible, third-order time derivatives. These are then evaluated from the governing partial differential equation, so as to obtain simple two-level time-stepping schemes. The resulting time-discretized equation is successively discretized in space using the standard Bubnov-Galerkin finite-element formulation.
Herein, the method is applied to Burgers' equation and it is shown to be effective in all cases studied, including ones with discontinuous solutions.

1. Introduction

The numerical solution of initial-boundary value problems systematically involves two distinct discretization processes: temporal discretization and spatial discretization. In the last years, considerable interest has developed in the potential advantages of the finite-element method for the spatial discretization of fluid mechanics problems. The high spatial accuracy attainable with Galerkin methods is now well established, but the difficulty comes in coupling the spatial approximation provided by finite-element methods to the time discretization. To illustrate the situation, consider the initial-boundary value problem defined by

$$u_t = Lu \tag{1.1}$$

where L is a differential operator.

In the semidiscrete Galerkin formulation, the approximate solution $U(\underline{x},t)$ to eq.(1.1) is represented as product of shape functions $N_I(\underline{x})$ that are independent of time and nodal values $U_I(t)$ which are independent of x and incorporate the time dependence:

$$U(\underline{x},t) = N_I(\underline{x})\, U_I(t)$$

The Galerkin finite element approximation to eq.(1.1) is then obtained by setting

$$< U_t - LU, N_I > = 0 \quad \text{, for all I.} \tag{1.2}$$

where $<u,v>$ denotes the L_2 inner product $\int uvdx$ over the domain of the problem. If the N_I's are linear shape functions on a uniform mesh of size h, the semidiscrete nodal equations resulting from (1.2) are accurate to $0(h^4)$ when the operator L represents a first-order hyperbolic system, either linear or quasi-linear. However, the system of ordinary differential equations (1.2) still has to be integrated forward in time to produce the transient response, and the fourth-order accuracy can be quickly eroded if the finite difference approximation to the time-derivative term U_t is not of comparable accuracy.

In order to generate an accurate time-differencing to be associated with the high spatial resolution attainable with the semidiscrete Galerkin formulation, the Taylor-Galerkin method was introduced in [1]. The essence of the method is briefly recalled in section 2. It consists of developing improved versions of the usual time-stepping methods (Euler, leap-frog, Crank-Nicolson) on the basis of Taylor series expansions including second- and third-order terms in the time step. Then, to preserve the simplicity and ease of implementation of the usual two-level time-stepping methods, the second and third-order time derivatives are evaluated from the original partial differential equation. This process yields a generalized governing equation which is discretized in time only, the spatial variable being left continuous. Such equation is successively discretized in space using the conventional Bubnov-Galerkin finite element method. The effectiveness of the Taylor-Galerkin method has been illustrated in [1] for the case of pure advection problems in one and two space dimensions. In [2], the method was applied successfully to mixed advection-diffusion problems. Herein, it will be illustrated for the solution of the non-linear Burgers' equation in both its inviscid and viscous forms.

2. Generalized time-stepping methods

To produce high-order accurate methods for the time discretization of eq.(1.1), consider the following Taylor series expansions in the time step Δt:

$$u^{n+1} = u^n + \Delta t\, u_t^n + \tfrac{1}{2}\Delta t^2 u_{tt}^n + \tfrac{1}{6}\Delta t^3 u_{ttt}^n + \ldots \tag{2.1}$$

$$u^{n-1} = u^n - \Delta t\, u_t^n + \tfrac{1}{2}\Delta t^2 u_{tt}^n - \tfrac{1}{6}\Delta t^3 u_{ttt}^n + \ldots \tag{2.2}$$

where superscript n indicates the time level, so that $t^n = n\Delta t$.

To generalize the usual forward-time, or Euler time-stepping method, we combine eqs.(1.1) and (2.1) and write

$$u_t^n = \frac{u^{n+1} - u^n}{\Delta t} - \tfrac{1}{2}\Delta t\, u_{tt}^n - \tfrac{1}{6}\Delta t^2 u_{ttt}^n = Lu^n \tag{2.3}$$

Assuming for simplicity that the operator L is linear, successive time-differentiations of eq.(1.1) give

$$u^n_{tt} = L u^n_t = L^2 u^n \quad \text{and} \quad u^n_{ttt} = L^2 u^n_t \tag{2.4}$$

so that, approximating u^n_t in (2.4) by $u^{n+1}-u^n/\Delta t$, we may rewrite eq.(2.3) in the form

$$(1 - \tfrac{1}{6}\Delta t^2 L^2)\left(\frac{u^{n+1}-u^n}{\Delta t}\right) = Lu^n + \tfrac{1}{2}\Delta t\ L^2\ u^n \tag{2.5}$$

which represents a generalized, third-order accurate, Euler time-stepping method. When the operator L represents a first-order hyperbolic system, the scheme (2.5) involves first- and second-order spatial derivatives only, and, therefore, finite elements with C^0 continuity may be employed for the spatial discretization. On the contrary, when L represents a parabolic problem (e.g. advection-diffusion), the scheme (2.5) cannot be used in conjunction with C^0 elements, since third-order spatial derivatives are involved. In this case, a second-order accurate Euler method may be employed in the form

$$\frac{u^{n+1}-u^n}{\Delta t} = Lu^n + \tfrac{1}{2}\Delta t\ L\ u^n_t \tag{2.6}$$

or

$$(1 - \tfrac{1}{2}\Delta t\ L)\left(\frac{u^{n+1}-u^n}{\Delta t}\right) = L\ u^n \tag{2.7}$$

Similarly, a generalised leap-frog method may be derived by subtracting (2.2) from (2.1). The result reads

$$\frac{u^{n+1}-u^{n-1}}{2\Delta t} = u^n_t + \tfrac{1}{6}\Delta t^2 u^n_{ttt} = L u^n + \tfrac{1}{6}\Delta t^2 L^2 u^n_t \tag{2.8}$$

or, approximating u^n_t by $u^{n+1}-u^{n-1}/2\Delta t$,

$$(1 - \tfrac{1}{6}\Delta t^2 L^2)\left(\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right) = L u^n \tag{2.9}$$

Scheme (2.9) is fourth-order accurate in the time step.

Finally, a generalised Crank-Nicolson time-stepping method can be derived by writing

$$\frac{u^{n+1}-u^n}{\Delta t} = \tfrac{1}{2}(u^n_t + u^{n+1}_t) + \tfrac{1}{4}\Delta t(u^n_{tt} - u^{n+1}_{tt}) + \tfrac{1}{12}\Delta t^2(u^n_{ttt} + u^{n+1}_{ttt}) \tag{2.10}$$

and replacing the time derivatives by spatial derivatives, as indicated in eq.(2.4). The resulting scheme is

$$(1 - \tfrac{1}{6}\Delta t^2 L^2)\left(\frac{u^{n+1}-u^n}{\Delta t}\right) = \tfrac{1}{2}L(u^n + u^{n+1}) + \tfrac{1}{4}\Delta t L^2 (u^n - u^{n+1}) \tag{2.11}$$

and represents a fourth-order accurate generalization of the Crank-Nicolson method.

3. Application to Burgers' equation

Let us first consider the solution of the inviscid form of Burgers' equation in one dimension:

$$u_t = -\partial_x(\tfrac{1}{2}u^2) \tag{3.1}$$

with initial data $u(x,0) = 1$ for $x \leqslant 0$, $u(x,0) = 0$ for $x > 0$, and the boundary condition $u(0,t) = 0$.

By successive time differentiations of eq.(3.1), we obtain

$$u_{tt} = -\partial_x(uu_t) = \partial_x(u^2 u_x) \tag{3.2}$$

and

$$u_{ttt} = \partial_x(u^2 u_{tx}) - \partial_x[2u^2 (u_x)^2] \quad (3.3)$$

so that the generalized Euler time-stepping method for the inviscid Burgers' equation is obtained in the form

$$\{1 - \partial_x(\frac{u^2 \Delta t^2}{6}\partial_x)\} (\frac{u^{n+1} - u^n}{\Delta t}) = - \partial_x(\tfrac{1}{2}u^2)^n + \partial_x(\frac{u^2 \Delta t}{2}(1 - \tfrac{2}{3}\Delta t u_x)u_x)^n \quad (3.4)$$

To obtain a fully discrete equation, the Galerkin formulation was applied to eq.(3.4) assuming a uniform mesh of piecewise linear elements with size h. The numerical results obtained with $\Delta t = 0.5h$ and $\Delta t = h$ are shown in Fig. 1a. By comparison with the analytical solution, it is noted that the shock speed is well predicted with both time steps and that the shock remains very sharp, with virtually no oscillations in the case $\Delta t = h$, i.e. when the generalized Euler scheme is run at its stability limit.

As a second test problem, we consider Burgers' equation with dissipation:

$$u_t = - \partial_x(\tfrac{1}{2}u^2) + \nu u_{xx} \quad (3.5)$$

The initial data at time $t = 1$ are chosen in tne form [3]

$$u(x,1) = x/[1 + t_0^{-1/2} \exp(x^2/4\nu)] \quad (3.6)$$

where $t_0 = \exp(1/8\nu)$. Assuming that $u(0,t) = 0$, the problem has the following analytical solution:

$$u(x,t) = \frac{x/t}{1 + \exp(x^2/4\nu t)\,(t/t_0)^{1/2}} \quad (3.7)$$

As explained in section 2, the presence of a diffusion operator in eq.(3.5) prevents us from using a third-order accurate Euler scheme in conjunction with C^0 finite elements. To circumvent this difficulty, we have proposed in [2] to use a splitting-up method in which advection and diffusion are treated separately in two distinct phases of the time integration procedure. The advection phase is treated first by the third-order Euler method in eq.(3.4) and yields intermediate values $u^{n+1/2}$. These are then used for the diffusion phase which is based on the following second-order Euler method:

$$(1 - \frac{\nu \Delta t}{2}\partial_x^2)(\frac{u^{n+1} - u^{n+1/2}}{\Delta t}) = \nu u_{xx}^{n+1/2} \quad (3.8)$$

The above splitting-up method was applied to solve eqs.(3.5) – (3.6) on a uniform mesh of 100 linear elements ($h = 0.01$) with a time step $\Delta t = h$. Fig. 1b shows the computed propagation of the initial shock for $\nu = 0.005$, and Fig. 1c that for the more advective case, $\nu = 0.0005$. In both cases the numerical predictions are seen to be in excellent agreement with the analytical solutions. For the purpose of comparison, we have reported in Fig. 1d the solution obtained for $\nu = 0.0005$ with the standard Euler-Galerkin formulation. Here, the computed shock speed is slightly in error and rather severe oscillations do appear at the shock front.

4. Conclusion

The Taylor-Galerkin method appears to be capable of producing numerical schemes of high accuracy for the solution of time-dependent convective and convective-diffusive transport problems, including non-linear situations. As shown in [1,2], the method is easily applied to multi-dimensional problems, and, in contrast to the Petrov-Galerkin weighted residual formulations of upwind-type, it does not require the use of special weighting functions, nor the determination of free parameters to maximise the accuracy.

References

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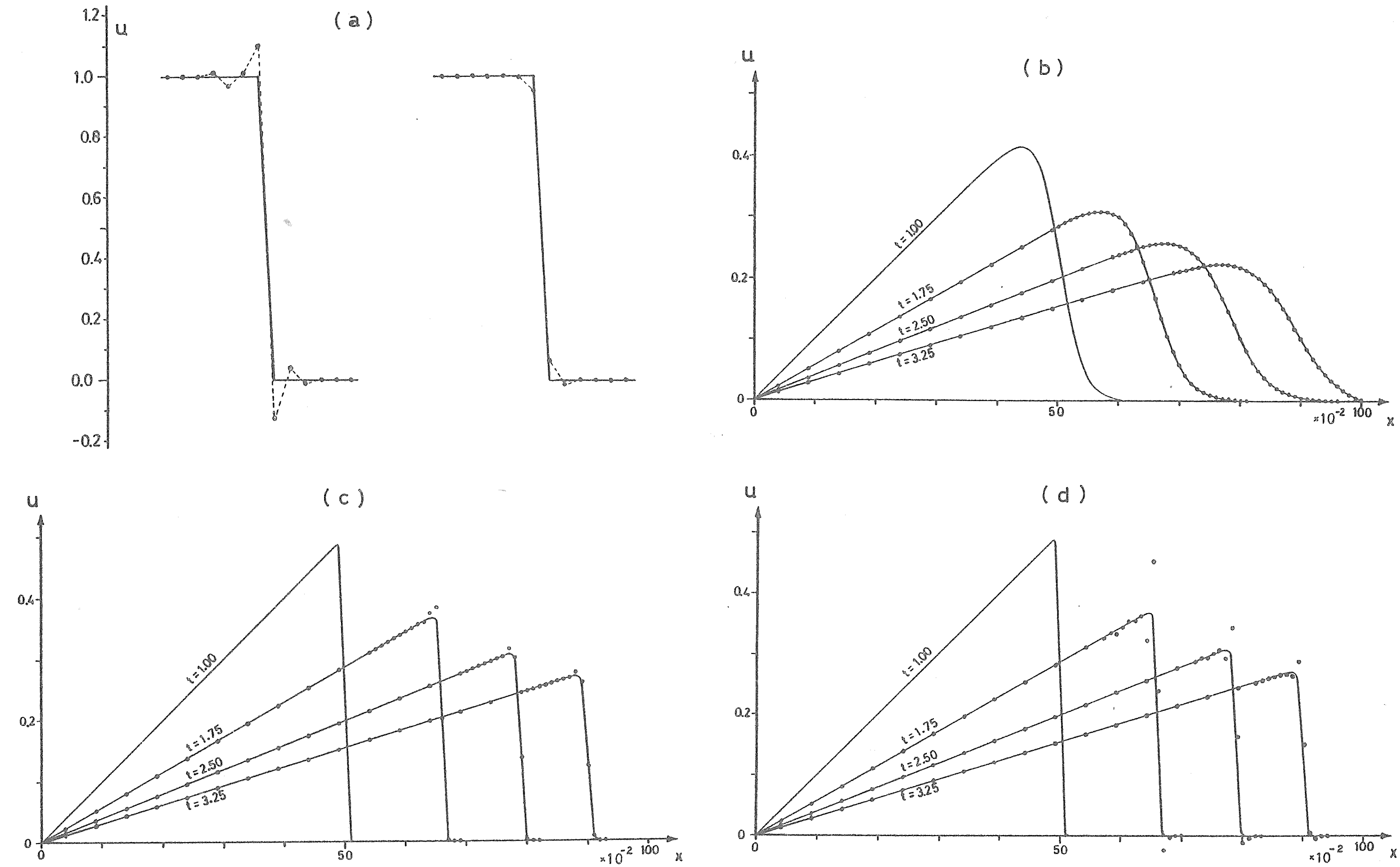

Fig. 1 : Solution of Burgers' equation by Taylor-Galerkin method : (a) inviscid form, $\Delta t/h = 0.5$ (left) and 1.0 (right) ; (b) viscous form, $\nu = 0.005$; $\Delta t/h = 1.0$; (c) id., $\nu = 0.0005$, $\Delta t/h = 1.0$; (d) Euler-Galerkin, $\nu = 0.0005$, $\Delta t/h = 1.0$. (——) : analytical solution ; (•••) : numerical solution.