# A Comparison of Two Algorithms for Creep Calculations of Shells

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ABSTRACT

For high-temperature applications, the effects of creep become important in design. For thin shells, creep problems are solved by solving successive boundary value problems for a shell at discrete values of time. Various algorithms have been proposed. The simplest algorithm (Penny and Marriott, Design for Creep, pp. 61-76) assumes that the creep strain rates can be calculated from the stress at the beginning of a time interval and then kept constant within the interval. Another algorithm, proposed by Kalnins and Updike (SMiRT-6, Paper 11/4), expands the creep strain rates and the stresses in a Taylor's series about the beginning of the interval, and keeps the first-order terms in these expansions in the calculation of, now time-dependent, creep strain rates within the interval. Each of these two algorithms has its advantages and disadvantages. The objective of this paper is to obtain solutions to specially designed problems from which the merits of each of the algorithms can be assessed.

In designing the problems, the first priority was placed on those for which analytic solutions are available. This is the case for a cylindrical shell, placed between two rigid walls, and heated to and held at a given temperature. As the material creeps, stresses relax with time. The change of stress with time is studied numerically using various strategies for selection of the time step interval for both algorithms and is compared with the exact analytical solution.

Creep calculations play an important role in the evaluation of damage that is caused by edge and discontinuity stresses in shells. As an example, an edge and pressure loading is chosen to simulate the crotch region of two intersecting cylindrical shells. Finally, an example of a geometrically nonlinear problem is treated that exhibits the phenomenon of snap-through-by-creep instability of a spherical cap under external pressure.

A general conclusion has been reached that, unless the stress state is statically determinate, the advantage of the simpler algorithm, which keeps the stresses constant throughout the time interval, is far outweighed by the disadvantage of having to take much smaller time steps for adequate accuracy. In all cases, for the same accuracy, the computer time that was used by the simpler algorithm was much more than that used by the second algorithm.

## 1. Introduction

For high-temperature applications, the creep of a structure may become the deciding design parameter. When calculating creep in steps of time, it is common practice [1] to determine the creep strain rate at the beginning of the time step and keep it constant within the step. As shown in [1], this leads to a simple algorithm, in which the creep strain plays the same role as thermal strain in elastic analyses. The creep solution then consists of repetitive elastic solutions at points in time, with the creep strains accumulating at each step. This is one of the two algorithms considered in this paper, and is designated A1.

For a snap-through-by-creep problem [2], it was discovered that the A1 algorithm suffered from poor accuracy and instability, and a different algorithm, designated A2, had to be employed in order to solve the problem.

The objective of the present paper is to provide a basis for judging whether or not the A2 algorithm is preferable to A1 also for other creep problems. Three types of creep problems are considered: (1) Relaxation, where the stresses relax to zero in time; (2) Redistribution, in which the stresses first decrease and then reach a stationary state; and (3) Snap-through-by-creep, in which the creep strain rates increase without bound at some critical time. All examples are applied to thin, axisymmetric shells, because a computer program for creep of shells was available to the authors.

## 2. Algorithms

Both A1 and A2 have been described adequately in [1] and [2], respectively. Only the main features will be repeated here.

We consider the secondary creep range only and assume an exponential power law between equivalent creep strain rate and equivalent stress at a point. For this derivation, and the following examples, we assume a two-dimensional plane stress state that acts, for example, in an axisymmetric shell, subjected to axisymmetric loads. Then the creep strain rates (Finnie and Heller [3], page 172) in the meridional and circumferential directions of the shell are given by

$$\dot{\varepsilon}_{cx} = (\sigma_x - \frac{1}{2}\sigma_\theta)\left(\frac{U_o}{\sigma_o{}^n}\right)\sigma_{eq}^{n-1}$$
$$\dot{\varepsilon}_{c\theta} = (\sigma_\theta - \frac{1}{2}\sigma_x)\left(\frac{U_o}{\sigma_o{}^n}\right)\sigma_{eq}^{n-1} \qquad (1)$$

where $U_o$ is the reference creep strain rate, $\sigma_o$ is the reference stress, both of a one-dimensional creep test, $\sigma_x$ and $\sigma_\theta$ are the meridional and circumferential normal stress components, respectively, and the equivalent stress is given by

$$\sigma_{eq} = \sqrt{\sigma_x{}^2 + \sigma_\theta{}^2 - \sigma_x\sigma_\theta} \qquad (2)$$

The creep strains that are accumulated within a time interval $\Delta t$, from $t_1$ to $t_2$, are given by

$$\varepsilon_{cx_2} = \int_{t_1}^{t_2} \dot{\varepsilon}_{cx}\, dt \qquad \varepsilon_{c\theta 2} = \int_{t_1}^{t_2} \dot{\varepsilon}_{c\theta}\, dt \qquad (3)$$

The stresses at $t_1$ are known, and those at $t_2$ are among the unknown variables that will be determined from the solution of a boundary value problem at $t_2$. If the creep strain rates in the integrands of (3) are evaluated at $t_1$, and assumed to remain constant throughout the interval, then we have the A1 algorithm. This is actually the case for statically determinate problems.

To obtain the A2 algorithm, we expand the stresses in Taylor series about $t_1$ and get

$$\sigma_x = \sigma_{x1} + \left(\frac{\partial\sigma_x}{\partial t}\right)_1 (t - t_1) + \ldots$$

$$\sigma_\theta = \sigma_{\theta 1} + \left(\frac{\partial\sigma_\theta}{\partial t}\right)_1 (t - t_1) + \ldots \qquad (4)$$

Assuming the interval $\Delta t = t_2 - t_1$ to be sufficiently short, we keep only two terms on the right-hand side of (4). Next, we approximate the derivatives in (4) by finite differences and obtain

$$\sigma_x = \sigma_{x1} + \frac{\sigma_{x2} - \sigma_{x1}}{\Delta t} (t - t_1)$$

$$\sigma_\theta = \sigma_{\theta 1} + \frac{\sigma_{\theta 2} - \sigma_{\theta 1}}{\Delta t} (t - t_1) \qquad (5)$$

The integrands of (3), as given by (1) and (2), are nonlinear functions of the stresses. With the same argument that was used to go from (4) to (5), we expand the creep strain rates of (1) about $t_1$ and keep the linear terms only; thus, we get

$$\dot\varepsilon_{cx}(\sigma_x, \sigma_\theta) = \dot\varepsilon_{cx}(\sigma_{x1}, \sigma_{\theta 1}) + \left(\frac{\partial\dot\varepsilon_{cx}}{\partial\sigma_x}\right)_1 (\sigma_x - \sigma_{x1}) + \left(\frac{\partial\dot\varepsilon_{cx}}{\partial\sigma_\theta}\right)_1 (\sigma_\theta - \sigma_{\theta 1})$$

$$\dot\varepsilon_{c\theta}(\sigma_x, \sigma_\theta) = \dot\varepsilon_{c\theta}(\sigma_{x1}, \sigma_{\theta 1}) + \left(\frac{\partial\dot\varepsilon_{c\theta}}{\partial\sigma_x}\right)_1 (\sigma_x - \sigma_{x1}) + \left(\frac{\partial\dot\varepsilon_{c\theta}}{\partial\sigma_\theta}\right)_1 (\sigma_\theta - \sigma_{\theta 1}) \qquad (6)$$

The derivatives of $\dot\varepsilon_{cx}$ and $\dot\varepsilon_{c\theta}$, with respect to $\sigma_x$ and $\sigma_\theta$, are easily evaluated with the use of (1) and (2). We now replace the stresses in (6) by (5), substitute into (3) and carry out the integration with respect to time. The result for the accumulated creep strains within the interval $\Delta t$ is

$$\varepsilon_{cx} = \left(\frac{\partial\dot\varepsilon_{cx}}{\partial\sigma_x}\right)_1 \left(\frac{\sigma_{x2}}{2} - \sigma_{x1}\right) \Delta t + \left(\frac{\partial\dot\varepsilon_{cx}}{\partial\sigma_\theta}\right)_1 \left(\frac{\sigma_{\theta 2}}{2} - \sigma_{\theta 1}\right) + \dot\varepsilon_{cx}(\sigma_{x1}, \sigma_{\theta 1})$$

$$\varepsilon_{\theta x} = \left(\frac{\partial\dot\varepsilon_{c\theta}}{\partial\sigma_x}\right)_1 \left(\frac{\sigma_{x2}}{2} - \sigma_{x1}\right) \Delta t + \left(\frac{\partial\varepsilon_{c\theta}}{\partial\sigma_\theta}\right)_1 \left(\frac{\sigma_{\theta 2}}{2} - \sigma_{\theta 1}\right) + \dot\varepsilon_{c\theta}(\sigma_{x1}, \sigma_{\theta 1}) \qquad (7)$$

At the beginning of the interval, everything in (7) is known, except $\sigma_{X2}$ and $\sigma_{\theta 2}$. We recall now that the total strains at $t_2$ are given by

$$\varepsilon_{X2} = A_X \sigma_{X2} + A_{\theta X} \sigma_{\theta 2} + \varepsilon_{cX2}$$

$$\varepsilon_{\theta 2} = A_{\theta X} \sigma_{X2} + A_{\theta} \sigma_{\theta 2} + \varepsilon_{c\theta 2} \qquad (8)$$

Together with (7), (8) provide the new stress-strain law that includes creep. That is the A2 algorithm. When added to the equations of equilibrium and strain-displacement, the solution of the resulting boundary value problem provides the stresses, strains, and displacements at $t_2$.

3. Examples

They were chosen to represent typical creep problems. The KSHEL computer program was used to calculate the stresses and strains at a given value of final time. At first, the automatic step selection was adjusted to produce converged answers with the A2 algorithm. In the Relaxation example, the answers agreed with the exact solution. Then the step selection was adjusted for the A1 algorithm until acceptable accuracy was reached and the solution remained stable, and stresses and strains calculated up to the same final time. Often, with A1, the time steps had to be reduced because the solution became unstable. No instability with A2 was observed. The actual CPU computer time expended for each case on a CYBER 720 was recorded for each case and algorithm separately and is shown on the Figures. In all cases the creep properties were taken as $U_0 = 10^{-6}$ 1/sec and $\sigma_0 = 1000$ psi.

a. Relaxation of Thermal Stress

We consider a short cylindrical shell, inserted between rigid walls, with thickness=1, radius=10, length=2. Temperature is raised by 1000°F and held. There is only one axial stress acting which relaxes asymptotically to zero in time. This problem has an exact solution. The total axial strain rate is

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + U_0 \left(\frac{\sigma}{\sigma_0}\right)^n = 0$$

where $\sigma$ is the axial stress and E the elastic modulus of elasticity. $U_0$ and $\sigma_0$ have been already defined. This means that

$$\frac{d\sigma}{\sigma^n} = -\frac{EU_0}{\sigma_0^n} dt$$

Integration of both sides from t = 0 to some t gives a relation for $\sigma$ at any time t, in terms of $\sigma_i$, which is the stress at t = 0. For n = 1,

$$\sigma = \sigma_i \exp\left(-\frac{EU_0 t}{\sigma_0}\right)$$

while for n > 1,

$$\sigma = \sigma_i \, [1 + (n-1) \frac{E\sigma_i^{n-1} U_0 t}{\sigma_0^n}]^{\frac{1}{1-n}}$$

Figures 2 show the relaxation of stress in time for different creep exponents. The final times were selected for each n such that the stress relaxes to 20% of its initial value. The elastic properties were Modulus of Elasticity $E = 10^6$ psi, Poisson's Ration $\nu = .5$, and coefficient of thermal expansion $\alpha = 10^{-5}$ 1/F°.

b. Redistribution of Stress at Juncture

We consider a typical pressure vessel problem, with the geometry and forces as shown in Fig. 1. No exact creep solution is available, but the maximum stress is expected to decrease in time, until a stationary stress state is reached. The final time for each creep exponent was selected on the basis of the results given by the A2 algorithm, such that a stationary state had been reached. The results are shown in Figures 3. The elastic properties were: $E = 30 \times 10^6$ psi and $\nu = 0.3$.

c. Snap-through-by-Creep Problem

We consider a shallow spherical cap, with a simply supported edge, subjected to external uniform pressure. This shell has a limit load, at which it snaps through elastically. As stated in [2], if a lower than the elastic limit load is applied, it will snap through by creep, at some critical time. The results are shown in Figures 4. The critical time was signaled by a very large creep strain rate. The elasic properties were: $E= 10^6$ psi, $\nu = 0.3$. The radius is R = 5 in, thickness h = 1.0 in, and the cap angle is 10.0 degrees.

4. Results

In the Figures, the squares are for A2 and the circles for A1 algorithm. In Figure 1, the continuous curve shows the exact solution. In Figure 4, the dashed curve is drawn through the squares. The A1 algorithm failed to converge as the critical time (at about 10.8 sec) was approached. The decreasing of the step size did not avoid the nonconvergence. The CPU times shown in Figure 4 are not comparable, because A1 failed to produce the critical time.

5. Conclusions

The advantage of the A1 algorithm is that it does not require the recalculation of the shell stiffnesses at every time step, because the creep strains in (8) do not contain the stresses at $t_2$. The stiffnesses remain the same and are those of the elastic case. This means that if the structure stiffness matrix, or the homogeneous solutions, can be stored, then they need not be recalculated. The solution of the boundary value problem at every time step requires only the processing of the nonhomogeneous terms. The A1 algorithm is satisfactory for statically determinate cases, for which the stresses do not change.

As seen from (7), the A2 algorithm contains terms that multiply the stresses at $t_2$. This means that the stiffnesses change as the shell creeps, and they must be recalculated at every time step. Therefore, the A2 algorithm is bound to require more computer time to execute one step in time. Owing to the higher accuracy of the solution, it can afford to take larger time steps. The question that is raised, and answered, in this paper is whether the net computer time is more, or less, with the A2 algorithm. The number of steps taken and the computer time is shown on each Figure. For the type of creep problems considered in this paper, it appears that the A2 algorithm will cost less, in every situation. To be sure, the A1 algorithm will provide the stresses and strains at more points in time, and that sometimes may be useful. But for the problems that we have posed, that is, to find the stresses and strains at a given time, there is no doubt that the A2 algorithm will be cheaper.

## 6. References


[1] R. K. Penny and D. L. Marriott, Design for Creep, McGraw-Hill Book Co., London, 1974.

[2] A. Kalnins, D. P. Updike, and S. J. Yang, "Creep Analysis of Shells", Proceedings of the 6th Internations Conference on Structural Mechanics in Reactor Technology, Paper L-11/4, August 1981.

[3] I. Finnie and W. R. Heller, Creep in Engineering Materials, McGraw-Hill Book Co., New York, 1959.


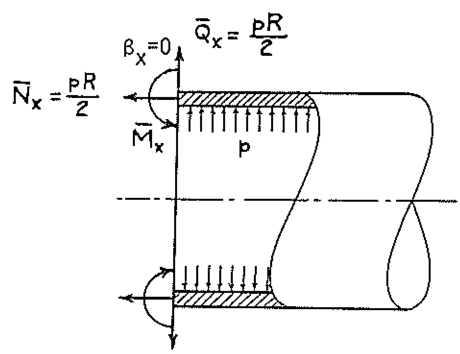


FIGURE 1. GEOMETRY FOR REDISTRIBUTION EXAMPLE.

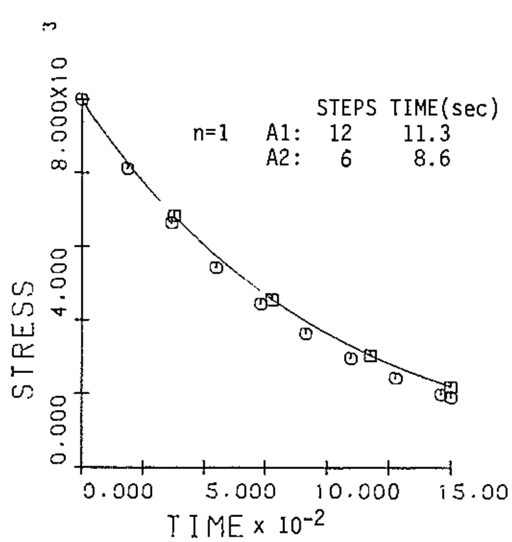


FIGURE 2a

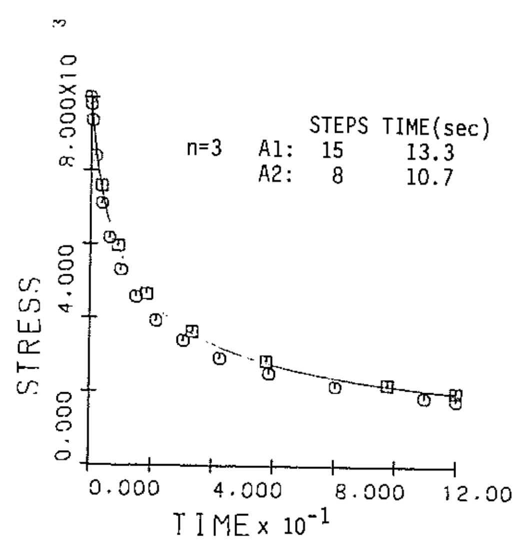


FIGURE 2b

AXIAL STRESS vs. TIME FOR RELAXATION EXAMPLE

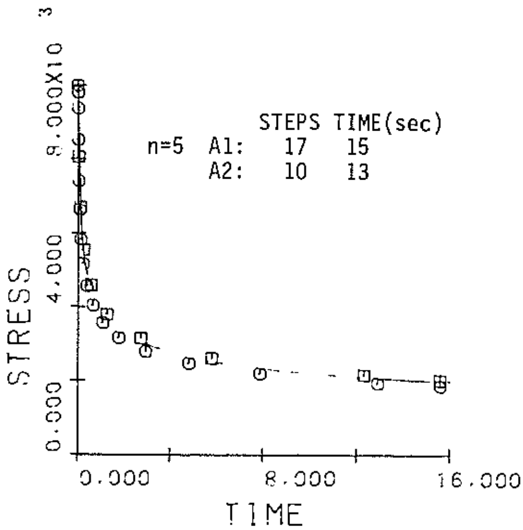


FIGURE 2c

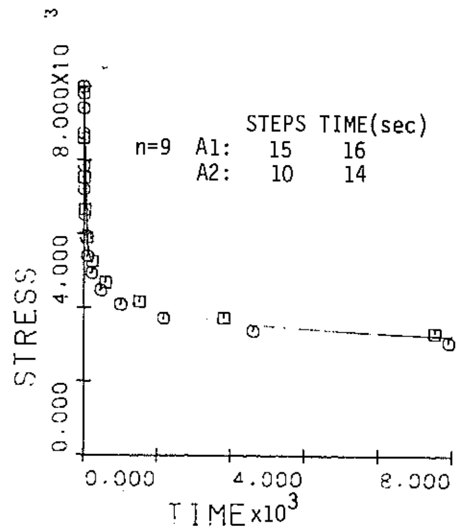


FIGURE 2d

AXIAL STRESS vs. TIME FOR RELAXATION EXAMPLE

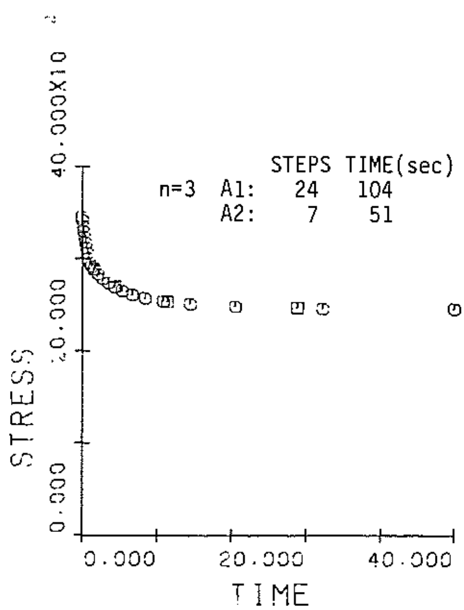


FIGURE 3a

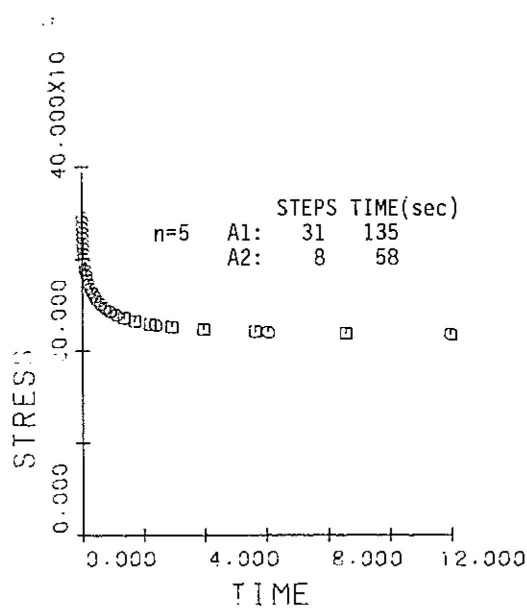


FIGURE 3b

EQUIVALENT STRESS vs. TIME FOR REDISTRIBUTION EXAMPLE

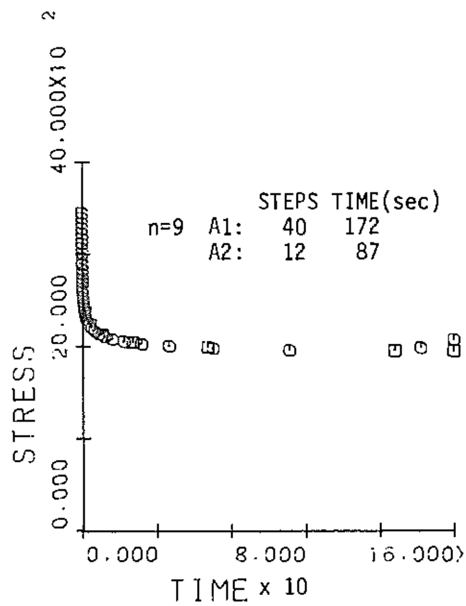


FIGURE 3c

EQUIVALENT STRESS vs. TIME FOR REDISTRIBUTION EXAMPLE

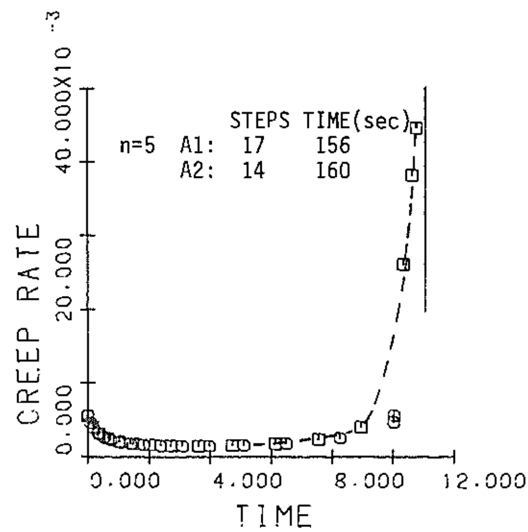


FIGURE 4

CREEP STRAIN RATE vs. TIME FOR SNAP-THROUGH BY CREEP EXAMPLE