A Simplified Method for the Inelastic Analysis of Structures Under Non Radial Cyclic Loadings

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A B S T R A C T

This paper discusses the present status of the simplified method which was proposed a few years ago by J. ZARKA and J. CASIER to obtain a straightforward evaluation of the total inelastic strains which accumulate in a structure under cyclic loadings when a linear kinematic hardening behaviour is assumed for the materials involved.

An iterative procedure has been developped within the past year to handle plastic shakedown under non radial anisothermal cyclic loadings.

The theoretical basis of this iterative procedure is presented. An application example is then given which highlights the efficiency of the method.

1. Introduction

An inelastic analysis of the structure response to cyclic loadings proves necessary in solving many industrial problems such as those which are encountered in the design of nuclear components.

The usual approach is to perform incremental computations over a large number of cycles. These expensive computations are no longer necessary : The simplified iterative method which is proposed hereunder gives (through elastic computations) a straightforward, accurate, evaluation of the limit state of the structure at each instant in the load cycle.

In a previous paper |1| we described this iterative procedure in the case of radial cyclic loadings. We now present how it has been extended to non-radial cyclic loadings. By way of illustration the whole method is then applied to a simple example which exhibits non radial plastic shakedown.

2. Principle of the Method

This method applies to linear kinematic hardening materials or to multilayer materials (since each layer behaves as a linear kinematic hardening material).

The yield stress is considered as a temperature dependent quantity Sy (θ) but the YOUNG's modulus and the tangent modulus are assumed to be constant within the temperature range involved.

The principle of the method is as follows :

- from the actual hardening parameter α ($\alpha = C\ \epsilon_p$) we derive a modified hardening parameter which we denote by $\hat{\alpha}$

- in the space of the modified hardening parameters (actually the space of the deviatoric stresses) simple algorithms, based on the elastic response of the structure, enable us to reach an estimate of the limit state of $\hat{\alpha}$ i.e. $\hat{\alpha}_{lim}$

- we then come back to the actual hardening parameters to obtain the limit value for α i.e. α_{lim} and thus for ϵ_{plim} ($= C^{-1}\ \alpha_{lim}$).

3. The Theoretical Bases of the Proposed Method

The proposed method is applied in four main successive steps.

3.1 Step One : Elastic Analysis

We first of all compute the structure response assuming pure elastic behaviour, denoted by $\sigma^{el}(t)$ and $s^{el}(t)$ the resulting stress tensor and its deviatoric part.

We then consider for the rest of the procedure what is going on in the space of the deviatoric stresses.

3.2 Step Two : A Piecewise Linear Elastic Response

The intermediate states of the elastic response of the structures are suppressed : we only keep the load points which correspond to the extremities of the piecewise linear elastic response of the structure.

For the sake of simplicity we shall only consider below the case where these extremities are reached at the same times simultaneously for all points in the structure.

3.3 Step Three : Evaluation of the modified hardening parameters $\hat{\alpha}_{lim}$ at every point of the load path

The response of the structure for inelastic analysis would be

. the stress tensors $\sigma(t)$

. the plastic strain tensors $\epsilon_p(t)$

. the tensors of the hardening parameters $\alpha(t)$

One of the key points of the method consists in writing the actual stress tensor $\sigma(t)$ as the sum of two terms : $\sigma^{el}(t)$ and a complementary term called the residual stress which is denoted by ρ

$$\sigma(t) = \sigma^{el}(t) + \rho(t) \qquad (1)$$

In the space of the deviatoric stresses this equation is written :

$$s(t) = s^{el}(t) + \text{dev}\ \rho(t) \qquad (2)$$

where devρ denotes the deviatoric part of the residual stress tensor.

Let us now introduce the modified hardening parameter $\hat{\alpha}$ which we define as follows:

$$\hat{\alpha}\ (t) = \alpha(t) - \text{dev}\ \rho(t) \qquad (3)$$

This new tensor $\hat{\alpha}$ has no physical significance but it enables us to express the yield criterion

$$f(s-\alpha) - S_y \leqslant 0 \qquad (4)$$

in a new form which is based on the elastically computed deviatoric stresses.

$$f\ (s^{el} - \hat{\alpha}) - S_y \leqslant 0 \qquad (5)$$

In the above equation (4) and (5) S_y stands for the yield stress and f for the VON MISES effective stress which is proportional to the euclidian distance in the space of the deviatoric stresses :

$$f\ (s) = \sqrt{\frac{3}{2}\ sij\ sij} \qquad (5)$$

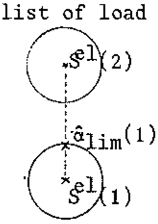
We have now to build the limit values of $\hat{\alpha}(i)$ at all extremities i of the load segments, according to the respective positions of the spheres the centers of which are $s^{el}(i)$ and the radii of which are $S_y(i)$. This work is performed through the following iterative procedure.

3.3.1 <u>Sub Step One : Initiation of the Algorithm</u>

We start by calculating the scalar quantities

$$\gamma(i) = \frac{f\ (s^{el}(i+1) - s^{el}(i))}{S_y\ (i+1) + S_y(i)} \qquad (6)$$

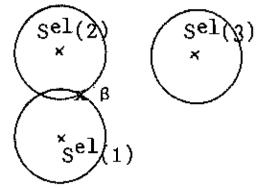
We choose, as the initial point of the cycle, load point j corresponding to the maximum value of $\gamma(i)$ on the whole structure. The list of load points is then renumbered with point j as point 1



<u>Case 1</u> : On the elements for which $\gamma(1)$ is greater than unity we assume that $\hat{\alpha}_{lim}(1)$ is equal to

$$\hat{\alpha}_{lim}(1) = s^{el}(1) + S_y(1)\left[s^{el}(2) - s^{el}(1)\right] \qquad (7)$$

Case 2 : On the elements for which $\gamma(1)$ is lower than unity we assume that $\hat{\alpha}_{lim}(1)$ is equal to β where β is defined by

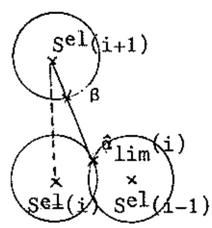


$$\left|\begin{array}{l} \beta \;=\; s^{el}(1) + \lambda\,(S^{el}(2) - s^{el}(1)) + \mu\,(s^{el}(3) - s^{el}(2)) \\ \lambda > o \\ \mu > o \\ f(\beta - s^{el}(1)) \;=\; S_y(1) \\ f(\beta - s^{el}(2)) \;=\; S_y(2) \end{array}\right. \qquad (8)$$

3.3.2 Sub Step Two : Change in the modified hardening Parameter $\hat{\alpha}_{lim}$ on the Load Segment from i to i+1

We have to consider three cases depending on the respective positions of the yield spheres the center of which are $s^{el}(i)$ and $s^{el}(i+1)$ and on the position of $\hat{\alpha}_{lim}(i)$

Case 1 : These spheres have no common area : There will necessarily be yielding from i to i+1. we locally build the point β such that :



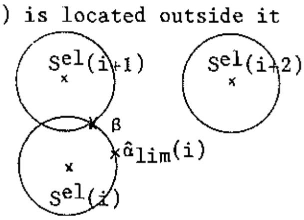
$$\beta \;=\; S^{el}(i+1) + S_y(i+1)\frac{[\hat{\alpha}_{lim}(i) - S^{el}(i+1)]}{f[\hat{\alpha}_{lim}(i) - S^{el}(i+1)]} \qquad (9)$$

The limit value $\hat{\alpha}_{lim}(i+1)$ is assumed to be equal to β and thus the change in $\hat{\alpha}$ from i to i+1 is known

$$\Delta\hat{\alpha} \;=\; \hat{\alpha}_{lim}(i+1) - \hat{\alpha}_{lim}(i) \qquad (10)$$

Case 2 : These spheres have a common area and $\hat{\alpha}_{lim}(i)$ is located outside it

There will also be yielding from i to i+1 We locally build the point β according to equations (8) and defined $\hat{\alpha}_{lim}(i+1)$ equal to β



There again the change in $\hat{\alpha}$, from i to i+1, is known (equation (10)).

Case 3 : These spheres have a common area and $\hat{\alpha}_{lim}(i)$ is located inside it. We then assume that there may be no yielding on that point along the load path from i to i+1: $\Delta\varepsilon_p = o$ (But there will be a change in $\hat{\alpha}$ due to the other parts of the structure, change which is to be determined).

3.3.3 Sub Step Three : Change in the residual stress field ρ on the load segment from i to i+1

We have to consider one the one hand the elements for which yielding will necessarily occur and on the other hand the elements for which there may be no yielding between i and i+1.

Case 1 : Elements for which yielding will necessarily occur

The response of the structure at load point i, according to elastic and inelastic analyses are :

. for elastic analysis

$$\begin{cases} \sigma^{el}(i) & \text{. Statically admissible (S.A.) with force data } f(t) \\ \varepsilon^{el}(i) = M\,\sigma^{el}(i) & \text{. Kinematically admissible (K.A.) with displacement data } u(t) \end{cases} \quad (11)$$

. for inelastic analysis

$$\begin{cases} \sigma(i) & \text{S.A. with forces data } f(t) \\ \varepsilon(i) = M\,\sigma(i) + \varepsilon_p(i) & \text{K.A. with displacement data } u(t) \end{cases} \quad (12)$$

. the mere difference of those two states of the structure leads to :

$$\begin{cases} \rho(i) & \text{S.A. with nul applied forces} \\ M\rho(i) + \varepsilon_p(i) & \text{K.A. with nul applied displacements} \end{cases} \quad (13)$$

But $\varepsilon_p(i)$ may be expressed in a different way as

$$\varepsilon_p(i) = C^{-1}\,\alpha(i) = C^{-1}\,(\hat{\alpha}_{lim}(i) + \text{dev}\,\rho(i)) \quad (14)$$

The residual stress field $\rho(i)$ in the structure is thus defined by

$$\begin{cases} \rho(i) & \text{S.A. with nul applied forces} \\ (M+C^{-1}\,\text{dev})\,\rho(i) + C^{-1}\,\hat{\alpha}_{lim}(i) & \\ \quad \text{K.A. with nul applied displacements} \end{cases} \quad (15)$$

We, of course, obtains the same equations at the load point i+1 and thus the (15) change in the residual stress field along the load path from i to i+1 is defined by

$$\begin{cases} \Delta\rho & \text{S.A. with nul applied forces} \\ (M+C^{-1}\,\text{dev})\,\Delta\rho + C^{-1}\,\Delta\hat{\alpha} & \text{K.A. with nul applied displacements} \end{cases} \quad (16)$$

It can easily be seen that the operator $(M + C^{-1}\,\text{dev})$ is derived from the HOOKE's matrix M by replacing the YOUNG's modulus E and the POISSON's ratio ν by two modified quantities denoted by E' and ν', which are defined as follows :

$$E' = \frac{Eh}{E+h} \quad (17)$$

$$\nu' = \frac{Eh}{E+h}\left(\frac{\nu}{E} + \frac{1}{2h}\right) \quad (18)$$

where h is the hardening modulus

$$h = \frac{E\,E_T}{E - E_T} \quad (19)$$

Case 2 : elements for which there may be no yielding between i and i+1.
In the same way we obtain that the change in the residual stress field between i and i+1 will meet, on these elements, the following conditions :

$$\begin{cases} \Delta\rho & \text{S.A. with nul applied forces} \\ M\Delta\rho & \text{K.A. with nul applied displacements} \end{cases} \tag{20}$$

It ensues from above, as a conclusion to this sub step three, that the whole field of the change in the residual stresses between i and i+1 will be obtain by a structural elastic analysis performed :

. with forces data and displacements data set to zero

. with initial strains set :

. to $C^{-1}\ \Delta\hat{\alpha}$ on the elements for which yielding will necessarily occur

. to zero on the elements for which there may be no yielding between i and i+1

. with a stiffness matrix built up

. with the modified elastic properties E' (equ 17) and ν' (equ 18) on the elements for which yielding will necessarily occur

. with the actual elastic properties E and ν on the other elements

3.3.4 Sub Step Four : Sub Iteration Test

We now have to check that the hypothesis ($\Delta\varepsilon_p$) we assumed on the elements concerned was correct.

The output of the previous structural (elastic) analysis was the field $\Delta\rho$.

We now consider for the elements which are concerned the tensor X

$$X = -\operatorname{dev} \Delta\rho \tag{21}$$

and build the stress point β

$$\beta = \hat{\alpha}_{lim}(i) + X \tag{22}$$

If $f(\beta - S^{el}(i+1))$ is lower than S_y (i+1) our hypothesis was right and : $\hat{\alpha}_{lim}$ (i+1) is set equal to β.

If this last condition is not met the corresponding hypothesis was wrong : There will necessarily be yielding on the faulted elements between i and i+1.

Let us introduce :

. the stress tensor $t = \dfrac{X}{f\ (X)}$ (23)

. the stress tensor u

$$\begin{cases} u = at + b\,(S^{el}(i) - S^{el}(i+1)\,) \\ f\ (u) = 1 \\ t : u = o \end{cases} \tag{24}$$

. the scalar quantity $k_{max} = \dfrac{Sy\ (i)}{Sy(i) + Sy(i+1)}$ (25)

We locally build the limit value $\hat{\alpha}_{lim}$ (i+1) on these elements as follows : Two cases are to be considered :

Case 1 : If $\hat{\alpha}_{lim}$ (i) is inside the yield sphere the center of which is $S^{el}(i+2)$ we look for the maximum value of λ such that β :

$$\beta \;=\; \hat{\alpha}_{lim}(i) \;+\; \lambda\, t \;+\; \mu(\lambda)u \tag{26}$$

meets $$f(\beta - S^{el}(i+1)) = Sy\ (i+1) \tag{27}$$

and $$f(\beta - S^{el}(i+2) \leqslant Sy\ (i+2) \tag{28}$$

Case 2 : If $\hat{\alpha}_{lim}$ (i) is outside this previous sphere we look for the maximum value of λ such that β (equ. 26) only meets equation (27).
In both cases we then consider the value λo

$$\lambda o \;=\; \text{Min}\ (\lambda_{max},\ k_{max}) \tag{28}$$

and locally build $\hat{\alpha}_{lim}$ (i+1) :

$$\hat{\alpha}_{lim}\ (i+1) \;=\; \hat{\alpha}_{lim}(i) \;+\; \lambda o(t) + \mu(\lambda o)u \tag{29}$$

Thus on the faulted elements, the yielding between i and i+1 will occur with a change in $\hat{\alpha}$ equal to

$$\Delta\hat{\alpha} \;=\; \lambda o\ t \;+\; \mu(\lambda o)u \tag{30}$$

The procedure must then be rerun from sub step three until the hypothesis $\Delta\epsilon_p$ is proved to be correct on all the elements involved.
When this is achieved we have obtained the whole field $\hat{\alpha}_{lim}$ (i+1) and we may proceed to the next load point i.e. : (i+2)

When the whole loading cycle has been described we have a full set of $\hat{\alpha}_{lim}\,(i)_1$ for the first iteration step.
Subsequent iterations are run until we reach (usually with no more than 2 iterations) stabilized results everywhere in the structure and at any point i of the load path.
Which means :

$$\hat{\alpha}_{lim}\,(i)_j \;-\; \hat{\alpha}_{lim}\,(i)_{j-1} \;<\; C \tag{31}$$

Where C is a convergence criteria.
This step of the procedure is now completed : we know the limit value of $\hat{\alpha}$ at every point i of the load path

$$\hat{\alpha}_{lim}\,(i) \;=\; \hat{\alpha}_{lim}\,(i)_j$$

3.4 Step Four : Way Back to the Plastic Strains

The change in ϵ_p ($\Delta\epsilon_p$) between load point i and i+1 is directly computed through a structural analysis performed.

. with force data and displacement data set to zero

. with initial strains set everywhere in the structure to $C^{-1}\ \Delta\hat{\alpha}$ (now known everywhere)

. with a stiffness matrix built up with modified elastic properties E' and ν'

This structural analysis gives the change in the residual stress field $\Delta\rho$.

We then calculate locally

$$\Delta\alpha \;=\; \Delta\hat{\alpha} \;+\; \text{dev}\ \Delta\rho \tag{32}$$

and $$\Delta\epsilon_p \;=\; C^{-1}\ \Delta\alpha$$

For the first load point $\Delta\hat{\alpha}$ is set to $\Delta\hat{\alpha} \;=\; \hat{\alpha}_{lim}(1) - 0 \;=\; \hat{\alpha}_{lim}(1)$

and then $\epsilon plim(i) \;=\; \sum_{k=1}^{i} \Delta\epsilon_p$

4. An Application to a Simple Example

We consider hereunder a thin tube loaded with out-of-phase-alternating tensile and torque in order to have a non radial loading (fig. 1).

The numerical analyses of this problem were made with two 2D plane stress finite elements connected by their four common nodes.

The boundary conditions of this simple structure are given on figure 2. Different linear kinematic hardening behaviours were assumed on these 2 elements (fig. 3).

The elastic responses of both of them in the space of the deviatoric stresses s_{xx} and s_{xy} look like rectangles (dot and dashed lines on figures 4 and 5).

The proposed procedure was applied and led to the limit values of the plastic axial and shear strains at the four points of the loading path which are given in tables 1 and 2 respectively for the first and the second element.

In order to check these results we performed a full detailed inelastic analysis of the structure over a large number of cycles. Figures 4 and 5 show (dashed lines) the evolution of the modified hardening parameters ($\hat{\alpha}_{xx}$ and $\hat{\alpha}_{xy}$) of the first and second elements. The limit values of the plastic strains on both elements, according to this incremental analysis are reported in tables 1 and 2.

Let us consider for instance the limit value of the plastic axial strain of the first element at the third point of the loading path.

The proposed method gives $\epsilon p_{xx} = 88{,}43\ 10^{-4}$

and thê inèlastic analysis gives $\epsilon p_{xx} = 89{,}93\ 10^{-4}$

The agreement between both evaluations (simplified method and incremental analysis) is fairly good.

5. Conclusion

An iterative elastic method was proposed which enable us to obtain, without any incremental inelastic analysis, the limit values of the plastic strains which accumulate in a structure under non radial cyclic loading when plastic shakedown is to occur.

This method proved to be very efficient.

6. Acknowledgment

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7. References

[1] TRIBOUT J. , INGLEBERT G. and CASIER J.
"A simplified Method for the inelastic analysis of structures under cyclic loadings" PVP Conference ORLANDO, July 1982 (PVP VOlume 59)

TABLE I : Plastic Strains on the First Element

	axial strains 10^{-4}				shear strains 10^{-4}			
Load path	Ext 1	Ext 2	Ext 3	Ext 4	Ext 1	Ext 2	Ext 3	Ext 4
Limit value inelastic analysis	100,55	71,43	59,93	119,05	-31,41	13,92	31,41	-13,92
Limit value simplified method	95,55	72,25	88,43	114,73	-31,24	14,78	31,24	-14,78

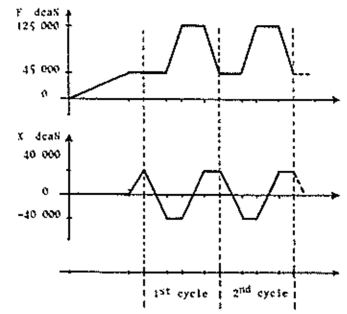

FIGURE 1 LOAD HISTOGRAMME

TABLE 2 : Plastic Strains on the Second Element

	axial strains 10^{-4}				shear strains 10^{-4}			
Load path	Ext 1	Ext 2	Ext 3	Ext 4	Ext 1	Ext 2	Ext 3	Ext 4
Limit value inelastic analysis	84,02	59,80	70,40	95,01	-23,64	8,41	23,64	- 8,41
Limit value simplified method	84,38	62,49	71,02	93,07	-23,18	9,30	23,18	- 9,30

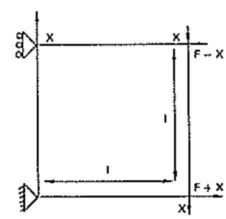

FIGURE 2 LOADING AND BOUNDARY CONDITIONS

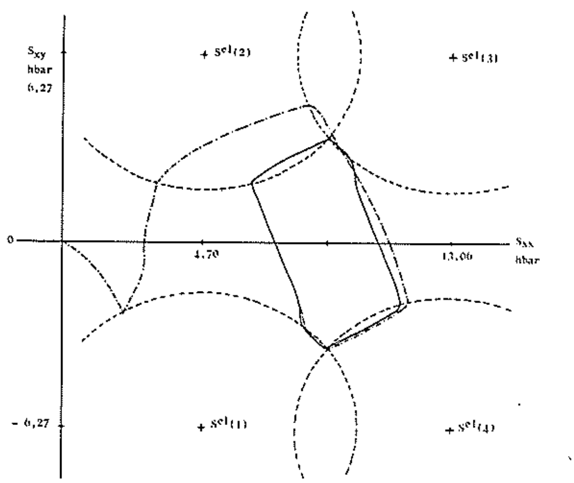

FIGURE 4 PATH FOLLOWED BY $\hat{\sigma}$ ON THE FIRST LAYER ACCORDING TO THE INELASTIC ANALYSIS

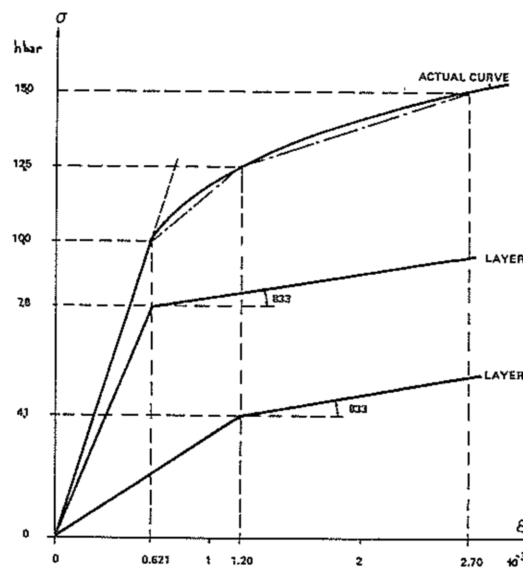

FIGURE 3 STRESS-STRAIN CURVES

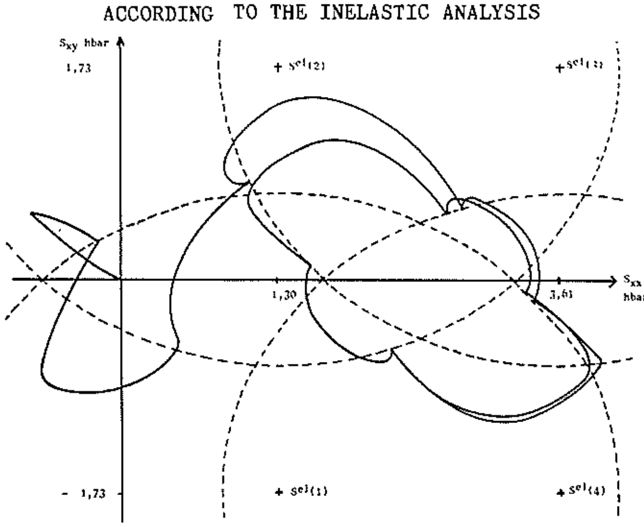

FIGURE 5 PATH FOLLOWED BY $\hat{\sigma}$ ON THE SECOND LAYER ACCORDING TO THE INELASTIC ANALYSIS