

Quadratic Eigenvalue Solver for Modal Response Analysis of Non-Proportionally Damped Elastic Structures

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Summary

The modal superposition method is widely used for the analysis of structural response. The standard method uses the undamped (i.e. linear) eigenvalue solution. The undamped natural modes are superposed to give the response to arbitrary excitation time histories or spectra. For small damping and for proportional (Rayleigh) damping this is a satisfactory approximation.

In the case of essentially non-proportional damping the damped (i.e. quadratic) eigenvalue solution leads to more accurate results. Moreover it gives the reliable possibility to optimize the effect of dashpots externally built into the structure to control its dynamical behavior. Based on the new approach of J-symmetry the damped modal analysis is presented. The quadratic eigenvalue solver is a powerful tool, which is in storage and computer time of the same order as classical undamped (linear) eigenvalue solvers.

1. Introduction

The structural response analysis of elastic structures under different dynamical excitations such as earthquake ground motion, aircraft impact, driving vibration etc., generally is performed by the modal superposition method. There is a nice feature of the modal superposition method that by knowing the natural frequencies and modes the structure becomes dynamically evident, independently of the type of excitation applied. This enables the engineer to design the structure in such a way that the natural frequencies of the structure lie outside the essential spectral content of the excitation.

The modal superposition method allows without great loss of accuracy to reduce the number of DOF (degrees of freedom), which is originally very high in the most practical cases, to a small number of essential natural modes. This is an obvious computational advantage.

Consequently, once the eigenvalue problem for a structural system is solved the response to any type of excitation can easily be found by little computational effort using always the same few natural modes.

Consider the governing equation of vibration

$$M\ddot{r} + C\dot{r} + Kr = p \quad (1)$$

where

r = time dependend vector of modal displacement

p = time dependend vector of outer excitation

M = structural mass matrix

C = structural viscous damping matrix

K = structural stiffness matrix

n = number of DOF

At the present state of the art the modal superposition method involves the solution of the undamped eigenvalue problem

$$(\omega^2 M + K) x = 0 \quad (2)$$

where ω denotes the unknown circular frequency and x the corresponding natural mode.

Denoting $\lambda = \omega^2$ the eq. (2) presents a "linear" eigenvalue problem for the eigenvalue λ . Let $\omega_1, \dots, \omega_n$ denote the circular eigenfrequencies which are solution of eq. (2) and let X be the matrix the columns of which are the eigenvectors x_1, \dots, x_n of eq. (2). Then the orthogonality conditions hold

$$X^T M X = I$$

$$X^T K X = \Omega^2$$
(3)

where Ω is the diagonal matrix with elements ω_i on diagonal, I is the unit matrix and X^T is the transpose of X . Using eq. (3) the coordinate transformation

$$r = Xy$$
(4)

results in the decoupling of the undamped (i.e. $C = 0$) system in eq. (1). The uncoupled system

$$\ddot{y} + \Omega^2 y = P$$
(5)

where $P = X^T p$, is easy to solve over the range of eigenfrequencies ω_i of interest.

Because the damping is always present in any form in the structure the solution of undamped eigenvalue problem can only approximate the exact solution of (1) with $C \neq 0$. The question arises for which type and amount of damping this approximation is good enough.

This is normally the case:

if the damping is small or

if the damping is nearly Rayleigh (proportional) i.e. it is of the form

$$C \approx \alpha M + \beta K$$
(6)

(Clough-Penzien /1/). In the first case the damping term $2 \xi_i \omega_i \dot{y}_i$ will be added to the decoupled equations in eq. (5), leading to

$$\ddot{y}_i + 2 \xi_i \omega_i \dot{y}_i + \omega_i^2 y_i = P_i$$
(7)

where ξ_i is composite modal damping based on a weighted average of strain energies in each material and for each mode.

In the second case there is a relation between α and β on the one side and ξ_i on the other, which is used for their determination, Clough-Penzien /1/.

In the design of nuclear power plants there are many cases in which the structure comprehends an essentially non-proportional damping. Such damping can either be inherit by the nature of the structural system (e.g. soil-structure interaction) or is given from outside by means of point-dashpots (examples are seismic isolation of the reactor building on helical springs and dashpots, or point-dashpots for design of pipes etc.).

The task of the design engineer in the first case is to verify if the peak values of dynamical stresses are less as permitted without changing the structure. In the second case there is the possibility to reduce the dynamic response. The effect of damping can be optimized by appropriate location of point dashpots and selection of their damping characteristics. For both tasks - verification and (or) optimization - it is necessary to solve the quadratic eigenvalue problem

$$(M\omega^2 + C\omega + K) x = 0 \quad (8)$$

subordinate to the eq. (1), where now C is neither "small" nor Rayleigh.

At the first glance it is theoretically possible to solve (8) by "direct" computation, which is exhausting already in the case with two or three DOF, Hurty Rubinstein /2/. It is easy to see that under assumptions on C made above the solution of the quadratic eigenvalue problem is essentially complex, i.e. the eigenvalues and modes are complex. (In overdamped modes the solution is "double" real, but these modes give no contribution to the steady state solution. However they may be relevant for local design of dashpots.)

The procedures available have all some of the disadvantages like the loss of symmetry or (and) bandwidth in structural matrices and generally they include the complex arithmetic. This causes an increase in the computer time or (and) computer storage, i.e. generally an increase of the computer costs. Some of the calculations become impossible at all.

This paper presents the new quadratic eigenvalue solver and its effect on structural response analysis for strongly non-proportional damping.

There are lots of papers considering the more realistic treatment of (non-proportional) damping in the computation of structural response. The papers Clough, Mojtahedi /3/, Duncan, Taylor /4/, Novak, El-Hifnawi /5/, Traill-Nash /6/, Tsai /7/ and Warburton /8/ represent a small extract.

2. "Block-Diagonalization" of Structural Matrices

To give rough illustration of the idea let us represent the conjugate complex pair $\alpha \pm i\beta$ as 2x2 matrix

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Note that this representation is a "good" one, i.e. the arithmetic of complex numbers corresponds to the arithmetic of 2x2 specified matrices. Due to this fact the following two equations of 1-dimensional vibration

$$\ddot{x} - 2\alpha \dot{x} + (\alpha^2 + \beta^2) x = 0$$

and

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

describe the same vibration state, i.e.

$$x(t) = u(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$

where $\lambda_{1,2} = \alpha \pm i\beta$ (suppose here for simplicity that the vibration is underdamped). Herefrom it follows that the real matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ governs

the vibration having complex pair $\alpha \pm i\beta$ as eigenfrequencies. Clearly the diagonal of this matrix is complex i.e.

$$\begin{pmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{pmatrix}$$

which we try to avoid. Therefore we reformulate the task and try to "block-diagonalize" the system, each (2x2)-block representing one natural mode.

To this aim let us modify the problem (1) by the use of substitution

$$\left. \begin{aligned} y &= L_2 r \\ L_2^{-T} L_1^T z &= y + L_2^{-T} C L_2^{-T} y \\ M &= L_2^T L_2 \\ K &= L_1^T L_1 \end{aligned} \right\} \quad \text{(Cholesky)} \quad (9)$$

The new system is now

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ L_1^{-T} p \end{pmatrix} \quad (10)$$

where

$$A = \begin{pmatrix} -L_2^{-T} C L_2^{-1} & L_2^{-T} L_1^T \\ -L_1 & L_2^{-1} & 0 \end{pmatrix} \quad (11)$$

referred to as a "big" matrix of the system (1).

It has the double size and the symmetry is lost. Nevertheless there is another type of symmetry of (11) called "J-symmetry" (Veselić /9/). The matrix A in (11) is of type

$$A = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix} \quad (12)$$

where A and D are symmetric. It has the property that

$$JAJ = A^T \quad (13)$$

with

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (14)$$

I being the unit matrix of order n.

There is an obvious storage advantage of J-symmetric matrices in comparison to general (2n) x (2n) matrices: it needs roughly $2n^2$ storage which is about half as for a non J-symmetric case.

In addition the use of the real arithmetic reduces the storage for another half.

Let us shortly review some of the properties of the J-symmetric matrices, cf. Mal'cev /10/. There is an analogue of an orthogonal matrix in terms of the J-orthogonal defined by the property

$$U^{-1} = J U^T J \quad (15)$$

J being the matrix in eq. (14). It is easy to verify that if U is J-orthogonal the similarity transformation

$$A \longrightarrow U^{-1} A U \quad (16)$$

preserves J-symmetry. This is an important fact for the diagonalization procedure.

The most important result is concerned with the decoupling of the system (10). A "brutal" diagonalization of the matrix A would cause introducing complex arithmetic. The corresponding complex natural modes would have no kind of orthogonality property which would itself destroy J-orthogonal structure of the iterates of A.

Therefore as we already mentioned it before we are going to "block-diagonalize" the big matrix A i.e. to transform it to the form

$$T^{-1} A T = \begin{pmatrix} \alpha & \beta \\ -\beta & \delta \end{pmatrix} \quad (17)$$

in which α , β and δ are real diagonal matrices, for instance

$$\alpha = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} \quad (18)$$

The transforming matrix T appears to be J-orthogonal, cf. eq. (15).

Each of the small 2x2 blocks in (17)

$$\begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \delta_k \end{pmatrix} \quad (19)$$

corresponds to the one natural mode.

The corresponding modal parameters are:

- eigenvalues

$$\lambda_{1,2}^{(k)} = \frac{\alpha_k + \delta_k}{2} \pm i \sqrt{\beta_k^2 - \left(\frac{\alpha_k - \delta_k}{2}\right)^2} \quad (20)$$

- eigenfrequencies (Hz)

$$f_k = \frac{1}{2\pi} \sqrt{\beta_k^2 - \left(\frac{\alpha_k - \delta_k}{2}\right)^2} \quad (21)$$

- modal damping (ratio)

$$D_k = \frac{\text{abs}(\alpha_k + \delta_k)}{2\sqrt{\alpha_k \delta_k + \beta_k^2}} \quad (22)$$

where $k = 1, \dots, n$. Let us now discuss some consequences from the formula (20), (21), (22).

In the case of an underdamped system the eigenvalues have a form of essentially complex conjugate pair as seen from (20). This changes to real pair of different numbers if the system is overdamped and to double real number if the system vibrates at the critical damping.

In the last case the big matrix A can not be diagonalized at all, even not in complex arithmetic, in contrast to the "block-diagonalization". This is another advantage of the "block-diagonalization".

It is worthwhile to remark that the modal parameters in (20) - (22) contain all the information of the mechanical system such as for example non-properly supported system (singular stiffness), non-dissipative energy in the system etc. Some of them appear at the first glance in (20) - (22) as a spitefull singularity but in fact describe an essential system property.

3. Damped Modal Analysis

Without going into technical details let us now assume the big matrix A to be block-diagonalized by the J-orthogonal T (cf. (17)) and that all modal parameters are known (cf. (20) - (22)).

Transforming the coordinates in (10) by

$$\begin{pmatrix} y \\ z \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix} \quad (23)$$

where u and v are time dependent u-vectors the equations (10) change to the block-decoupled ones

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g \\ h \end{pmatrix} \quad (24)$$

where

$$\begin{pmatrix} g \\ h \end{pmatrix} = T^{-1} \begin{pmatrix} 0 \\ L_1 \\ -T \\ p \end{pmatrix} \quad (25)$$

For each k there is an explicite solution of (24)

$$\begin{pmatrix} u_k(t) \\ v_k(t) \end{pmatrix} = \exp(A_k t) \left[\begin{pmatrix} c_{1k} \\ c_{2k} \end{pmatrix} + \int_0^t \exp(-A_k \tau) \begin{pmatrix} g_k(\tau) \\ h_k(\tau) \end{pmatrix} d\tau \right] \quad (26)$$

where

$$A_k = \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \delta_k \end{pmatrix}$$

and c_{1k} , c_{2k} are constants depending on initial conditions. It is easy to construct the explicite formula for $\exp(A_k t)$, which then inherits all the characteristics coming from the overdamped, underdamped or critically damped natural mode.

Knowing vectors u and v (cf. (26)) the final result for structural response (cf. (1)) is given by the expression

$$r(t) = L_2^{-1} \sum_{k=1}^s (u_k(t) \cdot t_k + v_k(t) \cdot t_{k+n}) \quad (27)$$

where L_2 is given by (9) and t_i denotes the first n components of the i -th column in T . The summation bound s takes care of the fact that normally only a few frequencies (smallest s frequencies) are essential for the dynamical response, $s \leq n$. The rest of them can then be neglected for calculating $r(t)$ by (27), cf. also Trail-Nash /6/.

4. Conclusion

There are lots of details we didn't consider here. Some of them are very similar to the corresponding classical case, as for example the reduction for zero masses, while the others like orthogonality check are specific for the "block-diagonalization". We also didn't discuss the technical details of the quadratic eigenvalue solver. It is not only capable to treat the damping exactly (zero, small and Rayleigh damping are included) but it is also less sensible for multiple or nearly multiple eigenvalues as far as the big matrix doesn't include any Jordan box. But this case has practically no relevance for structural dynamics.

The important fact is that the new quadratic solver is in the same order of effort as the classical QR or Lanczos procedures with respect to storage memory and time consumed. It can therefore be seen as a possible branch-point in the classical approach to the natural mode analysis.

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