Transient Analysis of Two-Dimensional Structures Using a Coupled Thermoelastic Finite Element Procedure

W.-H. Chen, K. Ting
Department of Power Mechanical Engineering, National Tsing Hua University, 855 Kuang Fu Road, Hsinchu, Taiwan 300, China

Summary

The transient response of realistic structures subjected to thermal shocks is becoming increasingly important in the design of nuclear reactor structural components, such as fuel-cladding, pressure vessel and piping. In these problems, as is known, both the thermoelastic and inertia effects need be taken into account. Due to the complexity of the thermo-mechanical interactions, as is seen in literature, the problem is usually solved based on the uncoupled thermoelastic theory. Although several finite element procedures have been devoted to one-dimensional coupled thermoelastic analysis, the work which is done on two-dimensional analysis is very limited. The aim of this work is thus to develop an efficient and accurate finite element procedure to deal with two-dimensional coupled thermoelastic problems subjected to thermal shocks. The achievement of this work becomes of practical interest in assessing the performance of the nuclear reactor components.

In the finite element formulation developed herein, the entropy displacement introduced by Biot is employed and a relevant coupled thermoelastic dynamic variational principle is established. Based on this principle, the mass, thermal damping and stiffness materices are derived respectively. A eight-node, coupled-thermoelastic isoparametric element is specially designed. In this element, each node contains four degrees of freedoms, two of them are mechanical displacements and others are entropy displacements. Therefore, the mechanical and entropy displacements can be solved directly. The thermal stress can then be computed by Duhamel-Neumann constitutive equations and the temperature distribution is inferred from the mechanical and entropy displacements. To solve the resulting equations of motion, Newmark's directed intergration scheme is taken in this work.

To demonstrate the accuracy and validity of the solution technique developed, two examples are first analyzed. Good correlations between the computed results with the referenced solutions available in literatures can be noted. To evaluate the influence of the coupled thermoelastic and inertia effects respectively, the analysis includes: (1) dynamic uncoupled thermoelastic approach, (2) dynamic coupled thermoelastic approach, (3) quasi-static uncoupled thermoelastic approach. and (4) quasi-static coupled thermoelastic approach. Finally, the transient thermoelastic behaviors of a finite plate with a circular hole subjected to a thermal shock are investigated in detail. The variation of thermoelastic stress concentration factors around the circular hole for each case is also drawn respectively.

1. Introduction

The true transient response of reactor components subjected to thermal shocks is governed by the thermo-mechanical interactions and inertia effects. However, due to the complexity of such effects, only little attention has been paid to study those problems. Thus, an efficient and accurate finite element procedure which can be used to solve realistic thermoelastic problems subjected to thermal shocks is established and presented here. To study the role of the coupled and inertia effects on the computation of the temperature as well as stress distributions in the structures, several illustrative examples are solved.

In the design of actual machine and engineering structural members, the evaluation of thermoelastic stress concentration factors is essential and seldom found in the literature. To study these factors, the example of a square plate with a central circular hole subjected to thermal shocks is also devised and investigated in this work.

2. Finite Element Formulation for Dynamic, Coupled Thermoelastic Problems

Consider a homogeneous, isotropic body which undergoes a small elastic deformation due to the action of applied loading and temperature change. The temperature change is assumed to be small compared with the reference temperature. T_0 and does not lead to any appreciable change in the elastic and thermal properties. That is the thermoelastic material properties are treated as temperature-independent. To properly model the transient thermo-mechanical coupled effects of relevant thermoelastic materials, the concept of entropy displacements introduced by Biot [1] is incoporated into the Hamilton's principle [2-3] adoped here. Consequently, the modified Hamilton's principle governing the dynamic equilibrium of the structure between instants t^N and t^{N+1} can be viewed as [4-6]:

$$\delta\pi = \delta\int_{t^N}^{t^{N+1}} [A(\epsilon_{ij},\theta) + D(H_i) - K(U_i) - W]dt = 0 \qquad (1)$$

where $A(\epsilon_{ij},\theta)$ is the Biot's thermoelastic potential, $D(H_i)$ is the dissipation energy function, $K(U_i)$ is the kinematic energy and W is the external work done by body forces, surface tractions and temperature changes. ϵ_{ij} denote the strain tensor and U_i are the components of mechanical displacements. θ is the temperature difference between the instantaneous absolute temperature T and the reference temperature T_0, i.e. $\theta = T-T_0$; H_i represent the entropy displacements which are defined by Biot [1] as:

$$\dot{H}_i = h_i$$

where h_i are the components of heat flux. The relationship between strains ϵ_{ij} and displacements U_i under the assumption of small deformation theory is

$$\epsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \qquad (2)$$

Hence, the thermalstresses σ_{ij} can be computed by the Duhamel-Neumann constitutive relations:

$$\sigma_{ij} = E_{ijkl}\,\epsilon_{kl} - \beta_{ij}\theta \qquad (3)$$

where E_{ijkl} is the elastic modulus tensor and β_{ij} are the thermal stress constants which are assumed to be independent of time. For the isotropic material

$$\beta_{ij} = \beta\delta_{ij}$$

Here, δ_{ij} is the Kronecker delta and β is defined as:

$$\beta = (3\lambda + 2\mu)\alpha$$

where λ and μ are Lamé constants and α represent the linear thermal expansion coefficient.

The Biot's thermoelastic potential $A(\varepsilon_{ij}, \theta)$ and the dissipation energy function $D(H_i)$ in eq.(1) can be expressed as the followings [4-6]:

$$A(\varepsilon_{ij}, \theta) = \int_V \frac{1}{2} (E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{\rho C_v}{T_0} \theta^2) \, dV \qquad (4)$$

and

$$D(H_i) = \frac{d}{dt} \int_V \frac{1}{2} \frac{\eta_{ij}}{T_0} H_i H_j dV \qquad (5)$$

where V represents the volume of the body considered, ρ is the density, C_v is the specific heat per unit mass and η_{ij} is the thermal resistivity matrix. ρ and C_v are independents of time. For isotropic material, η_{ij} can be expresses as follows:

$$\eta_{ij} = \eta\delta_{ij}$$

η is the thermal resistivity constant which is the inverse of the coefficient of thermal conductivity. The coupling relation between the deformation and temperature change is linked by the divergence of the entropy displacements, i.e.:

$$H_{i,i} = -\rho C_v \theta - T_0 \beta_{ij} \varepsilon_{ij}$$

Again, for the isotropic case, the above eq. can be written as :

$$H_{i,i} = -\rho C_v \theta - T_0 \beta U_{i,i} \qquad (6)$$

Substituting eqs. (4) and (5) into eq. (1), analogous to the derivation of the variational principle for isothermal elastodynamic problem [2,3] , the relevant functional governs the present dynamic, coupled thermoelastic problems between isntants t^N and t^{N+1} can be written as (in finite element version):

$$\pi (U_i, H_i) = \int_{t^N}^{t^{N+1}} \sum_m \{ \int_{V_m} [\frac{1}{2}(E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{\rho C_v}{T_0} \theta^2) + \frac{\eta_{ij}}{T_0} \dot{H}_i H_j - \frac{1}{2} \rho \dot{U}_i \dot{U}_i - \bar{F}_i U_i] \, dV - \int_{S\sigma_m} \bar{T}_i U_i dS - \int_{S\theta_m} \frac{\bar{\theta}}{T_0} n_i H_i dS \} \, dt \qquad (7)$$

In the above, V_m is the volume of the mth element (M=1, 2,...),

$S\sigma_m$ and $S\theta_m$ represent the portions of the boundary surface of the mth element where prescribed tractions $\bar{T}_i$ and temperature $\bar{\theta}$ are applied respectively,

$\dot{U}_i$ is the velocity within V_m,

$\bar{F}_i$ are the body forces in V_m,

n_i are the direction cosines of the outward unit normal on $S\theta_m$.

Now, making the finite element approximations in matrix form, the mechanical element interior displacements U_i and entropy element interior displacement H_i can be assumed as:

$$\{U\} = [N] \{q_M\} \quad \text{in } V_m \qquad (8)$$

and

$$\{H\} = [N] \{q_E\} \quad \text{in } V_m \qquad (9)$$

respectively. [N] is the matrix of shape functions [7], $\{q_M\}$ is a column vector of mechanical nodal displacements and $\{q_E\}$ is a column vector of entropy nodal displacements. Thus,

$$\{\dot{U}\} = [N] \{\dot{q}_E\} \quad \text{in } V_m \qquad (10)$$

and

$$\{\dot{H}\} = [N]\{\dot{q}_E\} \quad \text{in } V_m \tag{11}$$

The corresponding matrix form of eqs. (2) and (6) can thus be expressed in an alternative form:

$$\{\varepsilon\} = [B_M]\{q_M\} \tag{12}$$

and

$$\theta = -\frac{\beta T_0}{\rho C_v}[B_T]\{q_M\} - \frac{1}{\rho C_v}[B_E]\{q_E\} \tag{13}$$

For isotropic case, by substituting eqs.(8)-(13) into eq.(7) and expressing $\{q_M\}$ and $\{q_E\}$ in terms of generalized global nodal vectors $\{q_M^*\}$ and $\{q_E^*\}$, the functional π of eq.(7). can be written as:

$$\pi = \int_{t^N}^{t^{N+1}} \Big(-\frac{1}{2}\{\dot{q}_M^*\}^T[M]\{\dot{q}_M^*\} + \frac{1}{2}\{q_E^*\}^T[D]\{\dot{q}_E^*\}$$

$$+ \frac{1}{2}\{q_M^*\}^T[K_M]\{q_M^*\} + \frac{1}{2}\frac{\beta^2 T_0}{\rho C_v}\{q_M^*\}^T[K_E]\{q_M^*\}$$

$$+ \frac{1}{2}\frac{\beta T_0}{\rho C_v}\{q_E^*\}^T[K_E]\{q_M^*\} + \frac{1}{2}\frac{\beta T_0}{\rho C_v}\{q_M^*\}^T[K_E]\{q_E^*\}$$

$$+ \frac{1}{2}\frac{1}{\rho C_v T_0}\{q_E^*\}^T[K_E]\{q_E^*\} - \{q_M^*\}^T\{Q_M\} - \{q_E^*\}^T\{Q_E\} \Big)\, dt \tag{14}$$

where $\{\ \}^T$ is the transpose of $\{\}$,

$[M] = \sum_m \int_{V_m} \rho [N]^T [N]\, dV$ (mass matrix),

$[D] = \sum_m \int_{V_m} \eta [N]^T [N]\, dV$ (damping matrix),

$[K_M] = \sum_m \int_{V_m} [B_M]^T [E_M][B_M]\, dV$ (mechanical stiffness matrix),

$[K_E] = \sum_m \int_{V_m} [B_E]^T [B_E]\, dV$ (thermal stiffness matrix),

$\{Q_M\} = \sum_m \Big(\int_{V_m} [N]^T\{\bar{F}\}dV + \int_{S_{\sigma m}} [N]^T\{\bar{T}\}\, ds\Big)$,

$\{Q_E\} = \sum_m \int_{S_{\theta m}} [N]^T\{\bar{\theta}\}\{n\}\, ds$,

$\{n\}$ is the column vector of direction cosines and $[E_M]$ is the elastic modulus matrix which is assumed to be constant throughout the computation process.

The stationary condition of π in equation (14) with respect to $\{q_M^*\}$ and $\{q_E^*\}$ yield the following final simultaneous algebraic equations:

$$[M]\{\ddot{q}_M^*\} + \Big([K_M] + \frac{\beta^2 T_0}{\rho C_v}[K_E]\Big)\{q_M^*\} + \frac{\beta T_0}{\rho C_v}[K_E]\{q_E^*\} = \{Q_M\} \tag{15}$$

$$[D]\{\dot{q}_E^*\} + \frac{\beta T_0}{\rho C_v}[K_E]\{q_M^*\} + \frac{1}{\rho C_v T_0}[K_E]\{q_E^*\} = \{Q_E\} \tag{16}$$

Introducing the coupling parameter, γ, which is defined as [8]:

$$\gamma = \frac{\beta^2 T_0}{\rho C_v (\lambda + 2\mu)}$$

Alternatively, eqs.(15) and (16) can be combined as:

$$\begin{bmatrix} [M] & [D] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}_M^*\} \\ \{\ddot{q}_E^*\} \end{Bmatrix} + \begin{bmatrix} [0] & [0] \\ [0] & [D] \end{bmatrix} \begin{Bmatrix} \{\dot{q}_M^*\} \\ \{\dot{q}_E^*\} \end{Bmatrix}$$

$$+ \begin{bmatrix} [K_M] + \gamma(\lambda+2\mu)[K_E] & \frac{\gamma}{\beta}(\lambda+2\mu)[K_E] \\ \frac{\gamma}{\beta}(\lambda+2\mu)[K_E] & \frac{1}{\rho C_v T_0}[K_E] \end{bmatrix} \begin{Bmatrix} \{q_M^*\} \\ \{q_E^*\} \end{Bmatrix} = \begin{Bmatrix} \{Q_M\} \\ \{Q_E\} \end{Bmatrix}$$

or briefly,

$$[M^*]\{\ddot{q}^*\} + [D^*]\{\dot{q}^*\} + [K^*]\{q^*\} = \{Q^*\} \qquad (17)$$

Thus, for two dimensional problems, each node contains four degrees of freedom, two of them are mechanical displacements and others are entropy displacements. Hence, the mechanical and entropy displacements are solved directly. As a result, the temperature distribution can be inferred from the mechanical and entropy displacements, as shown in eq.(13), and the thermal stress can be deduced by Duhamel-Neumann constitutive equations of eq.(3). To solve the equations of motion, Newmark's directed integration scheme [9] is taken in this work. The parameters δ and β used in Newmark's method are chosen as δ=0.5 and β=0.25 respectively.

For quasi-static thermoelastic problems, eq .(17) can be simplified as:

$$[D^*]\{\dot{q}^*\} + [K^*]\{q^*\} = \{Q^*\}$$

For the uncoupled thermoelastic problems, say γ=0 [8], the equations of motion are governed by the two independent sets of simultaneous linear ordinary differential equation as:

$$[M]\{\ddot{q}_M^*\} + [K_M]\{q_M^*\} = \{Q_M\} - \frac{\beta T_0}{\rho C_v}[K_E]\{q_E^*\}$$

and

$$[D]\{\dot{q}_E^*\} + \frac{1}{\rho C_v T_0}[K_E]\{q_E^*\} = \{Q_E\}$$

3. Results and Discussions

To show the effectiveness of the proposed technique, the solutions of two illustrative problems are first Presented. The first problem considered is known as the Danilovskaya's problem in thermoelasticity [10] as shown in Fig.1. The semi-infinite space is modeled with 25 eight-node isoparametric elements. The nondimensional variables are defined as:

$$\xi = \frac{vX}{d}, \quad \tau = \frac{v^2 t}{d}, \quad \theta^* = \frac{\theta}{\bar{\theta}} \quad \text{and} \quad U_x^* = \frac{(\lambda+2\mu)Ux}{d\beta\bar{\theta}}$$

where d is the thermal diffusivity and $v=\sqrt{(\lambda+2\mu)/\rho}$ denotes elastic wave speed.

Fig.2 and Fig.3 depict the time histories of the temperature and mechanical displacement variations at the point ξ =1.0. For comparison purposes, the uncoupled analytic solution obtained by Laplace transformation method [10] and the coupled numerical solution computed by finite element method [11-13] are also shown. Excellent agreement between the present results and referenced solutions can be found.

The second problem solved illustrates the transient response of a two-dimensional square plate with length 2L which is initially at a uniform temperature state with $T=T_0$. As shown in Fig.4, the outer two boundaries of $X=\pm L$ are subjected to a thermal shock $\bar{\theta}$, whereas other boundaries of $Y=\pm L$ are insulated. The uncoupled theoretical quasi-static solution of the same problem was solved by Takeuti [14] . The coupling parameters γ taken into account are $\gamma=0$ and $\gamma=0.25$ respectively. Fig.5 displays the variations of transient dimensionless stress $\sigma_x^*=\sigma_x/E\alpha\bar{\theta}$ along the boundary of y=L as shown in Fig.4 at Fr=0.01 ($Fr=dt/L^2$, the dimensionless time). Again, the agreement between the present results and the theoretical solution given by Takeuti is excellent. It is worthwhile to note that the stress obtained based on the coupled approach are smaller than the uncoupled results. This implies that more conservative results are obtained by the coupled approach. This is of parctical importance for the design of efficient and economic structural parts. To evaluate the influence of the inertia effects on the computation of transient thermal stresses, the quasi-static solutions based on coupled and uncoupled approaches are then computed respectively and shown in Fig.5. It appears that no significant difference between the dynamic and quasi-static solutions can be found.

In actual machine and engineering structural members, the highest stresses usually exist in the region involving fillets, holes, notches and other section discontinuities. In engineering practice, however, the calculation of thermoelastic stress concentration factors are seldom found. To evaluate the thermoelastic stress concentration factors in dynamic thermoelastic problems, the example considered is that of the square plate with a central circular hole subjected to a thermal shock as indicated in Fig.6. The thermoelastic stress concentration factor $K\sigma$ discussed here is defined as:

$$K\sigma = \sigma_d / \sigma_m$$

where σ_d is the maximum stress occurred at the geometric discontinuity and σ_m is the maximum stress evaluate in the plate without hole. Fig.7 shows the variations of the calculated thermoelastic stress concentration factors $K\sigma$ with various sizes of circular holes. It is seen that the values of the computed $K\sigma$ nearly proportional to the sizes of the circular holes at an instant and decrease monotonically as the time passes. Again, smaller thermoelastic stress concentration factors are obtained by coupled approach rather than by uncoupled approach.

4. Conclusions

This paper provides a rigorous dynamic, coupled thermoelastic finite element procedure which has been demonstrated to be a highly accurate means to evaluate the temperature as well as stress distributions within the structure with complicated geometries subjected to thermal shocks. The role of the coupled effects on the computation of transient thermal stresses can be drawn. However, from the results for the problems solved in this work, it appears that the inertia effects are not that important. Futher, the thermoelastic stress concentration factors are also investigated in this work. It will be of practical use for estimating the safety of structural parts subjected to thermal shocks.

References

[1] Biot, M.A., "Thermoelasticity and Irreversible Thermodynamics, "Journal of Applied Physics 27, 240-253 (1956).

[2] Chen, W.H., "An Assumed Hybrid Displacement Finite Element Model for Elastodynamic Cracked Problems", Engineering Structures 3, 249-255 (1981).

[3] Chen, W.H. and Wu, C.W., "On Elastodynamic Fracture Mechanics Analysis of Bi-material Structures Using Finite Element Method", Engineering Fracture Mechanics 15, 155-168 (1981).

[4] Ben-Amoz, M., "On a Variational Theorem in Coupled Thermoelasticity", Journal of Applied Mechanics 32, 943-945 (1965).

[5] Herrmann, G., "On Variational Principles in Thermoelasticity and Heat Conduction", Quarterly of Applied Mathematics 21, 151-155 (1963).

[6] Dhaliwal, R.S. and Sherief, H.H., "Generalized Thermoelasticity for Anisotropic Media", Quarterly of Applied Mathematics 38, 1-8 (1980).

[7] Zienkiewicz, O.C., "The Finite Element Method", McGraw-Hill, London (3rd ed.), (1977).

[8] Nowacki, W., "Dynamic Problems of Thermoelasticity", Noordhoff International Publishing, Leyden, The Netherlands, (1975).

[9] Bathe, K.J. and Wilson, E.L., "Numerical Methods in Finite Element Analysis", Prentice-Hall, Englewood Cliffs, (1976).

[10] Sternberg, E. and Chakravorty, J.G., "On Inertia Effects in a Transient Thermoelastic Problems", Journal of Applied Mechanics 26, 503-509, (1959).

[11] Ting, E.C. and Chen, H.C., " A Unified Numerical Approach for Thermal Stress Waves", Computers & Structures 15, 165-175 (1982).

[12] Keramidas, G.A. and Ting, E.C., "A Finite Element Formulation for Thermal Stress Analysis. Part I: Variational Formulation" Nuclear Engineering and Design 39, 267-275, (1976).

[13] Keramidas, G.A. and Ting, E.C., " A Finite Element Formulation for Thermal Stress Analysis Part II: Finite Element Formulation", Nuclear Engineering and Design 39, 277-287, (1976).

[14] Takeuti, Y. and Tanigawa, Y., "Improved Numerical Methods for Unsteady-state Thermal Stress Problems Using with Numerical Integrations", Proceedings of the 1st Intl. Conf. on Numerical Methods in Thermal Problems, Swansea, July 2-6, 1979, 644-653.

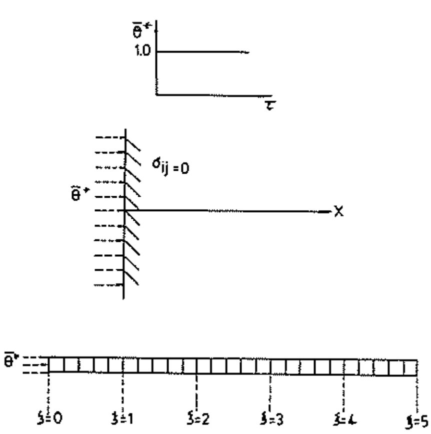

Fig.1. Danilovskaya's Problem and Finite Element Idealization

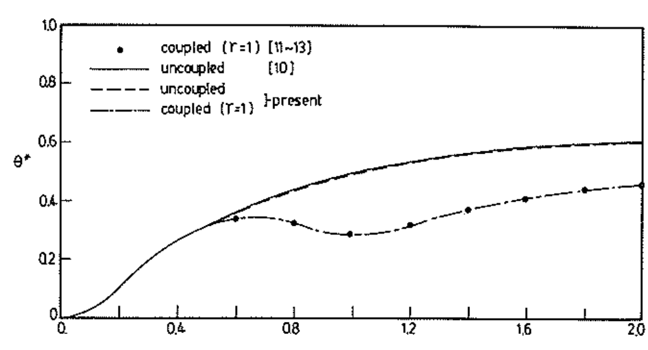

Fig.2. Variations of θ^* at ξ=1 for Danilovskaya's Problem

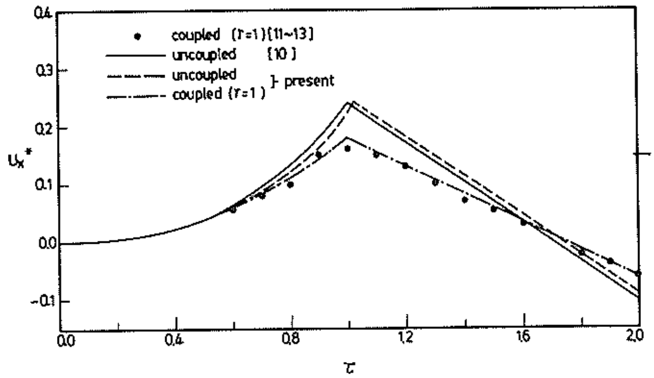

Fig.3. Variations of U_x^* at ξ=1 for Danilovskaya's Problem

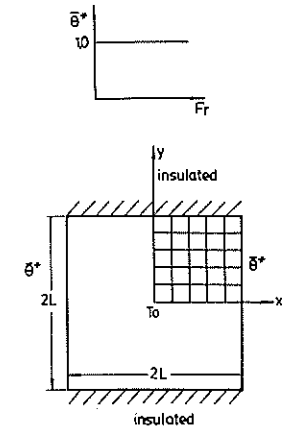

Fig.4. Finite Element Modeling for Takeuti's Problem

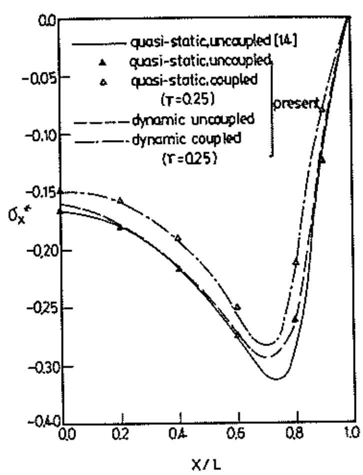

Fig.5. Variations of σ_x^* along Y=L at Fr=0.01

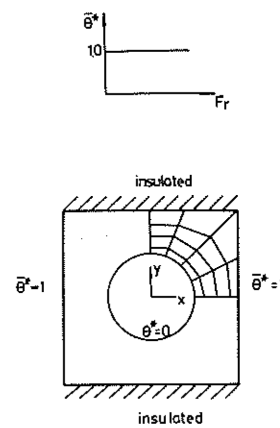

Fig.6. Finite Element Modeling for Thermal Stress Concentration Analysis

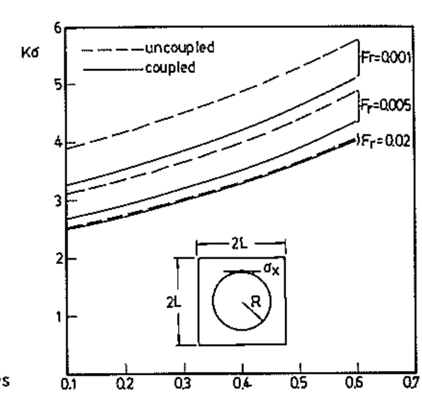

Fig.7. Variations of Thermoelastic Stress Concentration Factors with Various Sizes of Circular Holes