

## Creep Analysis of Orthotropic Shells of Revolution Under Asymmetrical Loading

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### SUMMARY

The paper describes the numerical analysis of creep problems of orthotropic shells of revolution under asymmetrical loads with application to a cylindrical shell.

Since a creep problem of shells of revolution is closely related to the analysis and the design of pressure vessels and pressure vessel components used in high temperature, there are many investigations on the creep for not only simple geometries such as cylindrical shells (Cozzarelli and others, Byrne and Mackenzie, Murakami and Iwatsuki), conical shells (Cozzarelli and Vito), spherical shells (Penny), but also general shells of revolution (Penny, Takezono) such as pressure vessel heads and expansion bellows, subjected to axisymmetrical loads or asymmetrical loads.

These investigations, however, are based on the assumption that the material is isotropic and remains so during creep. But many materials are anisotropic from the beginning, namely their mechanical properties vary depending upon the relative orientation of the stress system and some natural direction of the material, and many initially isotropic materials become anisotropic during prolonged deformation under creep.

In this paper the authors study the creep problem of rotationally axisymmetrical shells made of a homogeneous and orthotropic material under asymmetrical load. The equations of equilibrium and the relations between the strains and the displacements are derived from the Sanders theory for thin shells.

In the theory of creep it is assumed that in a given increment of time the total strain increments are composed of an elastic part and a part due to creep. The elastic strains are proportional to the stress by Hooke's law for orthotropic materials. The constitutive equations in the creep range are based on the orthotropic creep theory derived from the orthotropic plastic theory by Hill, and the effective creep strain is related to the effective stress by Norton-Vailey equation.

The basic differential equations for the incremental values with respect to time are numerically solved by a finite difference method and the solutions at any time are obtained by summation of the incremental values. Resultant forces and resultant moments are given from numerical integration by Simpson's 1/3 rule.

As a numerical example the creep deformation of simply supported cylindrical shell subject to asymmetrical, locally distributed load is treated. The numerical computations have been carried out for four cases of anisotropy. The effect of anisotropy has been studied. It is observed that the stress resultant and the displacement distributions are significantly affected by the anisotropy of the material.

The analysis of shells in steady creep becomes a subcase of the more general transient creep process described here.

# 1. INTRODUCTION

Since a creep problem of shells of revolution is closely related to the analysis and the design of pressure vessels and pressure vessel components used in high temperature, many investigations have been made on the creep for not only simple geometries such as spherical shells, conical shells, cylindrical shells, but also general shells of revolution [1-10] such as pressure vessel heads and expansion bellows.

These investigations, however, are based on the assumption that the material is isotropic and remains so during creep. But many materials are anisotropic from the beginning, namely their mechanical properties vary depending on the relative orientation of the stress system and some natural direction of the material, and many initially isotropic materials become anisotropic during prolonged deformation under creep.

In this paper the authors study the creep problem of rotationally axisymmetrical shells made of a homo-geneous and orthotropic material under asymmetrical load by extension of the method for the isotropic creep [6]. The equations of equilibrium and the relations between the strains and the displacements are derived from the Sanders thin shell theory [11]. The constitutive equations in the creep range are based on the orthotropic creep theory derived from the orthotropic plastic theory by Hill [12], and the effective creep strain is assumed to be related to the effective stress by Norton-Vailey equation.

The basic differential equations for the incremental values with respect to time are numerically solved by a finite difference method.

As a numerical example the creep problem of simply supported cylindrical shell subject to asymmetrical, locally distributed load is analyzed, and the effect of anisotropy has been studied.

# 2. FUNDAMENTAL EQUATIONS

If the middle surface of axisymmetrical shells is given by  $r=r(s)$ , where  $r$  is the distance from the axis and  $s$  is the meridional distance measured from a boundary along the middle surface, as shown in Fig. 1, the relations among the non-dimensional curvatures  $\omega_\xi (=a/R_\xi)$ ,  $\omega_\theta (=a/R_\theta)$  and the non-dimensional radius  $\rho (=r/a)$  become :

$$\omega_\xi = -(\gamma' + \gamma^2)/\omega_\theta, \quad \omega_\theta = \sqrt{1 - (\rho')^2/\rho}, \quad \omega_\theta' = \gamma(\omega_\xi - \omega_\theta),$$

$$\rho''/\rho = -\omega_\xi \omega_\theta, \quad \gamma = \rho'/\rho, \quad \xi = s/a, \quad ' = d/d\xi \tag{1}$$

where  $a$  is the reference length. An arbitrary point of the shell can be expressed in the orthogonal coordinate system  $(\xi, \theta, \zeta)$ .

Eliminating the transverse shear forces  $Q_\xi$  and  $Q_\theta$  in the equilibrium equations in the Sanders theory [11] and writing in the rate forms, the following equations are obtained :

$$a[\partial(\rho \dot{N}_\xi)/\partial \xi + \partial \dot{N}_{\xi\theta}/\partial \theta - \rho' \dot{N}_\theta] + \omega_\xi [\partial(\rho \dot{M}_\xi)/\partial \xi + \partial \dot{M}_{\xi\theta}/\partial \theta - \rho' \dot{M}_\theta]$$

$$+ (1/2)(\omega_\xi - \omega_\theta) \partial \dot{M}_{\xi\theta}/\partial \theta + a^2 \rho \dot{P}_\xi = 0$$

$$a[\partial \dot{N}_\theta/\partial \theta + \partial(\rho \dot{N}_{\xi\theta})/\partial \xi + \rho' \dot{M}_{\xi\theta}] + \omega_\theta [\partial \dot{M}_\theta/\partial \theta + \partial(\rho \dot{M}_{\xi\theta})/\partial \xi + \rho' \dot{M}_{\xi\theta}]$$

$$+ (\rho/2) \partial [(\omega_\theta - \omega_\xi) \dot{M}_{\xi\theta}]/\partial \xi + a^2 \rho \dot{P}_\theta = 0 \tag{2}$$

$$\partial[\partial(\rho \dot{M}_\xi)/\partial \xi + \partial \dot{M}_{\xi\theta}/\partial \theta - \rho' \dot{M}_\theta]/\partial \xi + (1/\rho) \partial[\partial \dot{M}_\theta/\partial \theta + \partial(\rho \dot{M}_{\xi\theta})/\partial \xi + \rho' \dot{M}_{\xi\theta}]/\partial \theta - a \rho (\omega_\xi \dot{N}_\xi + \omega_\theta \dot{N}_\theta) + a^2 \rho \dot{P}_\zeta = 0$$

where :

$$\dot{M}_{\xi\theta} = (\dot{M}_{\xi\theta} + \dot{M}_{\theta\xi})/2, \quad \dot{N}_{\xi\theta} = (\dot{N}_{\xi\theta} + \dot{N}_{\theta\xi})/2 + (\omega_\theta - \omega_\xi)(\dot{M}_{\xi\theta} - \dot{M}_{\theta\xi})/4a \tag{3}$$

and the notations are shown in Fig.2.

The membrane strain rates of the middle surface are given by :

$$\dot{\epsilon}_{\xi m} = (1/a)[\partial \dot{U}_\xi/\partial \xi + \omega_\xi \dot{U}_\zeta], \quad \dot{\epsilon}_{\theta m} = (1/a)[(1/\rho) \partial \dot{U}_\theta/\partial \theta + \gamma \dot{U}_\xi + \omega_\theta \dot{U}_\zeta] \tag{4}$$

$$\dot{\epsilon}_{\xi\theta m} = (1/2a)[(1/\rho) \partial \dot{U}_\xi/\partial \theta + \partial \dot{U}_\theta/\partial \xi - \gamma \dot{U}_\theta]$$

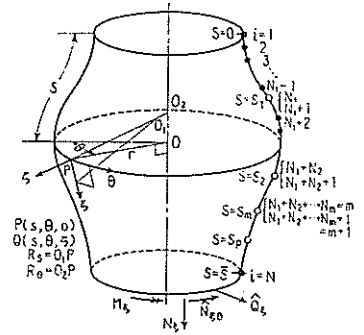


Fig.1. Geometry and mesh points.

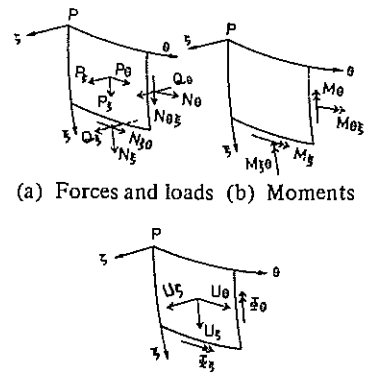


Fig.2. Notations.

where  $\dot{\epsilon}_{\xi\theta m}$  is half the usual engineering shear strain rate. The bending distortion rates are as follows :

$$\begin{aligned}\dot{\kappa}_{\xi} &= (1/a)\partial\dot{\Phi}_{\xi}/\partial\xi, \quad \dot{\kappa}_{\theta} = (1/a)[(1/\rho)\partial\dot{\Phi}_{\theta}/\partial\theta + \gamma\dot{\Phi}_{\xi}] \\ \dot{\kappa}_{\xi\theta} &= (1/2a)[(1/\rho)\partial\dot{\Phi}_{\xi}/\partial\theta + \partial\dot{\Phi}_{\theta}/\partial\xi - \gamma\dot{\Phi}_{\theta} + (1/2a)(\omega_{\xi} - \omega_{\theta})[(1/\rho)\partial\dot{U}_{\xi}/\partial\theta - \partial\dot{U}_{\theta}/\partial\xi - \gamma\dot{U}_{\theta}]]\end{aligned}\quad (5)$$

where rotation rates  $\dot{\Phi}_{\xi}$  and  $\dot{\Phi}_{\theta}$  are :

$$\dot{\Phi}_{\xi} = (1/a)[ -(\partial\dot{U}_{\xi}/\partial\xi) + \omega_{\xi}\dot{U}_{\xi}], \quad \dot{\Phi}_{\theta} = (1/a)[ -(1/\rho)\partial\dot{U}_{\xi}/\partial\theta + \omega_{\theta}\dot{U}_{\theta}]\quad (6)$$

Under the Kirchhoff-Love hypothesis and the neglect of terms of order  $\zeta/R_s$  and  $\zeta/R_o$  relative to unity, the strain rates at the distance  $\zeta$  from the middle surface are :

$$\{\dot{\epsilon}\} = \{\dot{\epsilon}_m\} + \zeta\{\dot{\kappa}\}\quad (7)$$

where  $\{\dot{\epsilon}\}^T = \{\dot{\epsilon}_{\xi}, \dot{\epsilon}_{\theta}, \dot{\epsilon}_{\xi\theta}\}$ ,  $\{\dot{\epsilon}_m\}^T = \{\dot{\epsilon}_{\xi m}, \dot{\epsilon}_{\theta m}, \dot{\epsilon}_{\xi\theta m}\}$ ,  $\{\dot{\kappa}\}^T = \{\dot{\kappa}_{\xi}, \dot{\kappa}_{\theta}, \dot{\kappa}_{\xi\theta}\}$  and  $\{\ }^T$  means the transposed matrix.

In the theory of creep it is assumed that in a given increment of time the total strain increments are composed of an elastic part and a part due to creep. Since the elastic strain rates are directly proportional to the stress rates by Hooke's law, the total strain rates may be expressed as follows :

$$\{\dot{\epsilon}\} = [D]^{-1}\{\dot{\sigma}\} + \{\dot{\epsilon}_T\} + \{\dot{\epsilon}_c\}\quad (8)$$

where  $\{\dot{\sigma}\}^T = \{\dot{\sigma}_{\xi}, \dot{\sigma}_{\theta}, \dot{\sigma}_{\xi\theta}\}$ ,  
 $\{\dot{\epsilon}_T\}^T = \{\alpha_{\xi}\Delta T, \alpha_{\theta}\Delta T, 0\}$ ,  $[D] = \frac{1}{1-\nu_{\xi\theta}\nu_{\theta\xi}} \begin{bmatrix} E_{\xi} & E_{\xi}\nu_{\theta\xi} & 0 \\ E_{\theta}\nu_{\xi\theta} & E_{\theta} & 0 \\ 0 & 0 & 2G_{\xi\theta}(1-\nu_{\xi\theta}\nu_{\theta\xi}) \end{bmatrix}$   
 $\{\dot{\epsilon}_c\}^T = \{\dot{\epsilon}_{\xi c}, \dot{\epsilon}_{\theta c}, \dot{\epsilon}_{\xi\theta c}\}$

and  $\alpha_{\xi}$ ,  $\alpha_{\theta}$  are thermal expansion coefficient,  $\Delta T$  is temperature rise,  $E_{\xi}$ ,  $E_{\theta}$  are Young's modulus,  $G_{\xi\theta}$  is the shearing modulus,  $\nu_{\xi\theta}$ ,  $\nu_{\theta\xi}$  are Poisson's ratios and  $\nu_{\xi\theta}/E_{\xi} = \nu_{\theta\xi}/E_{\theta}$ .

Now, it is assumed that in the transient creep the effective creep strain  $\bar{\epsilon}_c$  is related to the effective stress  $\bar{\sigma}$  (for constant stress) by the next equations [13] :

$$\bar{\epsilon}_c = A\bar{\sigma}^n t^m, \quad A = Ke^{-\beta/T}\quad (9)$$

where  $K$ ,  $m$ ,  $n$  and  $\beta$  are the material creep constants, and  $T$  is the absolute temperature.

If eqs.(9) is extended to the plane stress state assumed in the ordinary thin shell theory, then the creep strain rate,  $\{\dot{\epsilon}_c\}$ , may be written as follows for the time hardening and the strain hardening theories :

$$\{\dot{\epsilon}_c\} = \frac{3}{2}mA\bar{\sigma}^{n-1} t^{m-1} [B]\{\sigma\}, \quad \{\dot{\epsilon}_c\} = \frac{3}{2}mA\frac{1}{\bar{\sigma}} \left(\frac{n}{m}-1\right) \bar{\epsilon}_c^{(1-\frac{1}{m})} [B]\{\sigma\}\quad (10)$$

where

$$\begin{aligned}\bar{\sigma} &= \sqrt{\frac{3}{2(F+G+H)} [(G+H)\sigma_{\xi}^2 - 2H\sigma_{\xi}\sigma_{\theta} + (F+H)\sigma_{\theta}^2 + 2N\sigma_{\xi\theta}^2]} \\ \bar{\epsilon}_c &= \int_0^t \sqrt{\frac{2}{3(F+G+H)}} \left[ \frac{(F+H)\dot{\epsilon}_{\xi c}^2 + 2H\dot{\epsilon}_{\xi c}\dot{\epsilon}_{\theta c} + (G+H)\dot{\epsilon}_{\theta c}^2 + \frac{2\dot{\epsilon}_{\xi\theta c}^2}{N}}{FG+GH+HF} \right]^{1/2} dt, \quad [B] = \frac{1}{F+G+H} \begin{bmatrix} G+H, -H, 0 \\ -H, F+H, 0 \\ 0, 0, N \end{bmatrix}\end{aligned}\quad (11)$$

The creep strain rates in the uniaxial state may be written as for the two theories :

$$\begin{aligned}\dot{\epsilon}_{\xi c} &= mA \left(\frac{3}{2} \frac{G+H}{F+G+H}\right)^{\frac{n+1}{2}} \sigma_{\xi}^n t^{m-1} & \dot{\epsilon}_{\xi c} &= m \left\{ A \left(\frac{3}{2} \frac{G+H}{F+G+H}\right)^{\frac{n+1}{2}} \right\}^{\frac{1}{m}} \epsilon_{\xi c}^{(1-\frac{1}{m})} \sigma_{\xi}^{\frac{n}{m}} \\ \dot{\epsilon}_{\theta c} &= mA \left(\frac{3}{2} \frac{F+H}{F+G+H}\right)^{\frac{n+1}{2}} \sigma_{\theta}^n t^{m-1} & \dot{\epsilon}_{\theta c} &= m \left\{ A \left(\frac{3}{2} \frac{F+H}{F+G+H}\right)^{\frac{n+1}{2}} \right\}^{\frac{1}{m}} \epsilon_{\theta c}^{(1-\frac{1}{m})} \sigma_{\theta}^{\frac{n}{m}} \\ \dot{\epsilon}_{\xi\theta c} &= mA \left(\frac{3}{2} \frac{F+G}{F+G+H}\right)^{\frac{n+1}{2}} \sigma_{\xi\theta}^n t^{m-1} & \dot{\epsilon}_{\xi\theta c} &= m \left\{ A \left(\frac{3}{2} \frac{F+G}{F+G+H}\right)^{\frac{n+1}{2}} \right\}^{\frac{1}{m}} \epsilon_{\xi\theta c}^{(1-\frac{1}{m})} \sigma_{\xi\theta}^{\frac{n}{m}}\end{aligned}\quad (12)$$

Substituting eqs.(7) into eqs.(8) and solving them about stress rates, the stress rates are :

$$\{\dot{\sigma}\} = [D] (\{\dot{\epsilon}_m\} + \zeta\{\dot{\kappa}\}) - \{\dot{\sigma}_T\} - \{\dot{\sigma}_c\}\quad (14)$$

where  $\{\dot{\sigma}_c\} = [D]\{\dot{\epsilon}_c\}$ ,  $\{\dot{\sigma}_T\} = [D]\{\dot{\epsilon}_T\}$  (15)

From eqs.(14) the rates of change of the resultant forces and the resultant moments may be expressed by the following :

$$\begin{aligned}\{\dot{N}_{\xi}, \dot{N}_{\theta}, \dot{N}_{\xi\theta}\} &= \{\dot{\epsilon}_m\}^T \int_{-h/2}^{h/2} [D] d\xi - \{\dot{N}_{\xi T}, \dot{N}_{\theta T}, 0\} - \{\dot{N}_{\xi c}, \dot{N}_{\theta c}, \dot{N}_{\xi\theta c}\} \\ \{\dot{M}_{\xi}, \dot{M}_{\theta}, \dot{M}_{\xi\theta}\} &= \{\dot{\kappa}\}^T \int_{-h/2}^{h/2} [D] \xi^2 d\xi - \{\dot{M}_{\xi T}, \dot{M}_{\theta T}, 0\} - \{\dot{M}_{\xi c}, \dot{M}_{\theta c}, \dot{M}_{\xi\theta c}\}\end{aligned}\quad (16)$$

where  $\{\dot{N}_{\xi c}, \dot{N}_{\theta c}, \dot{N}_{\xi\theta c}, \dot{N}_{\xi T}, \dot{N}_{\theta T}\} = \int_{-h/2}^{h/2} \{\dot{\sigma}_{\xi c}, \dot{\sigma}_{\theta c}, \dot{\sigma}_{\xi\theta c}, \dot{\sigma}_{\xi T}, \dot{\sigma}_{\theta T}\} d\xi$

$$\{\dot{M}_{\xi c}, \dot{M}_{\theta c}, \dot{M}_{\xi\theta c}, \dot{M}_{\xi T}, \dot{M}_{\theta T}\} = \int_{-h/2}^{h/2} \{\dot{\sigma}_{\xi c}, \dot{\sigma}_{\theta c}, \dot{\sigma}_{\xi\theta c}, \dot{\sigma}_{\xi T}, \dot{\sigma}_{\theta T}\} \xi d\xi\quad (17)$$

and  $h$  is thickness of the shell. These creep components are calculated from numerical integration by Simpson's 1/3 rule.

A complete set of field equations for 27 independent variables:  $\dot{N}_\xi, \dot{N}_\theta, \dot{N}_{\xi\theta}, \dot{M}_\xi, \dot{M}_\theta, \dot{M}_{\xi\theta}, \dot{U}_\xi, \dot{U}_\theta, \dot{U}_\xi, \dot{\epsilon}_{\xi m}, \dot{\epsilon}_{\theta m}, \dot{\epsilon}_{\xi\theta m}, \dot{\kappa}_\xi, \dot{\kappa}_\theta, \dot{\kappa}_{\xi\theta}, \dot{\Phi}_\xi, \dot{\Phi}_\theta, \dot{N}_{\xi c}, \dot{N}_{\theta c}, \dot{N}_{\xi\theta c}, \dot{M}_{\xi c}, \dot{M}_{\theta c}, \dot{M}_{\xi\theta c}, \dot{N}_{\xi T}, \dot{N}_{\theta T}, \dot{M}_{\xi T}, \dot{M}_{\theta T}$  is now given by 27 equations, (2), (4)-(6), (16) and (17).

### 3. NONDIMENSIONAL EQUATIONS

In order to analyze the problem of the shells under arbitrary unsymmetrical loads, the 27 independent variables and the loads  $\dot{P}_\xi, \dot{P}_\theta, \dot{P}_\xi, \Delta \dot{T}$  are expanded into Fourier series. Substituting these into the above 27 equations and appropriately eliminating the variables, the simultaneous equations for  $\dot{u}_\xi^{(n)}, \dot{u}_\theta^{(n)}, \dot{u}_\xi^{(n)}$  and  $\dot{m}_\xi^{(n)}$  can be obtained as follows:

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_{12} & a_{15} & 0 \\ 0 & a_{21} & a_{24} & a_{27} \\ 0 & 0 & a_{33} & 0 \end{pmatrix} \dot{z}'' + \begin{pmatrix} a_2 & a_4 & a_6 & a_8 \\ a_{10} & a_{13} & a_{16} & 0 \\ a_{19} & a_{22} & a_{25} & a_{28} \\ a_{30} & 0 & a_{34} & 0 \end{pmatrix} \dot{z}' + \begin{pmatrix} a_3 & a_5 & a_7 & a_9 \\ a_{11} & a_{14} & a_{17} & a_{18} \\ a_{20} & a_{23} & a_{26} & a_{29} \\ a_{31} & a_{32} & a_{35} & a_{36} \end{pmatrix} \dot{z} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \quad (18)$$

where  $\dot{z} = \{ \dot{u}_\xi^{(n)}, \dot{u}_\theta^{(n)}, \dot{u}_\xi^{(n)}, \dot{m}_\xi^{(n)} \}^T$ , and  $a_1 - a_{36}$  and  $c_1 - c_4$  are:

$$\begin{aligned} a_1 &= f_1, a_2 = \gamma f_1, a_3 = -\nu_{\theta\xi} f_1 \omega_\xi \omega_\theta - f_2 \gamma^2 - \frac{f_3 n^2}{2\rho^2} - \lambda^2 [(1 - \nu_{\xi\theta} \nu_{\theta\xi}) \gamma^2 g_2 \omega_\xi^2 + \frac{(3\omega_\xi - \omega_\theta)^2 n^2}{8\rho^2} g_3], a_4 = \frac{(f_3 + 2f_1 \nu_{\theta\xi}) n}{2\rho} \\ &+ \frac{\lambda^2 g_3 n}{8\rho} (3\omega_\xi - \omega_\theta)(3\omega_\theta - \omega_\xi), a_5 = -\frac{(2f_2 + f_3) \gamma n}{2\rho} - \frac{\lambda^2 \gamma n}{\rho} [(3\omega_\xi - \omega_\theta)(3\omega_\theta - \omega_\xi) g_3 / 8 + (1 - \nu_{\xi\theta} \nu_{\theta\xi}) \omega_\xi \omega_\theta g_2] \\ a_6 &= f_1 (\omega_\xi + \nu_{\theta\xi} \omega_\theta) + \lambda^2 [(1 - \nu_{\xi\theta} \nu_{\theta\xi}) \gamma^2 \omega_\xi g_2 + \frac{n^2}{2\rho^2} (3\omega_\xi - \omega_\theta) g_3] \\ a_7 &= f_1 \omega_\xi' + \gamma (f_1 \omega_\xi - f_2 \omega_\theta) - \frac{\lambda^2 \gamma n^2}{\rho^2} \left[ \frac{(3\omega_\xi - \omega_\theta)}{2} g_3 + (1 - \nu_{\xi\theta} \nu_{\theta\xi}) g_2 \omega_\xi \right] \\ a_8 &= \lambda^2 \omega_\xi, a_9 = \lambda^2 (1 - \nu_{\theta\xi}) \gamma \omega_\xi, a_{10} = -\frac{(f_3 + 2f_2 \nu_{\xi\theta}) n}{2\rho} - \frac{\lambda^2 g_3 n}{8\rho} (3\omega_\xi - \omega_\theta)(3\omega_\theta - \omega_\xi), a_{11} = -\frac{\gamma n}{2\rho} (f_3 + 2f_2) \\ &+ \frac{\lambda^2 n}{\rho} \left[ -g_2 (1 - \nu_{\xi\theta} \nu_{\theta\xi}) \gamma \omega_\xi \omega_\theta + \frac{\gamma}{8} (6\omega_\xi \omega_\theta - 7\omega_\xi^2 - 3\omega_\theta^2) g_3 - \frac{\omega_\xi'}{4} (5\omega_\theta - 3\omega_\xi) g_3 \right], a_{12} = \frac{f_3}{2} + \frac{\lambda^2 g_3 (3\omega_\theta - \omega_\xi)^2}{8} \\ a_{13} &= \frac{f_3 \gamma}{2} - \frac{\lambda^2 g_3}{8} (3\omega_\theta - \omega_\xi) [2\omega_\xi' - \gamma (5\omega_\xi - 3\omega_\theta)], a_{14} = -\gamma a_{13} + \frac{1}{2} f_3 \omega_\xi \omega_\theta - \frac{f_2 n^2}{\rho^2} - \lambda^2 \left[ \frac{g_2 (1 - \nu_{\xi\theta} \nu_{\theta\xi}) \omega_\theta^2 n^2}{\rho^2} \right. \\ &\left. - \frac{\omega_\xi \omega_\theta}{8} g_3 (3\omega_\theta - \omega_\xi)^2 \right], a_{15} = \frac{\lambda^2 g_3}{2\rho} (3\omega_\theta - \omega_\xi) n, a_{16} = \frac{\lambda^2 n}{2\rho} [2g_2 (1 - \nu_{\xi\theta} \nu_{\theta\xi}) \gamma \omega_\theta + g_3 \{ -\omega_\xi' + 3\gamma (\omega_\xi - \omega_\theta) \}] \\ a_{17} &= -\frac{f_2 n}{\rho} (\omega_\theta + \nu_{\xi\theta} \omega_\xi) + \frac{1}{2\rho} \lambda^2 n \left[ -\frac{2}{\rho^2} g_2 (1 - \nu_{\xi\theta} \nu_{\theta\xi}) \omega_\theta n^2 + g_3 (\gamma \omega_\xi' - 2\gamma^2 \omega_\xi) + (3\omega_\theta - \omega_\xi) (\gamma^2 + \omega_\xi \omega_\theta) g_3 \right] \\ a_{18} &= -\frac{\nu_{\theta\xi} \lambda^2 \omega_\theta n}{\rho}, a_{19} = -f_1 \omega_\xi - f_2 \omega_\theta \nu_{\xi\theta} - \lambda^2 [(1 - \nu_{\xi\theta} \nu_{\theta\xi}) \gamma^2 \omega_\xi g_2 + \frac{n^2}{2\rho^2} (3\omega_\xi - \omega_\theta) g_3] \\ a_{20} &= -\gamma (f_2 \omega_\theta + f_1 \nu_{\theta\xi} \omega_\xi) + \lambda^2 [\gamma g_2 (1 - \nu_{\xi\theta} \nu_{\theta\xi}) (-\gamma \omega_\xi' + \gamma^2 \omega_\xi - \frac{n^2 \omega_\xi}{\rho^2} + 2\omega_\xi^2 \omega_\theta) + \frac{n^2}{2\rho^2} g_3 (\gamma \omega_\xi - \gamma \omega_\theta - 3\omega_\xi')] \\ a_{21} &= a_{15}, a_{22} = \frac{1}{2\rho} \lambda^2 n [3\gamma g_3 (\omega_\xi - \omega_\theta) - 2\gamma g_2 (1 - \nu_{\xi\theta} \nu_{\theta\xi}) \omega_\theta - \omega_\xi' g_3], a_{23} = -\frac{n}{\rho} (f_2 \omega_\theta + f_1 \nu_{\theta\xi} \omega_\xi) \\ &+ \frac{1}{2\rho} \lambda^2 n [2g_2 (1 - \nu_{\xi\theta} \nu_{\theta\xi}) (\omega_\xi \omega_\theta^2 - \gamma^2 \omega_\xi + 2\gamma^2 \omega_\theta - \frac{n^2 \omega_\theta}{\rho^2}) + g_3 \gamma \omega_\xi' + 3\gamma^2 (\omega_\theta - \omega_\xi) + \omega_\xi \omega_\theta (3\omega_\theta - \omega_\xi)] \\ a_{24} &= \lambda^2 \left[ \frac{2n^2}{\rho^2} g_3 + (1 - \nu_{\xi\theta} \nu_{\theta\xi}) \gamma^2 g_2 \right], a_{25} = -\lambda^2 [(2\gamma \omega_\xi \omega_\theta + \gamma^3) (1 - \nu_{\xi\theta} \nu_{\theta\xi}) g_2 + 2\gamma n^2 g_3 / \rho^2] \\ a_{26} &= -f_1 (\omega_\xi^2 + \nu_{\theta\xi} \omega_\xi \omega_\theta) - f_2 (\omega_\theta^2 + \nu_{\xi\theta} \omega_\xi \omega_\theta) + \frac{1}{\rho^2} \lambda^2 n^2 [g_2 (1 - \nu_{\xi\theta} \nu_{\theta\xi}) (\omega_\xi \omega_\theta - \frac{n^2}{\rho^2} + 2\gamma^2) + 2g_3 (\gamma^2 + \omega_\xi \omega_\theta)] \\ a_{27} &= \lambda^2, a_{28} = \lambda^2 \gamma (2 - \nu_{\theta\xi}), a_{29} = -\lambda^2 [(1 - \nu_{\theta\xi}) \omega_\xi \omega_\theta + \frac{\nu_{\theta\xi} n^2}{\rho^2}], a_{30} = g_1 \omega_\xi, a_{31} = g_1 (\omega_\xi' + \nu_{\theta\xi} \gamma \omega_\xi) \\ a_{32} &= \frac{1}{\rho} g_1 \nu_{\theta\xi} n \omega_\theta, a_{33} = -g_1, a_{34} = -g_1 \nu_{\theta\xi} \gamma, a_{35} = \frac{1}{\rho^2} g_1 \nu_{\theta\xi} n^2, a_{36} = -1 \end{aligned} \quad (19)$$

$$\begin{aligned}
c_1 = & -\dot{p}_\xi + \dot{n}'_{\xi c} + \gamma(\dot{n}'_{\xi c} - \dot{n}_{\theta c}) + \frac{n}{\rho} \dot{n}'_{\xi \theta c} - \lambda^2 \gamma \omega_\xi (\dot{m}_{\theta c} - \nu_{\theta \xi} \dot{m}'_{\xi c}) + \frac{n \lambda^2}{2\rho} (3\omega_\xi - \omega_\theta) \dot{m}'_{\xi \theta c} + \dot{n}'_{\xi T} + \gamma(\dot{n}'_{\xi T} - \dot{n}_{\theta T}) \\
& - \lambda^2 \gamma \omega_\xi (\dot{m}_{\theta T} - \nu_{\theta \xi} \dot{m}'_{\xi T}), c_2 = -\dot{p}_\theta + \dot{n}'_{\xi \theta c} + 2\gamma \dot{n}'_{\xi \theta c} - \frac{n}{\rho} \dot{n}_{\theta c} + \frac{1}{\rho} n \lambda^2 \omega_\theta (\nu_{\theta \xi} \dot{m}'_{\xi c} - \dot{m}_{\theta c}) + \frac{1}{2} (3\omega_\theta - \omega_\xi) \lambda^2 \dot{m}'_{\xi \theta c} \\
& + \frac{1}{2} \lambda^2 \{ \gamma(3\omega_\theta + \omega_\xi) - \omega'_\xi \} \dot{m}'_{\xi \theta c} - \frac{n}{\rho} \dot{n}_{\theta T} + \frac{1}{\rho} n \lambda^2 \omega_\theta (\nu_{\theta \xi} \dot{m}'_{\xi T} - \dot{m}_{\theta T}), c_3 = -\dot{p}_\xi - \omega_\xi \dot{n}_{\xi c} - \omega_\theta \dot{n}_{\theta c} \\
& + \lambda^2 (\omega_\xi \omega_\theta - \frac{n^2}{\rho^2}) (\dot{m}_{\theta c} - \nu_{\theta \xi} \dot{m}'_{\xi c}) - \gamma \lambda^2 (\dot{m}'_{\theta c} - \nu_{\theta \xi} \dot{m}'_{\xi c}) + \frac{2}{\rho} n \lambda^2 \dot{m}'_{\xi \theta c} + \frac{2}{\rho} \gamma n \lambda^2 \dot{m}'_{\xi \theta c} - \omega_\xi \dot{n}'_{\xi T} - \omega_\theta \dot{n}_{\theta T} \\
& + \lambda^2 (\omega_\xi \omega_\theta - \frac{n^2}{\rho^2}) (\dot{m}_{\theta T} - \nu_{\theta \xi} \dot{m}'_{\xi T}) - \gamma \lambda^2 (\dot{m}'_{\theta T} - \nu_{\theta \xi} \dot{m}'_{\xi T}), c_4 = \dot{m}'_{\xi c} + \dot{m}'_{\xi T}
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
f_1 = & \frac{1}{1 - \nu_{\xi \theta} \nu_{\theta \xi}} \frac{E_\xi h}{E_0 h_0}, f_2 = \frac{1}{1 - \nu_{\xi \theta} \nu_{\theta \xi}} \frac{E_\theta h}{E_0 h_0}, f_3 = \frac{2G_{\xi \theta} h}{E_0 h_0} \\
g_1 = & \frac{1}{12(1 - \nu_{\xi \theta} \nu_{\theta \xi})} \frac{E_\xi h^3}{E_0 h_0^3}, g_2 = \frac{1}{12(1 - \nu_{\xi \theta} \nu_{\theta \xi})} \frac{E_\theta h^3}{E_0 h_0^3}, g_3 = \frac{G_{\xi \theta} h^3}{6E_0 h_0^3}
\end{aligned} \tag{21}$$

Once the solutions  $\dot{u}_\xi$ ,  $\dot{u}_\theta$ ,  $\dot{u}_\xi$  and  $\dot{m}_\xi$  have been calculated, the stresses at any point in the shell can be found.

The rates of internal forces in eqs.(20) become the following by the use of eqs.(15), (17) etc.

$$\begin{aligned}
\sigma_0 h_0 \sum_{n=0}^{\infty} \{ \dot{n}'_{\xi c}{}^{(n)}, \dot{n}_{\theta c}{}^{(n)}, \dot{n}'_{\xi \theta c}{}^{(n)} \} [A_n] = \int_{-h/2}^{h/2} \{ \dot{\epsilon}_{\xi c}, \dot{\epsilon}_{\theta c}, \dot{\epsilon}'_{\xi \theta c} \} [D] d\xi \\
(\sigma_0 h_0^3 / a) \sum_{n=0}^{\infty} \{ \dot{m}'_{\xi c}{}^{(n)}, \dot{m}_{\theta c}{}^{(n)}, \dot{m}'_{\xi \theta c}{}^{(n)} \} [A_n] = \int_{-h/2}^{h/2} \{ \dot{\epsilon}_{\xi c}, \dot{\epsilon}_{\theta c}, \dot{\epsilon}'_{\xi \theta c} \} [D] \xi d\xi
\end{aligned} \tag{22}$$

The creep strain rates on the right-hand sides can be related to stresses by eqs.(10).

The differential equations (18) are not valid at points where discontinuities in geometry and / or mechanical property of the material exist. Accordingly, at such points equations of junction will be required which relate the solution and its derivative on either side of a discontinuity.

The differential equations (18) with the boundary conditions and the equations of junction will be cast into a unified set of appropriate finite difference equations.

#### 4. NUMERICAL METHOD

Suppose that  $p$  discontinuity locations  $s_1, s_2, \dots, s_p$  occur in the range  $(0, \bar{s})$  of the shell as shown in Fig.1, let the regions  $(0, s_1), (s_1, s_2), \dots, (s_p, \bar{s})$  be subdivided into  $N_1 - 1, N_2 - 1, N_{p+1} - 1$  equal segments, respectively, and give the end points of the segments two indices, running from 1 at  $s = 0$  to  $N (= \sum_{i=1}^{p+1} N_i)$  at  $s = \bar{s}$ , then the increments in the non-dimensional variable  $\xi$  are :

$$\Delta \xi_1 = s_1 / a(N_1 - 1), \Delta \xi_2 = (s_2 - s_1) / a(N_2 - 1), \dots, \Delta \xi_{p+1} = (\bar{s} - s_p) / a(N_{p+1} - 1) \tag{23}$$

in the successive regions bounded by discontinuities.

The differential equations (eqs.(18)) are written in finite difference form at all stations except the discontinuity stations and the boundaries ( $j = 1, N$ ) on the basis of the usual central difference formulas :

$$\dot{Z}'_j = (\dot{Z}_{j+1} - \dot{Z}_{j-1}) / 2\Delta \xi \quad \dot{Z}''_j = (\dot{Z}_{j+1} - 2\dot{Z}_j + \dot{Z}_{j-1}) / (\Delta \xi)^2 \tag{24}$$

where  $\dot{Z}^T = (\dot{u}_\xi, \dot{u}_\theta, \dot{u}_\xi, \dot{m}_\xi)$ .

For the boundary points ( $j = 1, N$ ) and discontinuity points (e.g.  $j = m, m + 1$ ) :

$$\begin{aligned}
\dot{Z}'_1 = (-3\dot{Z}_1 + 4\dot{Z}_2 - \dot{Z}_3) / 2\Delta \xi_1 \quad \dot{Z}'_N = (3\dot{Z}_N - 4\dot{Z}_{N-1} + \dot{Z}_{N-2}) / 2\Delta \xi_N \\
\dot{Z}'_m = (3\dot{Z}_m - 4\dot{Z}_{m-1} + \dot{Z}_{m-2}) / 2\Delta \xi^- \quad \dot{Z}'_{m+1} = (-3\dot{Z}_{m+1} + 4\dot{Z}_{m+2} - \dot{Z}_{m+3}) / 2\Delta \xi^+
\end{aligned} \tag{25}$$

are employed, where  $\Delta \xi^-$  and  $\Delta \xi^+$  are the intervals ahead and beyond the station  $j = m$ , respectively.

Applying the above difference formulas for the fundamental equations, the boundary equations and the junction equations, the following simultaneous equations with  $N$  unknowns with respect to  $\dot{Z}_j$  may be obtained :

$$A_j \dot{Z}_{j-1} + B_j \dot{Z}_j + C_j \dot{Z}_{j+1} = D_j \tag{26}$$

$$\begin{aligned}
(AM)_m \dot{Z}_{m-2} + (BM)_m \dot{Z}_{m-1} + (CM)_m \dot{Z}_m + (DM)_{m+1} \dot{Z}_{m+1} + (EM)_{m+1} \dot{Z}_{m+2} + (FM)_{m+1} \dot{Z}_{m+3} = 0 \\
(GM)_m \dot{Z}_{m-2} + (HM)_m \dot{Z}_{m-1} + (IM)_m \dot{Z}_m + (JM)_{m+1} \dot{Z}_{m+1} + (KM)_{m+1} \dot{Z}_{m+2} + (LM)_{m+1} \dot{Z}_{m+3} = (SM)_m - (SM)_{m+1}
\end{aligned} \tag{27}$$

$$(A1)\dot{Z}_1 + (B1)\dot{Z}_2 + (C1)\dot{Z}_3 = (D1) \quad (AN)\dot{Z}_{N-2} + (BN)\dot{Z}_{N-1} + (CN)\dot{Z}_N = (DN) \tag{28}$$

The above numerical method is for the calculation of variational rate of the solution at any calculating stage and the solutions are obtained by integration of the incremental values at each stage.

### 5. NUMERICAL EXAMPLE

As a numerical example of creep problem of axisymmetrical shells, a simply supported cylindrical shell subject to unsymmetrical, distributed loads as shown in Fig.3 is treated. The numerical computations have been carried out for four cases of anisotropy.

The geometrical parameters of this shell are as follows :

$$\Delta\xi = l/2R(N-1), \rho=1, \rho'=0, \gamma=0, \omega_\theta=1, \omega_\xi=\omega'_\xi=0 \quad (29)$$

Boundary conditions at the point A ( $j = 1$ ) and B ( $j = N$ ) are, respectively :

$$\dot{U}_\xi = \dot{U}'_\xi = \dot{M}_\xi = \dot{N}_{\xi\theta} = 0, \dot{N}_\xi = \dot{U}_\theta = \dot{U}'_\theta = \dot{M}_\xi = 0 \quad (30)$$

The material employed in the calculations is assumed to be isotropic in the elastic deformation and anisotropic in the creep deformation. If the material is cylindrically anisotropic and tensile test specimens are cut in axial, tangential, and radial directions and are subjected to the same uniaxial stress  $\sigma$ , the corresponding strain rates can be obtained from eqs. (12) for time hardening theory as :

$$\epsilon_{\xi c} = mA \left( \frac{3}{2} \frac{G+H}{F+G+H} \right)^{\frac{n+1}{2}} \sigma^n t^{m-1}, \epsilon_{\theta c} = mA \left( \frac{3}{2} \frac{F+H}{F+G+H} \right)^{\frac{n+1}{2}} \sigma^n t^{m-1}, \epsilon_{\zeta c} = mA \left( \frac{3}{2} \frac{F+G}{F+G+H} \right)^{\frac{n+1}{2}} \sigma^n t^{m-1} \quad (31)$$

The anisotropic constants have been calculated using eqs. (31) and assuming four kinds of anisotropy as shown in Table 1. These ratios of strain rates have been taken to be unity for the isotropic case.

Other material constants have been determined as follows :

$$N/F=3.0, E_\xi=E_\theta=166\text{GPa}, \nu_{\xi\theta}=\nu_{\theta\xi}=0.3, G_{\xi\theta}=64.1\text{GPa}, A=4.27 \times 10^{-4} (\text{MPa})^n \text{h}^{-m} \quad (32)$$

$$n=4.17, m=0.166 \text{ (0.19\% Carbon Steel, 450}^\circ\text{C)}$$

The meridional mesh point number  $N$  and the division number  $L$  through the thickness in the numerical integration are selected as  $N = 101$  and  $L = 19$  in this example. The number of terms of Fourier series [ $= (n+1)/2$ ] is decided so that  $n = 39$ . The time increment  $\Delta t$  is controlled as, in each calculating step, the increments of the internal force which varies remarkably with time become  $1/\beta$  of the initial elastic value of the point where the elastic value is maximum. The value of  $\beta$  in this calculation has been decided as 300 from consideration of the convergency of the solutions.

Some of the essential features of the creep solutions from the time hardening law are shown in Figs.4 – 10. The solutions based on the strain hardening law scarcely differ from them.

### 6. CONCLUSIONS

In this paper we have described the numerical analysis on creep problem of rotationally axisymmetrical shells made of a homogeneous and orthotropic material under asymmetrical load. The basic differential equations have been developed by the basis of the Sanders elastic shell theory. The constitutive equations in the creep range are based on the orthotropic creep theory derived from the orthotropic plastic theory by Hill, and the effective creep strain is related to the effective stress by Norton-Vailey equation. The basic differential equations for the incremental values with respect to time are numerically solved by a finite difference method and the solutions at any time are obtained by summation of the incremental values.

As a numerical example the creep deformation of simply supported cylindrical shell subject to asymmetrical, locally distributed load is treated. The numerical computations have been carried out for four cases of anisotropy. It is observed that the stress resultant and the displacement distributions are significantly affected by the anisotropy of the material.

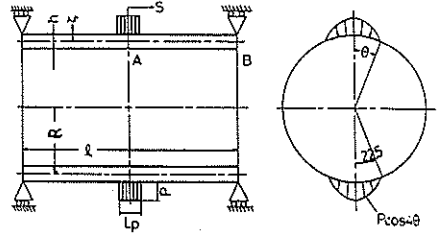


Fig.3. Numerical example.

Fig.3. Numerical example.

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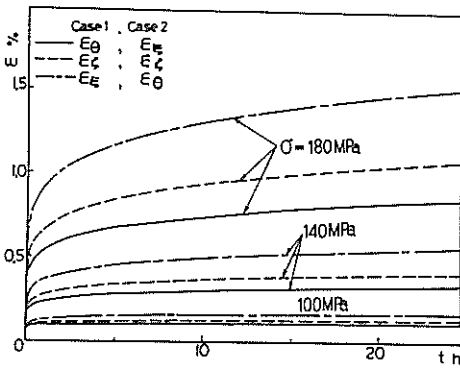


Fig.4. Creep curves.

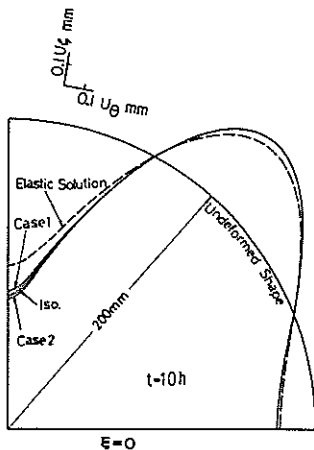


Fig.5. Deformation in mid cross-section with time.

Table 1. Anisotropic parameters.

	CASE 1	CASE 2	CASE 3	CASE 4
$\dot{\epsilon}_{\theta} : \dot{\epsilon}_{\phi} : \dot{\epsilon}_{\psi}$	1/0.7:1/1.3:1	1/1.3:1/0.7:1	1/0.5:1/1.5:1	1/1.5:1/0.5:1
G/F	1.647	0.6071	2,654	0,3767
H/F	1.392	0.8449	2,124	0,8001

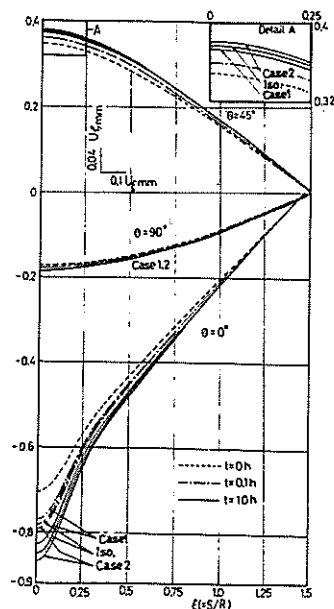


Fig.6. Deformation in meridional sections with time.

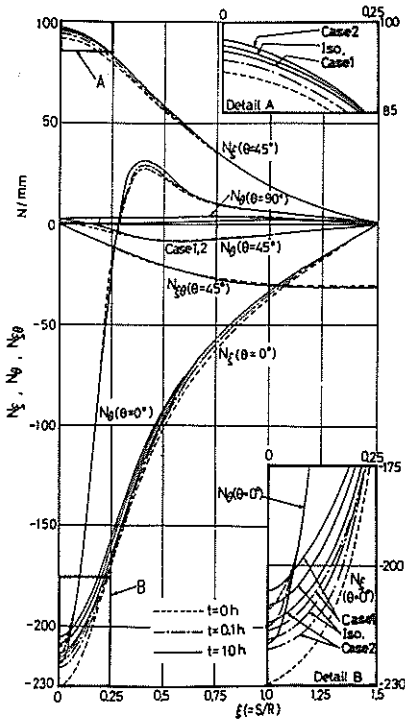


Fig. 7. Meridional distributions of resultant forces  $N_{\xi}$ ,  $N_{\theta}$ ,  $N_{\xi\theta}$ .

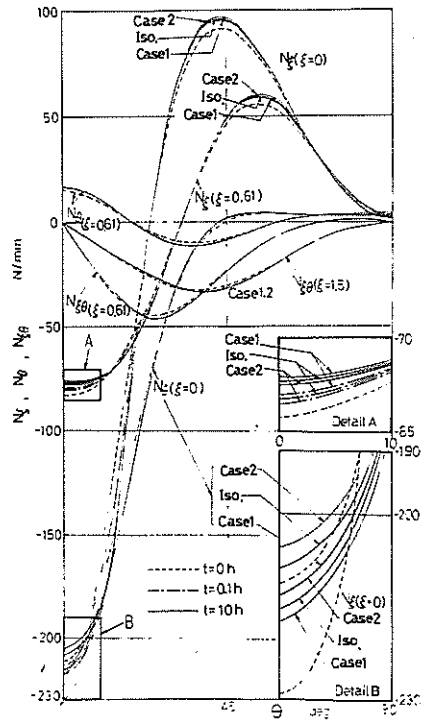


Fig. 8. Circumferential distributions of resultant forces  $N_{\xi}$ ,  $N_{\theta}$ ,  $N_{\xi\theta}$ .

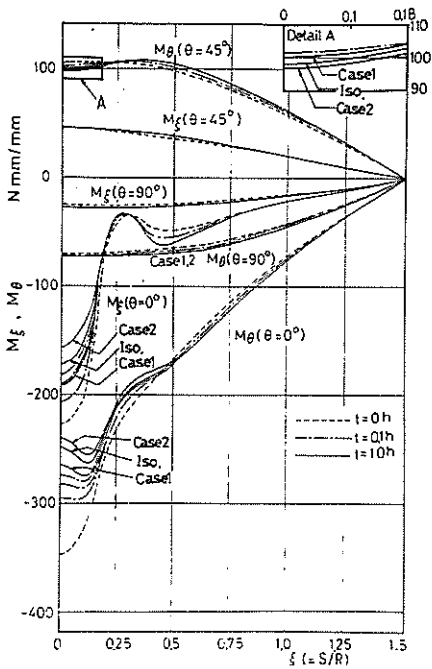


Fig. 9. Meridional distributions of resultant moments  $M_{\xi}$ ,  $M_{\theta}$ .

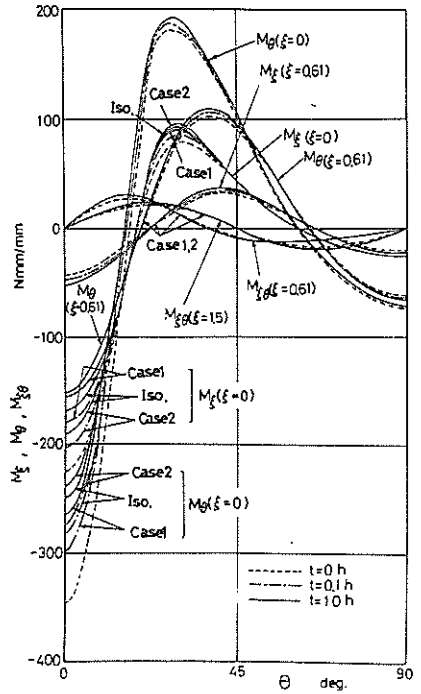


Fig. 10. Circumferential distributions of resultant moments  $M_{\xi}$ ,  $M_{\theta}$ ,  $M_{\xi\theta}$ .