A Simple Method of Combining Modal Responses

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A new method of combining modal responses in the response spectrum method of analysis is presented in Ref. 1, which is designated here as the Gupta's method. It is shown that the response spectrum method in conjunction with the Gupta's method of modal combination gives results which are very close to those obtained from the time-history analysis. Further, Gupta's method gives results which are more accurate than other methods of modal combination discussed here viz, SRSS, Kennedy's and Hadjian's.

In the Gupta's method response in any mode is split into two parts, the damped periodic part and the rigid part, which are mutually uncorrelated. This is done using a rigid response factor. A simple expression for the rigid response factor is presented. The key frequencies in the expression can be evaluated from the motion response spectrum.

1. Introduction

A new method of combining modal responses in the response spectrum method of analysis is presented in Ref. 1. It is well known that all the modal responses beyond a certain rigid frequency (33 Hz is the USNRC Regulatory Guide 1.60 [2] design spectra) are perfectly correlated. Therefore, those responses should be combined by algebraic sum. Responses in other modes with smaller frequencies, which are also sufficiently separated can be assumed to be mutually uncorrelated, and also uncorrelated with the rigid response beyond the rigid frequency. These responses can be combined by the SRSS (square root of the sum of the squares) rule. For the modes with the intermediate frequencies (say from 2 to 20 Hz in case of the USNRC spectra) the correlation between the modal response and the rigid response varies from zero to one. This correlation can be called the rigid response factor (designated by α) and is the basis of the method in Ref. 1. This method is referred to here as the Gupta's method.

To account for the fact that the modal responses in the modes having frequencies beyond 33 Hz are perfectly correlated Kennedy [3] has suggested a simple rule. In this rule all the modal responses in the modes having frequencies up to 33 Hz are combined by the SRSS method (if the frequencies are sufficiently apart) all the modal responses in the modes having frequencies beyond 33 Hz are summed algebraically. We will call this method as the Kennedy's method.

A some what elaborate method has been suggested by Hadjian to account for the questions similar to those raised above. He suggests making use of the concept of relative acceleration response spectrum. Details of the method are reported in Ref. 4, and is referred to here as the Hadjian's method.

A summary of the Gupta's method is given in the following section. A simple expression for the rigid response factor is derived in the next section. Four methods of modal combination, viz, SRSS, Kennedy's, Hadjian's and Gupta's methods are compared. It is shown that the Gupta's method gives results which are closest to the time-history results.

2. Summary of the Gupta's Method

In the Gupta's method [1], it is hypothesized that a response R_i in any mode i consists of two parts which are mutually uncorrelated, viz, the damped periodic response, R_i^p, and the rigid response, R_i^r.

$$R_i^2 = (R_i^p)^2 + (R_i^r)^2 \qquad (1)$$

Further, the rigid response part, is perfectly correlated with the high frequency (rigid) response, and that the damped periodic part is uncorrelated with the latter. Correlation factor between the response R_i and the rigid response is designated as the rigid response factor, α_i. Therefore,

$$R_i^r = \alpha_i R_i \qquad (2)$$

Equations (1) and (2) give

$$R_i^p = \sqrt{1-\alpha_i^2}\, R_i \qquad (3)$$

The rigid response factor α_i varies between 0 and 1 as a function of modal frequency f_i between certain frequencies f^1 and f^2. For frequencies $f_i \leq f^1$, α_i=0, $R_i^p = R_i$, $R_i^r = 0$; and for frequencies $f_i \geq f^2$, $\alpha = 1.0$, $R_i^p = 0$, $R_i^r = R_i$. More on the rigid response factor is discussed in the next section.

The damped periodic part from the various modes is combined by the conventional methods, SRSS or the double sum, depending on whether the frequencies are close or not.

$$(R^p)^2 = \sum_i (R_i^p)^2 + 2 \sum_i \sum_{j>i} \varepsilon_{ij} R_i^p R_j^p \tag{4}$$

The term ε_{ij} accounts for correlation (coupling) between the modes i and j depending on how close the frequencies are. An expression for ε_{ij} was given by Rosenblueth and Elorduy [5] which has been modified in Ref. 1 and is given below

$$\varepsilon_{ij} = \left\{ 1 + \left[\frac{f_j - f_i}{\zeta(f_j + f_i) + C_{ij}}\right]^2 \right\}^{-1} \tag{5}$$

in which

$$C_{ij} = (1-3\zeta)(.036- |f_j^2 - f_i^2|) \geq 0 \tag{6}$$

where f_i, f_j = frequencies for modes i and j, respectively, Hz.

ζ = critical damping ratio

In practical application ε_{ij} may be taken as zero, when f_i and f_j are not close and ε_{ij} is small, say less than 0.2. When ε_{ij} is small or zero for all the pairs of frequencies, Equation (4) degenerates into the standard SRSS equation.

The rigid modal responses are combined algebraically

$$R^r = \sum_i R_i^r \tag{7}$$

The total response is given by

$$R^2 = (R^p)^2 + (R^r)^2 \tag{8}$$

In the range of the Zero Period Acceleration a further simplification in the procedure is possible and is given in the Appendix.

3. Rigid Response Coefficient

By definition, the rigid response coefficient for mode i, α_i is equal to the correlation between the modal response at frequency f_i and the rigid response at frequency $f = \infty$. Theoretically, the rigid response is equal to scaled acceleration motion history. Using the data generated in Ref. 1, a complex equation for α_i was suggested. Such an expression is not universally applicable to other earthquake ground motions and to instructure motions. Given the motion time-history, the variation of α with frequency can be evaluated from the first principles. But then the method could not be applied to those cases where the motion time-history is not known, a serious draw-back in the use of the procedure in conjunction with response spectrum method.

It is observed that the Gupta's method, Eqs. (1 to 8), is not very sensitive to the minor errors in the values of α_i. Therefore, a reasonable estimation of α_i's would be adequate. In Figure 1, the curved line shows the variation of numerically calculated α with frequency f on a semi-log plot for San Fernando earthquake (Holywood Storage, 1971, EW) at 5% damping. The straight line represents a simple idealization. Between frequencies f^1 and f^2, the idealized line has the following equation

$$\alpha = \frac{\log f/f^1}{\log f^2/f^1} , \quad 0 < \alpha < 1 \tag{9}$$

For $f \leq f^1$, $\alpha=0$, and $f \geq f^2$, $\alpha = 1$.
This representation, although approximate, has the advantage that it only depends on two frequencies, which hopefully, can be determined from the corresponding response spectrum, without taking resort to the time history.

It was found that the frequency f^1 is related to the maximum spectral acceleration, S_{amax} and the maximum spectral velocity, S_{vmax} by the following equation

$$f^1 = \frac{S_{amax}}{2\pi \, S_{vmax}} , \text{ Hz} \tag{10}$$

If the input motion had one predominant frequency, S_{amax} and S_{vmax} will both occur at that frequency. The frequency f^1 will then be equal to the predominant frequency. In a typical motion history, which consists of several frequency motions, f^1 would represent a composite of those frequencies.

At the frequency f^2, the response history is almost perfectly correlated with the input history. Conceptually, it should occur at a certain frequency interval beyond the last significant frequency or beyond the frequency at the last significant peak in the response spectrum, designated here as the FLSP. A good correlation was obtained for the following equation.

$$f^2 = \text{FLSP} + 8 \text{ Hz} \tag{11}$$

The FLSP can be estimated from the response spectrum as follows: Draw a tangent on the right side of the response spectrum touching first a major peak (left) and then a minor peak (right) before finally intersecting the curve in the vicinity of zero period acceleration, as shown in Fig. 2. The frequency at the minor peak is the FLSP.

The above algorithm for α was verified using 14 motion histories, including 3 instructure motion histories in Ref. 6. It was found that the algorithm works equally well for the ground motion histories and the instructure motion histories. Further, Eqs. (11) and (12) were found to be applicable to damping ratios between 1 and 7%.

4. Comparison of the Time History and the Response Spectrum Results

One objective of the present study is to investigate the validity of the Gupta's method of modal combination in the response spectrum method of analysis. Other methods have been considered to establish their relative validity. Those are the SRSS method, the Kennedy's method and the Hadjian's method. The basis of validation is the closeness of the response spectrum results obtained by using one of the contending modal combination methods with the corresponding absolute maximum values obtained from the time history analysis, treating the latter as the standard.

Five 3-degrees-of-freedom buildings were analyzed. The five buildings are identical, except that their stiffnesses are proportionately increased, so that their fundamental frequencies vary from 2 to 64 Hz. The lumped mass model and the three mode shapes, which are same for the five buildings, along with the modal frequencies, three per building, are shown in Figure 3. The buildings are numbered 1 to 5 for this paper. These are identical to those analyzed by Hadjian in Ref. 4, in which they were numbered 6 to 10.

Each of the five buildings was subjected to six motion histories, three of those earthquake ground motions and the other three instructure motions. The three earthquake motions are San Fernando (Holywood Storage, 1971, EW), Taft (1952, S69E) and Olympia (1949, S86W). The three instructure motion histories were provided by a committee [7], and their source is unknown.

Five buildings each subjected to six motion histories resulted in a total of 30 cases. For each case, the following quantities were compared: displacements, inertia forces, shear forces and overturning moments at three story levels. Displacements from the time history analysis were almost identical with the displacement from the response spectrum analysis, irrespective of the combination technique. This was so because in all cases displacements from the first mode dominated the response. Therefore, displacements were eliminated from the comparison. Thus, for a total of 30 cases 9 quantities each were compared, a total of 270 quantities.

In the response spectrum analysis, a special consideration was made which is explained here. Any ordinate in a response spectrum represents the absolute maximum value of say, acceleration at a particular frequency. The assumptions of stationarity and ergodicity, used in the statistical modal combination, imply that theoretically the maximum positive and the maximum negative values obtained from the time history are the same. In reality, however, they are not. Absolute value of the higher of the two maxima - the positive and the negative, goes into the response spectrum. Whereas, for design purposes, using one response spectrum and assuming that the maximum positive and negative values are same, is quite proper, in making a comparison between the time history and response spectrum method it could cause a problem.

We introduce here the concept of positive and negative response spectra for the real life earthquake motions. The positive response spectrum ordinates are simply the maximum positive values, and the negative response spectrum ordinates represent the maximum negative values. Consider two modes at frequencies f_1, f_2 respectively, the maximum values of a response in those modes R_1^+ and R_2^+ from the positive response spectrum, and R_1^- and R_2^- from the negative response spectrum. The information about correlation between the two modes applies to R_1^+ and R_2^+ or to R_1^- and R_2^-, but not to R_1^+ and R_2^- nor to R_1^- and R_2^+. Therefore, the combined response should be based on one of the first two pairs. The absolute maximum time history response should be compared to the higher of the absolute values of the two combinations. In our calculations, this procedure was used. Note, when applying the method to a given design response spectrum, one can assume that the positive and negative spectra are identical to the design response spectrum.

The error in the response spectrum method is defined as follows

$$\text{Error (percent)} = \left(\frac{\text{Response spectrum value}}{\text{Time-history value}} - 1 \right) \times 100 \qquad (12)$$

A detailed comparison between the results from various methods is given in Ref. 6. A statistical summary of the errors in various methods of modal combination is given in Table 1. The error values were calculated for all the response values of interest, as explained above, for each of the five structures subjected to each of the motion histories. The mean error in all the methods is relatively insignificant. The most important error parameter for consideration is the standard deviation. Relative to the value for Gupta's method the standard deviation for Hadjian's method is 2.4, that for Kennedy's method is 2.9 and for the SRSS method is 4.6. Thus, the four methods are listed in the order of increasing accuracy in Table 1, viz, SRSS, Kennedy, Hadjian and Gupta. The Gupta's method gives response spectrum results which are closest to the time-history results, and are well within the acceptable range of accuracy.

TABLE 1 STATISTICAL SUMMARY OF PERCENT ERRORS

Method	SRSS	Kennedy	Hadjian*	Gupta
Maximum Absolute Error	41	37	38	21
Mean Error	-1.3	-1.5	4.1	-.8
Standard Deviation	16.9	10.7	8.8	3.7
Relative Standard Deviation	4.6	2.9	2.4	1.0

* These results are based on a table distributed by Hadjian at a committee meeting in Las Vegas in April 1982. Hadjian used a critical damping ratio of 5% for ground motions, and 2% for instructure motions. All our values (SRSS, Kennedy, Gupta) are based on 5% damping.

5. Conclusions

1. Gupta's method for combining modes in response spectrum method of analysis give results which are very close to the results obtained from the time-history method.
2. Other methods of modal combination can be ranked behind the Gupta's method in the following order: Hadjian's, Kennedy's and the SRSS. Relative to the Gupta's method, the standard deviation in the error due to these methods is as follows: 2.4, 2.9 and 4.6, respectively.
3. Gupta's method depends upon the rigid response coefficient, α, which can be estimated using a simple equation in terms of two frequencies f^1 and f^2. These frequencies can be evaluated from the response spectrum. The proposed method was tested against 11 earthquake ground motion histories and 3 instructure motion histories. The agreement between the proposed simple equation for α and the numerically calculated value is good.

References

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[2] United States Nuclear Regulatory Commission, "Design Response Spectra for Seismic Design of Nuclear Power Plants", Regulatory Guide 1.60, Revision 1, December 1973.

[3] KENNEDY, R. P., "Recommendations for Changes and Additions to Standard Review Plans and Regulatory Guides Dealing with Seismic Design Requirements for Structures, Report prepared for Lawrence Livermore Lab., Published in Nureg/CR-1161, June 1979. (Also personal communications with Kennedy).

[4] HADJIAN, A. H., "Seismic Response of Structures by the Response Spectrum Method," Nuclear Engineering and Design, Volume 66, No. 2, August, 1981, pp. (179-201).

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[6] GUPTA, A. K., CHEN, D. C., "Combination of Modal Responses - A Follow Up," Civil Engineering Research Report, North Carolina State University, November, 1982.

[7] ASCE Committee on Seismic Analysis Standard for Safety Class Nuclear Structures," Notes of the meeting on December 10 and 11, 1981, held in Atlanta, Georgia."

Appendix: An Alternate Method for Modes With Zero Period Response Acceleration

Beyond a certain frequency the spectral acceleration becomes approximately constant and is equal to what is called the zero period acceleration, ZPA. The particular frequency is called the rigid frequency, f^r. There are many structural problems in which significant response exists in the modes having frequencies greater than f^r. As will be shown here, it is unnecessary and uneconomical to evaluate responses in those modes separately. Further, there is always a danger that not all the modes have been included which have significant contribution to the total response.

An alternative is presented here. The method presented here is not new, but we don't know who introduced it first. The method is exact within the assumption of the response spectrum method.

Calculate mode shapes and frequencies for only those modes which have frequencies less than f^r; say there are 'm' such modes. The residual inertia force for the modes with frequencies > f^r is given by

$$\{F_o\} = [M]\ \{U_b - \sum_{i=1}^{m} \gamma_i\ \phi_i\}\ ZPA$$

where [M] = mass matrix; $\{U_b\}$ = displacement vector of the structure when a unit displacement is applied at the support in the direction of earthquake, in many cases $\{U_b\}$ becomes a unit vector {I}; $\{\phi_i\}$ = normalized modal displacement vector for mode 'i', $\{\phi_i\}^T\ [M]\ \{\phi_i\} = 1$; γ_i = participation factor for mode 'i'. Response to the residual inertia force can be calculated statically (since there is no dynamic amplification beyond f^r)

$$[K]\ \{U_o\} + \{F_o\}$$

Any response R_o can be calculated from $\{U_o\}$. Equation (8) of the text for R^r is modified to include R_o, $R^r = R_o + \Sigma\ R^r{}_i$. All the other steps remain the same as before.

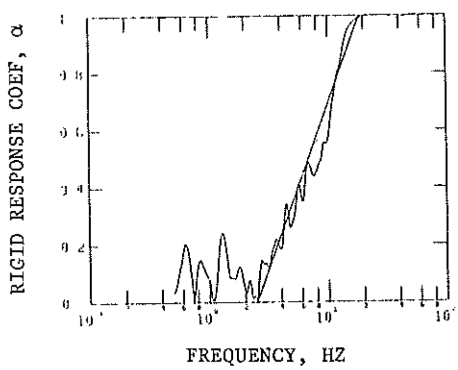


Figure 1 - Variation of Rigid Response Coefficient α with Frequency, San Fernando Earthquake (Hollywood Storage, 1971, EW), 5% Damping.

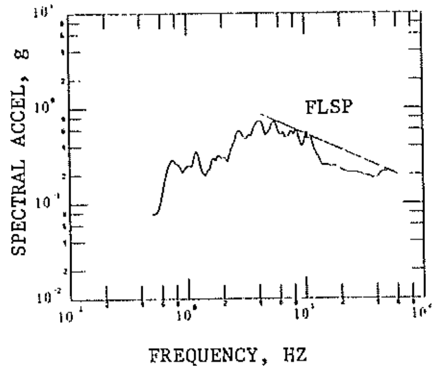
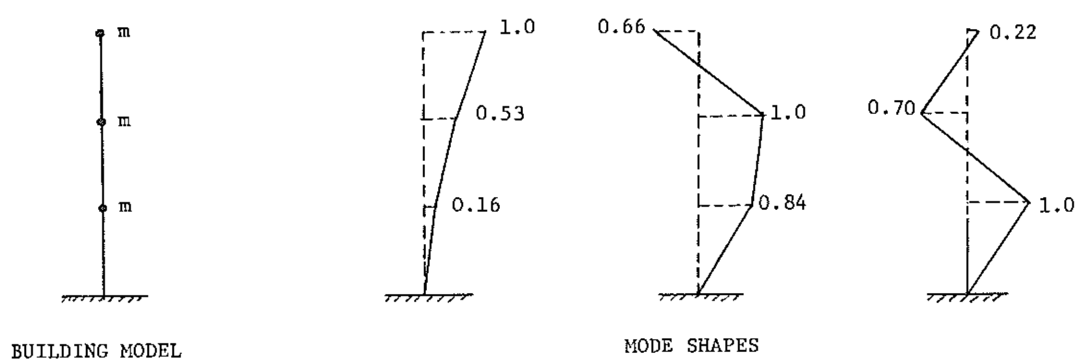


Figure 2 - Response Spectrum for San Fernando Earthquake (Hollywood Storage, 1971, EW), 5% Damping.



Building Frequencies, Hz

Modes	Freg. Ratio	Building 1	Building 2	Building 3	Building 4	Building 5
1	1.00	2.00	4.00	8.00	16.00	64.00
2	6.55	13.10	26.19	52.38	104.77	419.07
3	17.59	35.19	70.37	140.65	281.49	1125.95

Figure 3 - Summary of Dynamic Characteristics of the Example Problem (Based on Hadjian[4])