Blowdown Analyses Using Space-Time Finite Element Methods

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ABSTRACT
An analysis based on the least-square finite element method in space and time has been developed for transient flow boiling; particular emphasis has been placed on blowdown from subcooled liquid conditions. The numerical solutions obtained by the present algorithm are compared with those obtained by the method of characteristics. Agreement with experimental results is very good.
1. Introduction

The loss of coolant accident is a hypothetical transient in a nuclear power reactor which must be designed for before such a reactor can be licensed. The heat transferred to the coolant in the system is the major factor in determining whether the reactor could satisfactorily withstand such an accident. The accurate prediction of the velocity and temperature of the coolant during a hypothetical loss of coolant accident is essential in predicting the heat transfer. In general, the transient flow-boiling in a reactor heat transport system can be described by a system of one-dimensional partial differential equations (P.D.E.) of the hyperbolic type (Hancock and Banerjee [1], Mathers et al. [2]). To solve them, fast and accurate numerical procedures are needed.

There is a wide variety of solution procedures available for P.D.E. of the hyperbolic type (Richtmeyer and Morton [3], Roche [4]). Concerning the numerical solutions of the transient flow-boiling equations several numerical methods have been proposed during the past years. Lyczkowski et al. have given several explicit numerical schemes to solve the homogeneous equations of change for one-dimensional fluid flow and heat transfer [5]. The most successful technique proposed in [5] is the alternating gradient method which is based on the two-step Lax-Wendroff procedure. The characteristic finite difference (CFD) procedure has been introduced by Hancock et al. [6]. Recently, Van Goethem has generalized the ICE type finite elements for transient two-phase problems [7]. In some cases, however, very small space and time steps must be used to get good solution accuracy and stability.

In this paper, the numerical solutions of the transient flow-boiling equations are based on the least-square finite element method where the governing equations are written in the conservative forms and discretized with respect to both the space and time variables by means of an implicit scheme. The convective term and the source term are made implicit for more stable formulations and to permit larger time steps. It is shown that the algorithm is unconditionally stable against round-off error propagations and the finite element scheme obtained is second order accurate in the time variable and fourth order in the space variable. This high accuracy permits the use of coarser computational grids, and bigger time steps as compared to those required by traditional finite difference methods. The present numerical method has been applied with success on hyperbolic equations [8] and convection-diffusion problems [9]. Furthermore, this numerical procedure has been implemented in a computer program CFEM for the calculation of steady/unsteady, single-phase/two-phase flow [10]. The CFEM is a thermohydraulic module of the modular system EAC (European Accident Code) [11].

We will confine our attention in the present study to the simplest one-dimensional flow-boiling model, which is based on the assumption that the vapour and liquid phases move at equal pressure and equal temperature. Several standard problem solutions are presented. The standard problems were selected to isolate effects associated with analyses of the blowdown and emergency cooling phases of postulated loss-of-coolant accidents in nuclear power reactors. The blowdown problem is different from the shock tube problem in at least two respects: boundary conditions and steam tables. The boundary conditions are of vital interest in the blowdown problem because they significantly affect the pressure and mass in the system. The boundary conditions in the shock tube problem are insignificant because the boundaries are not reached. The blowdown problem uses water-steam properties which are much more complicated than the perfect gas properties frequently used in the shock tube problem.

2. Formulation of the Problem

2.1 Basic equations

In conservation law form, assuming constant flow area and that the contribution of gravitational effects on the total energy is small, the mass, momentum and energy equations are

\[
\frac{\partial X}{\partial t} + \frac{\partial S}{\partial z} = Q
\]  

(1)

where the quantities X, S and Q for homogeneous thermal equilibrium flow are defined as follows:

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>S</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>( \rho )</td>
<td>( G )</td>
<td>0</td>
</tr>
<tr>
<td>Momentum</td>
<td>( G )</td>
<td>( \frac{G^2}{\rho} + p )</td>
<td>(- \rho g - \frac{\ell v^2}{D_h} - \frac{G G_i}{2 \rho} )</td>
</tr>
<tr>
<td>Energy</td>
<td>( \rho H )</td>
<td>( GH )</td>
<td>( \phi + \frac{\partial p}{\partial t} + \frac{\rho}{\rho} \frac{\partial p}{\partial z} )</td>
</tr>
</tbody>
</table>
\[ \rho = \alpha \rho_v + (1-\alpha) \rho_g \]  
\[ H = xH_v + (1-x)H_g \]  

In the above equations, \( x \) and \( \alpha \) represent the vapour quality and the void fraction, respectively; they must be related by an empirical slip relationship. For \( 0 < x < 1 \) and \( 0 < \alpha < 1 \), the quantities \( \rho_v, \rho_g, H_v, H_g \) are evaluated as functions of the pressure on the saturation line. To close the system of equations, the following constitutive relations are required:

(i) State equation \( \rho = \rho(p, H) \)
(ii) \( f = \text{friction factor} \)
(iii) \( \psi^2 = \text{two-phase friction multiplier} \)
(iv) Empirical slip relationship

### 2.2 Constitutive Equations

The constitutive equations used in the present calculation are the following:

<table>
<thead>
<tr>
<th>Constitutive relation</th>
<th>Equation used</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation of state</td>
<td>( \rho = \rho(p, H) ) from steam-water properties</td>
<td>[12]</td>
</tr>
<tr>
<td>Friction factor</td>
<td>[ f = \begin{cases} 0.184 \frac{Re^{0.2}}{Re} &amp; \text{for } Re \gg 2000 \ \frac{256}{Re} &amp; \text{for } Re &lt; 2000 \end{cases} ]</td>
<td>[1]</td>
</tr>
<tr>
<td>Two-phase friction multiplier</td>
<td>First Lockhart-Martinelli’s correlation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \psi^2 = 1 + \frac{20}{x_{tt}} + \frac{1}{x_{tt}^2} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( x_{tt} = \left( \frac{1-x}{x} \right)^{0.9} \left( \frac{\rho_v}{\rho_g} \right)^{0.5} \left( \frac{\mu_g}{\mu_v} \right)^{0.1} )</td>
<td>[13]</td>
</tr>
<tr>
<td>Void fraction</td>
<td>Second Lockhart-Martinelli’s correlation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \alpha = 1 - \frac{1}{(1 + \frac{21}{x_{tt}})^{1/2}} )</td>
<td>[13]</td>
</tr>
</tbody>
</table>

### 3. Least-square Finite Element Weak Formulation in Time and Space

An alternative least-square weak formulation for equation (1) is to look for a minimum of the following functional

\[ I = \int_{t_1}^{t_2} \left( \frac{\partial X}{\partial t} + \frac{\partial S}{\partial z} - Q \right)^2 \, dz \, dt \]  

for arbitrary functions \( X(z,t) \) and \( S(z,t) \) satisfying the boundary conditions. The finite element discretization consists in dividing the domain \((t,z)\) into small rectangular elements (Fig. 1). The three quantities \( X, S \) and \( Q \) in each element can be approximated respectively by:

\[ X = [P_1, P_2, P_3, P_4] \begin{bmatrix} X_1, X_2, \Delta X_1, \Delta X_2 \end{bmatrix}^T \]  
\[ S = [P_1, P_2, P_3, P_4] \begin{bmatrix} S_1, S_2, \Delta S_1, \Delta S_2 \end{bmatrix}^T \]  
\[ Q = [P_1, P_2, P_3, P_4] \begin{bmatrix} Q_1, Q_2, \Delta Q_1, \Delta Q_2 \end{bmatrix}^T \]
where:
\[ X_1, S_1, Q_1 = \text{nodal values} \]
\[ \Delta X_1, \Delta S_1, \Delta Q_1 = \text{nodal increment values} \]

The \( P_i \) are called "element shape functions". Using the following transformations

\[ dt = \Delta t \, d\xi \]
\[ dz = \Delta z \, d\eta \]

the "element shape functions" have the form

\[ P_1 = 1 - \eta \quad P_2 = \eta \quad P_3 = (1-\eta)\xi \quad P_4 = \eta \xi \]

Substituting (5), (6) and (7) into (4), one obtains the contribution for one element to the functional

\[ I = \int \int \left( \frac{\partial P_1}{\partial \xi} \Delta X_1 + \frac{\partial P_2}{\partial \xi} \Delta X_2 \right) + \frac{\partial P_1}{\partial \eta} \Delta S_1 + \frac{\partial P_2}{\partial \eta} \Delta S_2 \]

\[ + \frac{\partial P_3}{\partial \xi} \Delta Q_1 + \frac{\partial P_4}{\partial \xi} \Delta Q_2 \right) \Delta t (P_1 Q_1 + P_2 Q_2 + P_3 \Delta Q_1 + P_4 \Delta Q_2) \, d\xi \, d\eta \]

(8)

The numerical solutions of the conservation equations are based on two types of solution.

"Direct solution"

The "direct solution" is obtained by minimizing the functional (8) with respect to \( \Delta X \), where the variations of the convective term and the source term as a function of the principal unknowns \( \Delta X \), have been introduced in order to render the system more explicit and stable:

\[ \Delta S_i = a_i \Delta X + \Delta b_i \]

(9)

\[ \Delta Q_i = c_i \Delta X + \Delta d_i \]

(10)

The minimization process for the functional of one element gives a matrix equation for each individual element:

\[ [A^e] \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \end{bmatrix} = [B^e] \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} + [C^e] \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} + [D^e] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + [E^e] \begin{bmatrix} \Delta d_1 \\ \Delta d_2 \end{bmatrix} \]

(11)

where the elements of the matrices \([A^e], [B^e], [C^e], [D^e], [E^e]\) depend on \( a_i, c_i \) and \( \alpha = \Delta t/\Delta z \).

Summing up over all elements yields the standard form of assembled equations

\[ [A] \begin{bmatrix} \Delta X \end{bmatrix} = [B] \begin{bmatrix} S \end{bmatrix} + [C] \begin{bmatrix} \Delta b \end{bmatrix} + [D] \begin{bmatrix} Q \end{bmatrix} + [E] \begin{bmatrix} \Delta d \end{bmatrix} \]

(12)

"Inverse solution"

In the "inverse solution" \( \Delta S \) constitute the principal unknowns, while the variations \( \Delta X \) and \( \Delta Q \) are given.

The minimization of the functional (8) with respect to \( \Delta S \) and the assembling of matrix equations for each individual element gives:

\[ [F] \begin{bmatrix} \Delta S \end{bmatrix} = - \frac{3}{2} [F] \begin{bmatrix} S \end{bmatrix} + [M] \left( - \frac{3}{4\alpha} \Delta X - \frac{3}{4\alpha} \Delta t Q - \frac{1}{2\alpha} \Delta t \Delta Q \right) \]

(13)

A stability analysis [9] using a Fourier type of analysis on a homogeneous mesh has shown the unconditional stability of the finite element equation. Using Taylor's series it can be shown that the error of discretization is of the order \( \mathcal{O}(\Delta t^2, \Delta t \Delta x^2, \Delta x^4) \).

Essentially, the solution of the transient flow-boiling equations is based on the following steps:

(i) "Direct solution" of energy equation

(ii) "Inverse solution" of mass equation

(iii) "Inverse solution" of momentum equation
4. Numerical Results

Two solutions obtained with the least-square finite element weak-formulation in time and space are presented in this section.

4.1 Standard Problem no. 1: Flow in a vertical pipe with heat addition

The problem is based on a vertical pipe, initially conducting an upward flow of water, consisting of four sections with the following specifications:

<table>
<thead>
<tr>
<th>Section</th>
<th>Length (m)</th>
<th>Flow-area (m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.181</td>
<td>127 x 10⁶</td>
</tr>
<tr>
<td>2</td>
<td>1.509</td>
<td>127 x 10⁶</td>
</tr>
<tr>
<td>3</td>
<td>3.281</td>
<td>127 x 10⁶</td>
</tr>
<tr>
<td>4</td>
<td>1.345</td>
<td>127 x 10⁶</td>
</tr>
</tbody>
</table>

The initial and boundary conditions for this problem are:

- Initial conditions:
  \[ u(z) = 1.089 \text{ m/s} \]
  \[ H(z) = 1184.7 \times 10^3 \text{ J/kg} \]
  \[ 0 \leq z \leq L \]

- Boundary conditions:
  \[ z = 0 \]
  \[ p = p_1 = 7.102 \times 10^6 \text{ Pa} \]
  \[ H(0,1) = 1184.7 \times 10^3 \text{ J/kg} \]
  \[ z = L \]
  \[ p = p_0 = 6.984 \times 10^6 \text{ Pa} \]

This problem involves a transient initiated by application of a uniform heat source of 9000 W/m instantaneously into the fluid in section 3. In this calculation, a rather coarse mesh is used \((\Delta z_{\text{min}} = 0.3658 \text{ m})\).

At the beginning of the transient the speed of the fluid is 1.089 m/sec and the time step is 0.2 sec. The use of big time steps is typical of the present implicit finite element formulation. In Figs. 2 and 3, the inlet mass-flowrate and the outlet mass-flowrate are plotted as a function of time. Boiling start at time \( t \approx 1 \text{ sec} \) and the outlet mass-flow rate increases rapidly. After \( \approx 7 \text{ sec} \), a new equilibrium state is established. Agreement with other numerical results [1] is satisfactory.

4.2 Standard Problem no. 2: Blowdown of a steam-water mixture

Consider blowdown of a closed cylindrical pipe, filled with high enthalpy water, which is suddenly opened at one end to the atmosphere. The geometry (4.09 m length and 73.2 mm dia) and initial conditions (7 MPa, 240°C) have been selected to agree with an experiment by Edwards and O’Brien [14].

The calculations have been made with the tube subdivided into 15 segments with a time step of 10³ sec.

Numerical solutions for the pressure obtained by the present method and experimental pressure histories at the closed end are shown in Figs. 4 and 5. As can be seen, the agreement is excellent. The long term void fraction transient results are compared with experiment results at the middle of the pipe in Fig. 6. The agreement is fair.

5. Conclusions

A least square weak formulation of the standard problems has been shown to result in an extremely stable and accurate finite element type algorithm. As a result a coarse meshes and large time steps can be used. The integration of these standard problems together with the supporting empirical relations give results which are in good agreement with the experiments.

References


Fig. 1 - Space-time finite element

Fig. 2 - Inlet flow

Fig. 3 - Outlet flow
Fig. 4 - Pressure at the closed end. Initial depressurization.

Fig. 5 - Pressure at the closed end. Long term depressurization.

Fig. 6 - Void fraction at the middle of the pipe.