Evaluation Procedures for Single Axis Sinusoidal Test to Design Spectrum Requirements

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SUMMARY

In the early 1970's, seismic qualification of electrical and mechanical equipment was often tested to a single frequency input along a single axis [1]. At that time, this practice was considered acceptable due to hardware limitations. Later, multi-frequency test input requirements were added to industry standards [2,3]. Single frequency test results are no longer acceptable unless they can be demonstrated as conservative. This paper outlines procedures which can be used to demonstrate the conservatism of a single frequency test.

Seismic floor response spectra, generated from building analysis, generally exhibit broadband characteristics. This is because the seismic ground spectrum is broadband [4]. Also, floor spectrum peaks are broadened to account for uncertainties in structural modeling and site soil properties [5]. On the other hand, the spectrum which corresponds to single frequency testing is a narrow band as shown in Figure 1. Hence, justification is needed to demonstrate the conservatism of a single frequency test if the test spectrum does not envelop the design floor spectrum entirely.

Two simple procedures are provided in this paper for the purpose of evaluating the adequacy of a single frequency single axis test. For the purpose of evaluating the adequacy of single frequency test to meet broadband response spectrum requirements, the proposed procedure is based on the equivalence of maximum response of a dynamic system when it is subjected to either type of design input. The required information used for the evaluation is usually recorded and available in the test report. This procedure is applicable to systems with or without closely-spaced modes. When evaluating against broadband design spectra and multi-axes requirements, an empirical procedure is proposed and it has been found conservative.

These two proposed procedures provide a quick assessment on the adequacy of a single frequency test performed earlier. The use of these procedures may eliminate the need of expensive and time consuming equipment re-testing.
1. Introduction

The evaluation procedure on the adequacy of single frequency test to a broad-band spectrum requirement is based on comparing the maximum response in a dynamic system to a given input. Consider a multi-degree-of-freedom system subject to a harmonic input as shown in Figure 2. The amplitude of the acceleration input is \( A_j \), and the input frequency is \( \omega_j \), where \( j \) is any arbitrary integer. It can be shown that the relative acceleration of any mass \( m_k \) is given by

\[
\ddot{x}_{k,j}(t) = \sum_{m=1}^{n} \left( k_k^m \xi_k^m \right) \frac{A_j e^{\text{i} \omega_j t}}{1 - \left( m - \omega_j \right)^2 + \text{i} \left[ 2 \beta (m - \omega_j) \right]}
\]  

(1)

where \( \ddot{x}_{k,j}(t) \) is the complex response of mass point \( k \) due to input acceleration with frequency \( \omega_j \), \( k_k^1, k_k^2, \ldots \) the mode shapes at mass point \( m_k \), \( \Gamma_1, \Gamma_2, \ldots \) the participation factor for the 1st, 2nd, \ldots modes, \( \omega_1, \omega_2, \ldots \) the circular frequencies, \( n \) is the number of significant modes, and \( i \) the complex root \( \sqrt{-1} \).

Let \( Q_{km} \) be the modal constant for mass point \( k \) and \( F_{mj} \) the complex frequency response acceleration for the \( m \)th mode, where

\[
Q_{km} = \sum_{m} \sigma_k^m \Gamma_m, \quad \text{and} \quad F_{mj} = \frac{A_j}{\left[ 1 - \left( m - \omega_j \right)^2 + \text{i} \left[ 2 \beta (m - \omega_j) \right] \right]}
\]  

(2)

(3)

The relative acceleration at mass point \( m_k \) in the complex domain is

\[
\ddot{x}_{k,j} = \sum_{m=1}^{n} Q_{km} F_{mj},
\]  

(4)

where \( \ddot{x}_{k,j} \) and \( F_{mj} \) are complex numbers and \( Q_{km} \) are real quantities.

The complex response acceleration \( F_{mj} \), which is the product of the transfer function and input acceleration amplitude is given in eq. (3). Assuming only the first \( n \) modes are significant and the effect of higher modes are ignorable, eq. (4) provides a sufficient number of equations for the number of unknowns \( Q_{km} \). The only unknowns in eq. (4) are \( Q_{km} \). In the same equation, \( F_{mj} \) relates to the dynamic properties of a system and to the input acceleration. The recorded acceleration response is \( \ddot{x}_{k,j} \). It is therefore possible to obtain a set of simultaneous equations from eq. (4), each corresponding to one single frequency test.

2. Maximum Response

The acceleration response is a complex number and it can be written in the form
(5)

\[ X_{k,j} = a_j + i b_j \]

where \(a_j\) and \(b_j\) are real. Coefficient \(a_j\) and \(b_j\) are related to the recorded acceleration amplitude and phase angle at mass point \(k\) and they can be calculated by one of the following methods:

2.1 Case 1 Both Phase Angles and Amplitudes of the Relative Acceleration Response Were Recorded.

In this case, modal constants \(Q_{km}\) can be solved from eqs. (4) and (5) directly. Let \(\ddot{X}_{k,j}\) be the recorded acceleration amplitude at mass point \(k\) due to sinusoidal input with frequency \(\omega_j\) and \(\theta_j^0\) the phase angle. Eqs. (4) and (5) can be re-written as \(n\) simultaneous equations:

\[ a_k + i b_k = \sum_{m=1}^{n} Q_{km} \hat{F}_{m,k} \] (6)

Also,

\[ (a_k^2 + b_k^2)^{1/2} = |\ddot{X}_{k,j}| \] (7)

and

\[ b_k/a_k = \tan \theta_j^0 \] (8)

where \(k = 1, 2, ..., n\), and \(a_k, b_k, Q_{km}\) are unknowns, \(|\ddot{X}_{k,j}|\) and \(\theta_j^0\) are the known amplitudes and phase angles. Modal constants \(Q_{km}\) are obtainable by solving eqs. (6) through (8).

2.2 Case 2 Only Maximum Acceleration Amplitude Was Recorded.

In this case, the phase angle is calculated indirectly. Since modal constants \(Q_{km}\) are real and response accelerations \(F_{mj}\) are complex, it is possible to write

\[ F_{mj} = f_{mj} + i g_{mj} \] (9)

where \(f_{mj}\) and \(g_{mj}\) are real and they are calculated from test data using eq. (3).

Using eqs. (5) and (9), it is possible to re-write eq. (4) into \(2n\) equations. Substituting eqs. (5) and (9) with eq. (4), we obtain

\[ a_k + i b_k = \sum_{m=1}^{n} Q_{km} (f_{m,k} + i g_{m,k}) \] (10)

eq. (10) can be divided into \(2n\) equations, relating the real and imaginary parts. These have the following forms:

\[ a_k = \sum_{m=1}^{n} Q_{km} f_{m,k} \] (11)

\[ b_k = \sum_{m=1}^{n} Q_{km} g_{m,k}, \text{ and} \] (12)
\[ (a_i^2 + b_i^2)^{1/2} = |x|_{k,i} \]

where \( f_{ni} \) and \( g_{ni} \) are given by eqs. (9) and (3) and \( |x|_{k,i} \) is the recorded maximum amplitude from testing.

Eqs. (11), (12), and (13) consist of 3n equations with 3n unknowns. It is possible to solve for modal constants \( Q_{kn} \) from these equations.

3. Response Spectrum Analysis

The response of equipment subjected to broad band response spectrum can be determined by analysis. The maximum acceleration response of a multi-degree-of-freedom system at mass point \( k \) by response spectrum techniques has the form:

\[
(Acc)_{k,\text{design}} = \left[ \sum_{n=1}^{n'} (\psi_k^m \Gamma_m^k a_m)^2 \right]^{1/2}
\]

\[
+ \sum_{q=1}^{p} \left( \sum_{m=1}^{n_q} (\psi_k^m \Gamma_m^q a_m)^2 \right)^{1/2}
\]

where \( n' = n - \sum_{q=1}^{p} n_q \), is the number of individual separated modes, \( n \) the number of significant modes considered, \( p \) the number of groups of closely spaced modes, \( n_q \) the number of closely spaced modes in group \( q \), and \( \psi_k^m \Gamma_m^q a_m \) the peak value of the response of the element attributed to the \( m^\text{th} \) mode of group number \( q \).

In the case when closely-spaced modes are absent, \( n' \) becomes \( n \) and the second term on the right-hand side vanishes.

Values of mode shapes \( \psi_k^m \) and modal participation factors \( \Gamma_m^k \) can be calculated according to methods described earlier by either eqs. (6) through (8) or eqs. (11) through (13). The design acceleration \( (Acc)_{k,\text{design}} \) is then calculated using eq. (14).

4. Evaluation Procedures

The single frequency test is adequate if the maximum acceleration or maximum stress at any point in the equipment under the test condition is greater than or equal to the maximum values determined from response spectrum analysis. The evaluation procedures are summarized below:

If only maximum acceleration is of interest,

a. Identify first the maximum acceleration response at critical locations in the equipment during single frequency single axis testing. The maximum relative acceleration at mass point \( k \), \( |x|_{k,ij} \), becomes:
\[ \tilde{\dot{X}}_{k,j} = (\text{Acc})_{k,\text{test}} - \tilde{\lambda}_j \]  

(15)

b. Identify modal dampings \( \beta_1, \beta_2, \ldots, \beta_n \) and modal frequencies \( \omega_1, \omega_2, \ldots, \omega_n \) for the significant modes.

c. Use either eq. (2) and eq. (6) through (8), or eqs. (11) through (13) to calculate modal constant \( Q_{km} \).

d. Calculate the required design acceleration value \( (\text{Acc})_{k, \text{design}} \) for a broad band BSS from eq. (14).

e. The single frequency test is adequate if \( |\tilde{\dot{X}}|_{k,j} > (\text{Acc})_{k, \text{design}} \) for any \( j \).

The product of mode shapes and modal participation factors is calculated using the method delineated in this report. This evaluation procedure is demonstrated by an example problem below.

**Example Problem**

Consider a reactor control benchboard which has been qualified by testing. The range switch, a device mounted near the top of the benchboard, has also been qualified to 8.5g. The dynamic model shown in Figure 3 represents the front-to-back motion for the benchboard. Figure 4 shows the required response spectrum to which the benchboard is to be qualified. The damping for the benchboard was determined by testing to be approximately 6% for all modes.

Natural frequencies for the benchboard are determined by testing and found to be \( f_1 = 13 \) Hz, \( f_2 = 19 \) Hz, and \( f_3 = 35 \) Hz.

In the qualification test, the input acceleration amplitude was selected at 0.75g and the input frequencies coincided with the first three natural frequencies of the system. Therefore,

\[ \ddot{\lambda} = \lambda_2 = \lambda_3 = 0.75g. \]

In each of the sinusoidal tests, the maximum absolute accelerations near the range switch location were also recorded. With these recorded values, the relative accelerations at the range switch location are found to be:

\[ |\tilde{\dot{X}}|_{3,1} = 5.16g \quad \text{for} \quad f_1 = 13 \text{ Hz}, \]

\[ |\tilde{\dot{X}}|_{3,2} = 4.18g \quad \text{for} \quad f_2 = 19 \text{ Hz}, \text{ and} \]

\[ |\tilde{\dot{X}}|_{3,3} = 3.24g \quad \text{for} \quad f_3 = 35 \text{ Hz}. \]
Since only maximum accelerations were recorded, Method 2 is used. The complex frequency acceleration responses are calculated by eqs. (3) and (5). With these values, eq. (13) is found to have the following form:

\[
\begin{align*}
    a_1^2 + b_1^2 &= (5.16)^2 \\
    a_2^2 + b_2^2 &= (4.20)^2 \\
    a_3^2 + b_3^2 &= (3.24)^2.
\end{align*}
\]

Eq. (10) becomes

\[
\begin{align*}
    a_1 &= Q_{31}(0) + Q_{32}(-0.68) + Q_{33}(-0.12) \\
    a_2 &= Q_{31}(1.45) + Q_{32}(0) + Q_{33}(-0.32) \\
    a_3 &= Q_{31}(0.87) + Q_{32}(1.07) + Q_{33}(0)
\end{align*}
\]

Eq. (11) becomes

\[
\begin{align*}
    b_1 &= Q_{31}(-6.25) + Q_{32}(-0.10) + Q_{33}(-0.01) \\
    b_2 &= Q_{31}(-0.22) + Q_{32}(-6.25) + Q_{33}(-0.03) \\
    b_3 &= Q_{31}(-0.05) + Q_{32}(-0.10) + Q_{33}(-6.25).
\end{align*}
\]

Eqs. (16), (17) and (18) consist of 9 equations with 9 unknowns. They are simultaneous equations in quadratic form and can be solved with iterative method [6]. The modal constants are found to be:

\[Q_{31} = 0.81, \quad Q_{32} = 0.63 \quad \text{and} \quad Q_{33} = 0.45.\]

The spectral accelerations for design purpose can be obtained from Figure 4. These accelerations are found to be \(a_1 = 3.0\text{g}, \quad a_2 = 1.25\text{g}, \quad \text{and} \quad a_3 = 0.75\text{g}.\)

Substituting these values into eq. (14), we obtain the design acceleration

\[
(\text{Acc})_3, \text{ design} = \left[ (0.81)^2 (3)^2 + (0.63)^2 (1.25)^2 + (0.45)^2 \right]^{1/2} (0.75)^2
\]

\[= 2.58\text{g}\]

Comparing the result given by eq. (19) to the recorded acceleration value discussed earlier, it is concluded that the single frequency testing is adequate.
5. Evaluation Procedures on the Adequacy of a Single Axis Test to Multi-Axes Requirements

For the purpose of evaluating the adequacy of a single axis test to a multi-axes design requirement, an empirical formula is proposed and has been found conservative. This formula is based on a comparison of the spectrum values from the test response spectrum (TRS) and the required response spectrum (RRS). It states that a single axis test is adequate provided the following inequality is satisfied:

\[
(TRS_A^2 + TRS_B^2 + \ldots)^{1/2} \geq 1.40 (RRS_A^2 + RRS_B^2 + \ldots)^{1/2} \tag{20}
\]

where the factor of 1.4 accounts for cross coupling and the factor of 1.5 for combined multi-mode effect. TRS_A, TRS_B, \ldots are the response spectrum values from the TRS while RRS_A, RRS_B, \ldots are from the RRS.

The conservatism of eq. (20) can be demonstrated by the same example given in Figure 5. In this case, the right-hand side of eq. (20) is found to be 10.83g while the left-hand side is 7g. Hence the single axis single frequency test is deemed adequate.

Conclusions

Evaluation procedures on the adequacy of a single frequency test to meet multi-frequency response spectrum requirements is provided. This procedure is obtained based on the equivalent response at critical locations in a dynamic system when it is subjected to single frequency or multi-frequency response spectrum input. The response of comparison can be either maximum acceleration or maximum stress. This procedure is equally applicable to a dynamic system with or without closely spaced modes.

An empirical procedure is also provided to evaluate the adequacy of a single axis test. Combining the empirical procedure with the procedure for single axis test, it is possible to justify the seismic adequacy without re-test the equipment which was test qualified to single frequency input along a single axis.

References

FIGURE 1  COMPARISON OF REQUIRED RESPONSE SPECTRUM AND
TEST RESPONSE SPECTRUM

FIGURE 2  MULTI-DEGREE-OF-FREEDOM
SYSTEM SUBJECTED TO
SINUSOIDAL INPUT

\[ \ddot{u}_g = \ddot{u} \sin \omega t \quad \text{or} \quad \ddot{u} \ e^{i \omega t} \]

FIGURE 3  DYNAMIC MODEL FOR
THE REACTOR CONTROL
BENCHBOARD

FIGURE 4  DESIGN SPECTRUM FOR
THE REACTOR BENCHBOARD