A Technique for Mixing General Shell Elements with Axisymmetric Shell Elements in Linear and Nonlinear Analysis

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Summary

This paper considers a general technique of mixing flat or curved thin shell elements with axisymmetric shell elements for the purpose of evolving an economical analysis of axisymmetrical shells with intersections or other structural irregularities. In the vicinity of any irregularity such as joints or cut-outs, the usual sort of shell elements with discrete nodal parameters at nodal points have to be used by necessity. Away from this region, the axisymmetric element having nodal circles can be used as the most economical idealisation. The stiffness matrices in each region are then condensed, to be represented by the stiffness corresponding to the parameters at the interface circles. The discrete nodal parameters of the irregular region are then transformed by a simple Fourier analysis, to be matched to the parameters corresponding to the Fourier harmonics of the regular region. The stiffness matrix of the irregular region can therefore be transformed likewise and added to the stiffness matrix of the regular region, enabling the Fourier displacement parameters at the interface circles to be finally solved. In general, the resulting stiffness matrix couples all the Fourier harmonic components at all the interface circles. The technique is applicable for any combination of general and axisymmetrical shell element types. It is particularly convenient for the nonlinear analysis, since stress concentration and hence plastic and large geometric deformation is only likely to occur near the irregularities. Over the majority of the structure, the elastic stiffness will suffice and can be kept unchanged. The structure of the resulting stiffness matrix remains much the same as before, and, apart from the necessity of an incremental procedure, the analysis for each step is no different from the linear calculation.

Introduction

Although the development of the finite element method as an approximate numerical procedure for structural analysis has already achieved considerable success, active research is still being engaged, not only in its perfection in the linear and its extension to the nonlinear fields, but also in the reduction of the computing cost, especially for practical engineering application. Consider the particular problem of analysing pressure vessels. They are generally axisymmetric, and hence the use of the axisymmetric type of thin shell elements, such as those given in references [1 - 3] are the most cost effective. However, the regular axisymmetric configuration is often disturbed by holes, reinforcements, or by intersections with other structural components such as pipes, which are themselves also axisymmetric with respect to their own local axes system, but the combined structure is irregular. This type of structure is normally dealt with by the use of the flat or curved shell elements for modelling its behaviour, resulting in a massive increase in the complexity of the problem [4 - 6]. With this in mind we present here a method which allows the use of both types of elements in different regions of the structure. In the vicinity of the structural irregularity such as joints or cut-outs, the flat or curved types of element with discrete nodal parameters at nodal points are used by necessity. Away from this region, the axisymmetric elements having nodal circles with displacement parameters associating with Fourier harmonics can still be used. At the junction of these two systems of structural idealisation, the discrete parameters are matched to the harmonic parameters by a simple discrete Fourier analysis. Thus the structure is divided into a number of sub-structures using different types of element, and the solution procedure follows the usual routine, eventually solving for the coupled Fourier parameters at the junctions of the regular and irregular regions simultaneously after the discrete nodal parameters have been transformed.

Since stress concentration and hence plastic deformation is likely to occur only near the structural irregularities, the discrete shell elements used in this region are also more suitable for this purpose and the nonlinear incremental analysis can be performed conveniently with a small number of unknowns. Over the majority of the regular axisymmetric part, the behaviour is likely to remain elastic throughout, and one elastic analysis of the sub-structures will suffice. In this way not only the number of unknowns but also the amount of computation can be kept relatively small, and computing cost can be greatly reduced.

To illustrate the principle involved, we select the example given in reference [7], a nozzle joining a pressure vessel with their centre-lines intersecting at 90°. The axisymmetric element given in reference [1] and the shell element of reference [4] are used in the computer program, and details of the matching will be given.

2. Matching of Nodal Freedoms of Discrete Elements with Axisymmetric Elements

The structure consists mainly of a regular axisymmetric shell with intrusions such as pipes, nozzles, or holes with or without reinforcements. Any regular locally axisymmetric part can be idealised by axisymmetric finite elements where the displacements u, v and w in each element can all be expressed by Fourier Series such as

$$u = u_o(\eta) + \sum u_j^s(\eta) \sin j\theta + \sum u_j^c(\eta) \cos j\theta \tag{1}$$

where each coefficient of the harmonic series is a function of the meridional coordinate $\eta(=s/\ell)$ and can be interpolated in terms of the displacement parameters ρ_k at the nodal circles of the idealised structure. Typically for an element with displacements varying as a quintic function of η such as that given in [1], the parameters in $\rho_k = \{\ldots \rho_j \ldots\}_k$ where each subset associated with one particular sine or cosine harmonic (j) will contain

$$\rho_j = \{ u\ u'\ u''\ \ v\ v'\ v''\ \ w\ w'\ w'' \}_j \qquad (\ ' = \partial/\partial s\ ) \tag{2}$$

A stiffness matrix associating with each harmonic (j) corresponding to these displacement parameters can then be constructed in accordance with the shell theory, the details of which have been given elsewhere [1] .

When the regularity of the axisymmetric structure is disturbed by connections or cut-outs, this treatment is no longer possible and the region nearby will have to be idealised by the usual flat or curved shell elements. Depending on the actual element employed in the analysis, the behaviour of the element is usually expressed in terms of the displacements and rotations and perhaps their higher order derivatives at the nodes. For the element given in reference [4], for example, the displacement parameter of the i^{th} node consists of typically:

$$\rho_i = \{ u\ v\ w\ \phi\ \psi \}_i \tag{3}$$

where $\phi = \partial u/r\partial\theta - v/r$ and $\psi = \partial u/\partial s$ are the rotations.

Taking each of the regular and the irregular regions as a sub-structure, all the internal parameters in each region can be expressed in terms of the parameters at the interface circles between the regions by the usual condensation procedure. The problem is then reduced to matching the nodal parameters at the discrete nodes of the irregular region with the displacement parameters associating with the Fourier harmonics of the regular region at the interfaces. The exact details of course depend on the parameters involved, but in general the displacements and rotations at the discrete nodes can always be considered to be the discrete values at those points derived from a Fourier expansion. Hence we may write

$$\rho_D = T\rho_F \tag{4}$$

where $\rho_D = \{ \rho_1 \ldots \rho_i \ldots \rho_n \}$ is a collection of all the parameters at the n discrete nodes at the interface circle of the irregular region, and $\rho_F = \{ \rho_0\ \rho_1 \ldots \rho_j \ldots \}$ is a collection of the parameters associating with all the Fourier harmonics at the interface circle of the regular region. The transformation matrix $T = [T_{ij}]$ will depend on the nature of the parameters presented in the two sets, but the chief component of T_{ij} which transforms the displacement parameters of the j^{th} harmonic into discrete displacement or rotation at the i^{th} node, will be the values of the harmonic function at that node. The stiffness matrix K_D and the equivalent load vector R_D associated with these displacement parameters are transformed to correspond with the Fourier displacement parameters by the congruent transformation

$$\left.\begin{aligned} K_F &= T^t K_D T \\ R_F &= T^t R_D \end{aligned}\right\} \tag{5}$$

K_F contains the stiffness matrices corresponding to the displacement parameters of all the Fourier harmonics. In general, they are all coupled and the resulting set of equations will have to be solved simultaneously, instead of one harmonic at a time as in the usual linear elastic calculation of the harmonic components. The number of equations involved is $M \times N \times P$, where M is the number of displacement parameters on a nodal circle of the axisymmetric element (= 9 for the element in reference [1]), N is the number of Fourier harmonics to describe the displacement variation (normally anything between 5 to 10 will be sufficient), and P is the number of interface circles between the regions. This number is therefore of the order of hundreds at most for the usual engineering problems. However, the matrix is fully polulated.

3. Example

The T-junction under uniform pressure, taken from reference [7] where analytical and experimental results are given, is here used as an example of matching the axisymmetrical shell elements of reference [1] with the cylindrical shell elements of reference [4]. The structure has 2 planes of symmetry so that only a quarter of the junction region needs to be considered and there are 2 interface circles (Fig. 1).

Since the deformations will be symmetric, the displacements in the axisymmetric finite elements are given by

$$\begin{aligned} &\text{radial} & u &= u_o + \sum u_j \cos j\theta \\ &\text{circumferential} & v &= \sum v_j \sin j\theta \\ &\text{axial} & w &= w_o + \sum w_j \cos j\theta \end{aligned} \qquad (6)$$

and the displacement vector at the interface circle associating with each harmonic has 9 parameters as given in eq.(2). This vector is related to its corresponding force vector R_j by an elastic stiffness K_j from the regular axisymmetric structure, which is uncoupled with any other harmonics.

The element used for the idealisation of the junction region given in reference [4] has at each node 5 parameters as given in eq.(3). In this vector ψ is physically equivalent to u'. Notice that there is only one meridional gradient u' involved, so that a straightforward matching of this set of nodal parameters around the interface circle with the Fourier parameters will give no stiffness contribution from the junction region to the parameters v', w', u'', v'', and w''. To improve on this rather crude result, we may make use of the stiffness associated with displacements at the nodes one element length away from the interface and use the finite difference interpolation to create artificially the nodal displacement vector

$$\bar{\rho}_i = \{ u \ v \ w \ \phi \ \psi \ v' \ w' \ \psi' \}_i \qquad (7)$$

where $v' = (v - v_{-1})/\ell$ (8)

$$w' = (w - w_{-1})/\ell \qquad (9)$$

$$\psi' = (\psi - \psi_{-1})/\ell \simeq u'' \qquad (10)$$

The suffix (-1) denotes nodal values at a meridional length ℓ from the interface. Second order derivative v'' and w'' may also be created by a higher order finite difference scheme but it is thought to be unnecessary in this particular case.

The stiffness coefficients associated with the displacements

$$\rho_{i,i-1} = \{ u\ v\ w\ \phi\ \psi\ v_{-1}\ w_{-1}\ \psi_{-1} \}_i \tag{11}$$

can now be transformed to be associated with $\bar{\rho}_i$ of eq.(7) by the relations given in eqs. (8 - 10).

The parameters in the expanded displacement vector $\bar{\rho}_i$ of eq.(7) at node i, positioned at angle θ_i on the circumference of the interface circle, can easily be expressed in terms of the harmonic parameters ρ_j of eq.(2) at the interface circle by the use of the displacement interpolation expressions in eq.(6), and their gradients as

$$\bar{\bar{\rho}}_i = \sum T_{ij}\, \rho_j \tag{12}$$

where

$$T_{ij} = \begin{array}{c} u \\ v \\ w \\ \phi \\ \psi \\ v' \\ w' \\ \psi' \end{array} \overset{\begin{array}{ccccccccc} u & u' & u'' & v & v' & v'' & w & w' & w'' \end{array}}{\begin{bmatrix} \cos j\theta_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin j\theta_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos j\theta_i & 0 & 0 \\ -j\sin j\theta_i/r & 0 & 0 & -\sin j\theta_i/r & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos j\theta_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin j\theta_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos j\theta_i & 0 \\ 0 & 0 & \cos j\theta_i & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \tag{13}$$

Therefore, the expanded displacement vectors for the nodes on the two interface circles I and II may be collected together and grouped into:

$$\rho_D = \{ \bar{\rho}_1\ \bar{\rho}_2 \ldots\ \bar{\rho}_i \ldots \} = \{ \rho_I\ \rho_{II} \}_D \tag{14}$$

and expressed in terms of the Fourier displacement vectors on the interface circles

$$\rho_F = \{ \rho_{FI}\ \ \rho_{FII} \} = \left\{ \{\rho_j\}_I\ \{\rho_j\}_{II} \right\} \tag{15}$$

by the transformation

$$\rho_D = \begin{bmatrix} T_I & T_{II} \end{bmatrix} \begin{bmatrix} \rho_{FI} \\ \rho_{FII} \end{bmatrix} = T\, \rho_F \tag{16}$$

The associated stiffness and load matrices are then transformed according to eq.(5), giving

$$\left. \begin{aligned} K_F &= \begin{bmatrix} T_I^t K_D T_I & T_I^t K_D T_{II} \\ T_{II}^t K_D T_I & T_{II}^t K_D T_{II} \end{bmatrix} \\ R_F &= \begin{Bmatrix} T_I^t R_D \\ T_{II}^t R_D \end{Bmatrix} \end{aligned} \right\} \tag{17}$$

corresponding to Fourier harmonics of the two interface circles. Each of the four submatrices in K_F contains further submatrices such as

$$K_{k\ell} = T_{ik}^{t} K_D T_{i\ell} \tag{18}$$

which links the displacements of the ℓ^{th} harmonic to the forces of the k^{th} harmonic. In general, the harmonics on each interface circle are not only coupled to each other, but also to those on the other interface circles as well. The computational task to form the matrix K_F in eq.(17) appears to be formidable on the surface; but in fact not even the submatrices T_{ij} of eq.(13) need to be stored because of its sparseness. All the submatrices $K_{k\ell}$ can be computed by a simple subroutine and put into the correct position in K_F once the stiffness property of the discrete region has been reduced to be represented by K_D at the nodes on the interface circles. After that the stiffness matrices of the regular structure are also condensed to be represented by the modified stiffness corresponding to the Fourier harmonics at the interfaces and added to K_F , from which the Fourier displacement parameter can be solved.

Although the chief advantage of the Fourier representations of circumferential behaviour, namely that each harmonic may be solved independently in a linear analysis, appears to have been forfeited, the scheme nevertheless has the merit of reducing the number of unknowns in the regular axisymmetric region. Furthermore, when plastic deformation eventually takes place and/or geometric change becomes important, it is only necessary to modify the stiffness properties of the affected discrete elements, with no further complication to the procedure. Each step of an incremental nonlinear analysis proceeds in exactly the same way as the linear one presented here. Wherever the property remains elastic, the stiffness matrix can be retained and no recalculation is necessary, and the computational effort can be greatly economised.

4. Numerical Results

The T-junction (Fig. 1) taken from reference [7], has been analysed incrementally for 4 steps from a uniform internal pressure of 85 psi, when yielding takes place, with an increment of 10 psi per step. Peak stress occurs along the intersection between the vessel and the nozzle, so that it only needs to consider elements close to that region for plastic yielding and large deformation. The relevant element is divided into two over the thickness, and synthesized at 8 Gaussian points within each half. An example of the stress output is shown in Fig. 2. In these calculations, it was found that the geometric effect is minor (making a maximum difference of 4% on the equivalent stress) and may be ignored.

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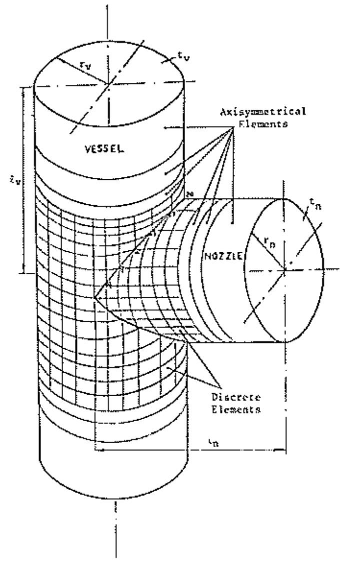

FIG. 1. GEOMETRY OF T-JUNCTION

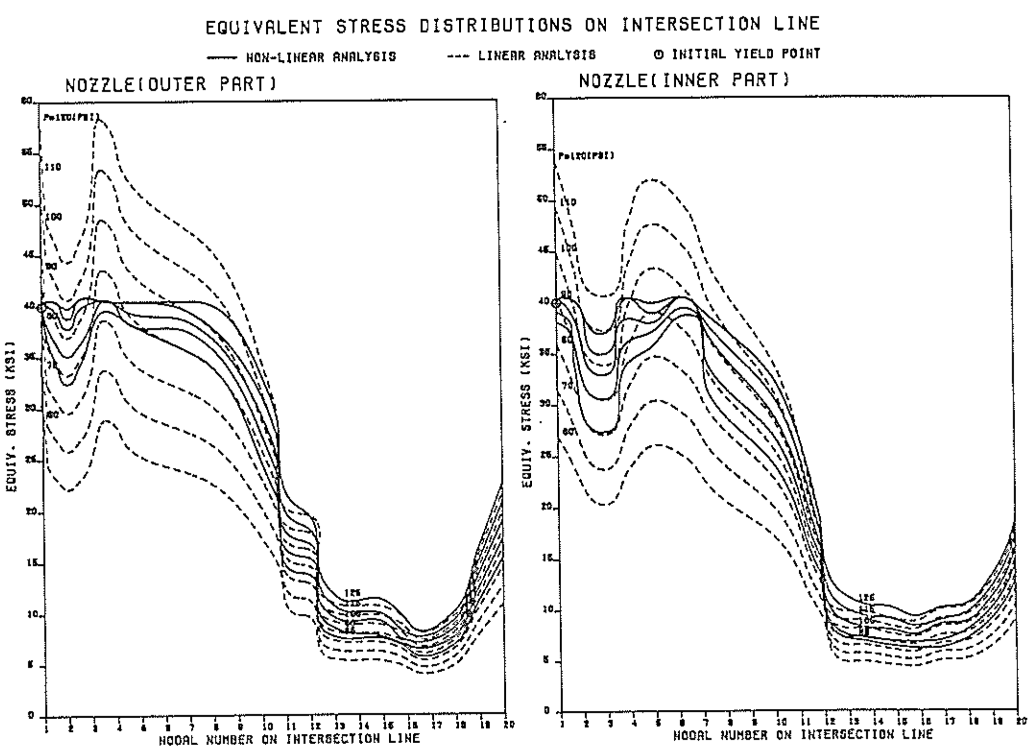

FIG. 2. EQUIVALENT STRESS DISTRIBUTIONS ON INTERSECTION LINE