Finite Element Analysis of Elastoplastic Contact Problems Without Friction

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Contact between two bodies often results in very high local stress concentrations, which causes local plastic deformation and permanent deformations after unloading. This permanent deformation can be at high importance for the functionality of a construction, e g static indentation in bearing races.

In this paper a method for analysis of elastoplastic bodies in contact with the finite element method will be presented. The procedure is based on the general finite element system ASKA, which permits access to all available facilities, e g a highly improved substructure technique. This permits that only the interesting areas will be handled as elastoplastic, which reduces the computer time and the amount of data on mass storage.

The analysis is separated into two different parts.

(i) Calculation of the yield load. To obtain the magnitude of applied external forces and the stress state at incipient yield for the following elastoplastic analysis an iterative procedure is needed due to the nonlinearity introduced from the contact conditions.

(i) Elastoplastic analysis. This part of the analysis is performed in a combined incremental and iterative algorithm calculating the load increments corresponding with the next discrete change of the contact surface automatically.

The paper includes presentation of results from yield load and elastoplastic analysis of a cylinder in contact with an infinite half plane and a smooth circular disc in an infinite plate with hole.

1. Introduction

The finite element method has become a common analysis procedure for the contact analysis of structures. The main part of the published material has reference to elastostatic contact problems with or without friction, [1], [2], [3] and [4]. Analysis of static or quasistatic elastoplastic contact problems seems not to be as well established. The main part of the work published deals with contact between rigid and deformable bodies [5], [6], [7]. In this paper an algorithm for analysis of two or several elastoplastic bodies in contact will be presented.

It is well established that unique relations do not exist in general between stress and strain components in the plastic region, the strain depends not only on the final state of stress, but also on the loading history. Therefore, the relations must be stated between increments of stress and strain. In case of contact problems this includes the requirement of determining the contact surface during the loading history. The aim of this paper is to present a procedure which calculates the loading increment corresponding with the next discrete change of the contact surface automatically. However, in the linear elastic region (excluding friction) a total formulation is utilized and an iterative procedure is developed to find the load for incipient yield of the structures in contact. This is in general a nonlinear problem.

Areas where it is interesting to study the influence of plastic phenomena is e g static indentation of ball or roller bearings and permanent deformations in mechanical assemblages.

The paper includes presentation of results from yield load and elastoplastic analysis of a cylinder in contact with an infinite half plate and a smooth circular disc in an infinite plate with hole.

2. Contact conditions

The conditions described below must be satisfied on the candidate contact surface Γ_c during the loading program. It is suitable to do two separate descriptions, a total and an incremental one. The total description is most suitable in the incipient yield load calculation, where the material is assumed to be linearly elastic. In the elastic plastic analysis the contact conditions must be formulated in terms of increments as the analysis as a whole. Furthermore, investigations about the sign of these increments must be performed in order to decide whether the contact is increasing, decreasing or neutral.

2.1 Displacement conditions

The initial distance δ between the surfaces

$$\delta = (\underset{\sim}{x}^1 - \underset{\sim}{x}^2) \cdot \underset{\sim}{i}_1 \tag{2.1}$$

The displacements must satisfy the inequality on Γ_c stipulating no penetration of material into the bodies.

$$(\underset{\sim}{u}^2 - \underset{\sim}{u}^1) \cdot \underset{\sim}{i}_1 - \delta \leq 0 \tag{2.2}$$

By introducing

$$u_c^\beta = \underset{\sim}{u}^\beta \cdot \underset{\sim}{i}_1 \qquad \beta = 1 \text{ to } 2 \tag{2.3}$$

Eq (2.2) reads

$$u_c^2 - u_c^1 - \delta \leq 0 \tag{2.4}$$

In terms of increments Eq (2.4) reads

$$\dot{u}_c^2 - \dot{u}_c^1 \leq 0 \tag{2.5}$$

The equality in Eq (2.5) is a sufficient and necessary condition for a point to remain in contact and the inequality indicates that points outside the contact are separating.

2.2 Normal contact stress condition

Excluding friction the only component of stress on the contact surface is the normal stress component, which, if adhesion is not present, only is allowed to be compressive (contact pressure).

$$p_c^{\beta} = \hat{\sigma}^{\beta} \cdot i_1 \leq 0 \quad \beta = 1 \text{ to } 2 \tag{2.6}$$

and the continuity condition

$$p_c^1 - p_c^2 = 0 \tag{2.7}$$

must hold. $\hat{\sigma}^{\beta}$ is the surface tractions.

In terms of normal stress increments we obtain

$$\dot{p}_c^{\beta} \leq 0 \qquad \beta = 1 \text{ to } 2 \tag{2.8}$$

and

$$\dot{p}_c^1 - \dot{p}_c^2 = 0 \tag{2.9}$$

The incremental inequality (2.8) stipulates that if a point will remain in contact these inequalities must be satisfied.

3. Yield load algorithm

In some applications it could be of interest to know the level of the applied static loads that initiates plastic flow in a system of bodies in contact. The problem of determining this load is in the general case nonlinear. The nonlinearity is introduced from the contact inequalities discussed in Chapter 2.

Introducing a proportional load factor α the objective of the analysis is to determine the value $\alpha_y = \max \alpha$ of this load factor that corresponds to incipient yield in any of the contacting bodies.

Using substructure technique all degrees of freedoms except for the presumptive contact freedoms can be condensed. This leads to a much smaller system to work with.

$$K\, u = \alpha\, P \tag{3.1}$$

We next specify two families of freedoms, u_c and u_l [8]. u_c are coupled freedoms, i e points in contact and u_l are local freedoms, i e points outside the contact.

$$\begin{bmatrix} u_c \\ u_l \end{bmatrix} = [a_c \; a_l]\, {}^{o}u = a\, {}^{o}u \tag{3.2}$$

The matrix a is a boolean transformation matrix.

The applied static load vector transforms as

$$\begin{bmatrix} P_c \\ P_l \end{bmatrix} = a^t\, {}^{o}P \tag{3.3}$$

where the superscript ^{o} indicates initial partitioning at the load vector. Hence Eq (3.1) reads

$$a^t\, K\, a\, u = \alpha\, a^t\, {}^{o}P \tag{3.4}$$

or

$$\begin{bmatrix} K_{cc} & K_{1c} \\ K_{1c}^t & K_{11} \end{bmatrix} \begin{bmatrix} u_c \\ u_1 \end{bmatrix} = \begin{bmatrix} P_c \\ P_1 \end{bmatrix} \alpha \tag{3.5}$$

The subvectors u_c, P_c and the submatrices contain elements related to body 1 and body 2

$$u_c = \begin{bmatrix} u_c^1 \\ u_c^2 \end{bmatrix} ; \; P_c = \begin{bmatrix} P_c^1 \\ P_c^2 \end{bmatrix} ; \; K_{jk} = \begin{bmatrix} K_{jk}^1 & 0 \\ 0 & K_{jk}^2 \end{bmatrix} \tag{3.6}$$

When condensing all freedoms not in contact and introducing the displacement condition Eq (2.4) the system will be reduced to

$$\begin{bmatrix} \hat{K}_{cc}^1 & 0 \\ 0 & \hat{K}_{cc}^2 \end{bmatrix} \begin{bmatrix} u_c^1 \\ u_c^1 \end{bmatrix} = \alpha \begin{bmatrix} \hat{P}_c^1 \\ \hat{P}_c^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{K}_{cc}^2 \; \delta \end{bmatrix} \tag{3.7}$$

The ^ denotes modified submatrices and subvectors due to the condensation operation. Finally, with an assembling procedure $\hat{K}_{cc} = \Sigma\hat{K}_{cc}^{\beta}$ and $\hat{P}_c = \Sigma\hat{P}_c^{\beta}$ we obtain

$$\hat{K}_{cc}\, u_c^1 = \alpha\hat{P}_c - \hat{K}_{cc}^2\, \delta \tag{3.8}$$

To be an admissible transformation the inequality Eq (2.2) must be satisfied for all freedoms in the subvector u_1 and the inequality Eq (2.6) must be satisfied for all freedoms in the subvector u_c^1 after backward transformations.

$$u_1^2 - u_1^1 - \delta < 0 \tag{3.9}$$

and

$$R_c^2 < 0 \tag{3.10}$$

where R_c^2 is the contact reaction force vector. Additionally, the yield conditions must be satisfied

$$\Phi \leq 0 \tag{3.11}$$

where

$$\Phi = \Phi_i; \quad i = 1, \ldots, n_y \tag{3.12}$$

n_y is the number of regions with different yield limits in the substructures assumed to be elastic plastic.

We solve this system of equations and inequality conditions with $\alpha^{(i)}$ as an estimation. An improved value for α can be obtained by developing $\max\{\Phi(\alpha^{(i)})\} = \Phi = 0$ in a Taylor series and Eq (3.11) takes the form

$$\Phi_{i+1} = \Phi_i + \frac{d\Phi}{d\alpha} (\alpha^{(i+1)} - \alpha^{(i)}) = 0 \tag{3.13}$$

We finally obtain

$$\alpha^{(i+1)} = \alpha^{(i)} - \frac{\Phi_i}{\Phi_i - \Phi_{i-1}} (\alpha^{(i)} - \alpha^{(i-1)}) \tag{3.14}$$

4. Elastic plastic contact algorithm

In the elastic plastic range, i e where $\alpha > \alpha_y$ an incremental algorithm has to be used. At an instant time τ after which the structure has reached first yield the problem is governed by the equation

$$ {}^{\tau}\underset{\sim}{K}\, {}^{\tau}\underset{\sim}{u} = {}^{\tau}\alpha\, {}^{\tau}\underset{\sim}{P} + {}^{\tau}\underset{\sim}{J} $$

where ${}^{\tau}\underset{\sim}{K}$, ${}^{\tau}\underset{\sim}{u}$ and ${}^{\tau}\underset{\sim}{P}$ are partitioned to establish an admissible contact solution. The vector ${}^{\tau}J$ contains initial loads due to plastic strains in the structure and is naturally partitioned in the same way as the other vectors. This can also be formulated using the boolean transformation matrix

$$ {}^{\tau}\underset{\sim}{a}^{t}\, {}^{o}\underset{\sim}{K}\, {}^{\tau}\underset{\sim}{a}\, {}^{\tau}\underset{\sim}{u} = {}^{\tau}\alpha\, {}^{\tau}\underset{\sim}{a}^{t}\, {}^{o}\underset{\sim}{P} + {}^{\tau}\underset{\sim}{a}^{t}\, {}^{o}\underset{\sim}{J} \tag{4.2} $$

This equation is valid until the time when load vectors are reaching a level that corresponds to a change in the contact status.

The purpose with this part of the paper is to describe an algorithm to obtain the load increment corresponding to a change in the contact status.

Applying a load increment

$$ \Delta\underset{\sim}{P} = \Delta\alpha\, \underset{\sim}{P} \tag{4.3} $$

Hence

$$ {}^{\tau}\underset{\sim}{a}^{t}\, {}^{o}\underset{\sim}{K}\, {}^{\tau}\underset{\sim}{a}\, ({}^{\tau}\underset{\sim}{u} + \Delta\underset{\sim}{u}) = ({}^{\tau}\alpha + \Delta\alpha)\, {}^{\tau}\underset{\sim}{a}^{t}\, {}^{o}\underset{\sim}{P} + {}^{\tau}\underset{\sim}{a}^{t}({}^{o}\underset{\sim}{J} + \Delta\, {}^{o}\underset{\sim}{J}) \tag{4.4} $$

The incremental solution can now be obtained from

$$ {}^{\tau}\underset{\sim}{a}^{t}\, {}^{o}\underset{\sim}{K}\, {}^{\tau}\underset{\sim}{a}\, \Delta u = \Delta\alpha\, {}^{\tau}\underset{\sim}{a}^{t}\, {}^{o}\underset{\sim}{P} + {}^{\tau}\underset{\sim}{a}^{t}\, \Delta\, {}^{o}\underset{\sim}{J} \tag{4.5} $$

Referring to Eq (3.7) this equation reads

$$ \begin{bmatrix} \hat{\underset{\sim}{K}}{}^{1}_{cc} & \underset{\sim}{0} \\ \underset{\sim}{0} & \hat{\underset{\sim}{K}}{}^{2}_{cc} \end{bmatrix} \begin{bmatrix} \Delta\underset{\sim}{u}^{1}_{c} \\ \Delta\underset{\sim}{u}^{2}_{c} \end{bmatrix} = \Delta\alpha \begin{bmatrix} \hat{\underset{\sim}{P}}{}^{1}_{c} \\ \hat{\underset{\sim}{P}}{}^{2}_{c} \end{bmatrix} + \begin{bmatrix} \Delta\hat{\underset{\sim}{J}}{}^{1}_{c} \\ \Delta\hat{\underset{\sim}{J}}{}^{2}_{c} \end{bmatrix} \tag{4.6} $$

In this case we have to use the contact conditions in the incremental form, Eq (2.5)

$$ \Delta\underset{\sim}{u}^{2}_{c} - \Delta\underset{\sim}{u}^{1}_{c} = 0 \tag{4.7} $$

To obtain the current value of the initial load increment vector ${}^{\tau}\underset{\sim}{a}^{t}\, {}^{o}\Delta\underset{\sim}{J}$ equilibrium iterations have to be performed. The obtained displacement and contact force increments have to be consistent with the no penetration condition

$$ {}^{\tau}\underset{\sim}{u}^{2}_{1} + \Delta\underset{\sim}{u}^{2}_{1} - {}^{\tau}\underset{\sim}{u}^{1}_{1} - \Delta\underset{\sim}{u}^{1}_{1} - \underset{\sim}{\delta}_{1} < \underset{\sim}{0} \tag{4.8} $$

and the compressive stress condition here in terms of nodal reaction forces

$$ {}^{\tau}\underset{\sim}{R}^{2}_{c} + \Delta\underset{\sim}{R}^{2}_{c} < \underset{\sim}{0} \tag{4.9} $$

It is obvious that the only possibility to get a change in the contact status is to satisfy either one or both of the following conditions

$$ \Delta\underset{\sim}{u}^{2}_{1} - \Delta\underset{\sim}{u}^{1}_{1} > \underset{\sim}{0} \tag{4.10} $$

or

$$ \Delta\underset{\sim}{R}^{2}_{c} > \underset{\sim}{0} \tag{4.11} $$

Now an iterative algorithm for determination of which level at the load factor increment that corresponds to a change of the contact status will be described.

4.1 Load factor algorithm

There exist three alternatives that must be investigated.

a) $$\Delta \underset{\sim}{u}_1^2 - \Delta \underset{\sim}{u}_1^1 < \underset{\sim}{0} \tag{4.12}$$

and

$$\underset{\sim}{R}_c^2 < \underset{\sim}{0} \tag{4.13}$$

This indicates that no change in the contact surface is possible, i e a neutral contact and the analysis can proceed as in a materially-nonlinear-only analysis.

b) There exist elements in the subvector satisfying Eq(4.10)

$$[\Delta \underset{\sim}{u}_1^2 - \Delta \underset{\sim}{u}_1^1]_k > 0 \qquad k \in 1 \tag{4.14}$$

All elements in $\underset{\sim}{R}_c^2$ in this case satisfy Eq (4.11). Eq (4.10) defines an increasing contact and new nodepairs will go into contact when

$$[{}^{\tau}\underset{\sim}{u}_1^2 + \Delta \underset{\sim}{u}_1^2 - {}^{\tau}\underset{\sim}{u}_1^1 - \Delta \underset{\sim}{u}_1^1 - \underset{\sim}{\delta}_1]_k = 0 \qquad k \in 1 \tag{4.15}$$

This will first be fulfilled by the nodepair which has

$$\max \left| \frac{\Delta \underset{\sim}{u}_1^2 - \Delta \underset{\sim}{u}_1^1}{{}^{\tau}\underset{\sim}{u}_1^2 - {}^{\tau}\underset{\sim}{u}_1^1 - \underset{\sim}{\delta}_1} \right|_k = \hat{\psi}_u \qquad k \in 1 \tag{4.16}$$

The denominator represents the remaining distance between the bodies at time $t = \tau$. A new estimate for the load increment is given by

$$\Delta\alpha^{(i+1)} = - \hat{\psi}_u^{(i)^{-1}} \Delta\alpha^{(i)} \tag{4.17}$$

The termination criteria

$$1 < \hat{\psi}_u \leq 1 + \gamma_u \tag{4.18}$$

makes it sure that a new nodepair has come into contact. γ_u is a small number and is choosen by the user.

c) All nodepairs outside the contact satisfy Eq (4.12) but inside the contact area there exist nodepairs which have

$$[\Delta \underset{\sim}{R}_c^2]_k > 0 \qquad k \in c \tag{4.19}$$

In this case we have a decreasing contact and nodepairs tend to leave the contact. Next pair leaving can be obtained from

$$\max \left| \frac{\Delta \underset{\sim}{R}_c^2}{{}^{\tau}\underset{\sim}{R}_c^2} \right|_k = \hat{\psi}_R \qquad k \in c \tag{4.20}$$

and leaves when

$$[{}^{\tau}\underset{\sim}{R}_c^2 + \Delta \underset{\sim}{R}_c^2]_k = 0 \qquad k \in c \tag{4.21}$$

In accordance with Eq (4.14) a new estimation for the load increment in the next iteration is given by

$$\alpha^{(i+1)} = - \hat{\psi}_R^{(i)^{-1}} \cdot \Delta\alpha^{(i)} \tag{4.22}$$

and the iterations will be terminated when

$$1 < \hat{\psi}_R^{(i)} \leq 1 + \gamma_R \tag{4.23}$$

is satisfied and makes it sure that this nodepair is out of contact.

5. Applications

In this section the preceeding developments are applied to two contact problems involving plasticity. The material behaviour is described by the von Mises yield condition with associated flow rule using the hypothesis of isotropic hardening. The hardening characteristic in the first case is linearly strain hardening and in the second case perfectly plastic.

5.1 Cylinder in contact with an infinite halfplate

As a first illustration the contact between an infinitely long cylinder and an finite halfplate plane strain is examined. The problem is described in Fig 5.1.

Introducing a loading parameter defined by

$$\alpha = \frac{P}{P^*} \tag{5.1}$$

the following relation in case of identical linearly elastic material properties yields [9]

$$\frac{p}{p^*} = \frac{b}{b^*} = \sqrt{\alpha} \tag{5.2}$$

where p is the maximum contact pressure, P the load per unit length, b half the contact length and the asterisk denotes values at incipient yield.

In Table I relations between calculated values with the numerical method discussed in Section 3 and by the equations above for the load level, maximum contact pressure, contact width and the location of the maximum equivalent stress at incipient yield. In the following elastoplastic analysis the contact surface was increased to 1.5 b* in seven increments with corresponding load increments. From this state the structure was unloaded to 0.3 b* by 13 increments. In Fig 5.2 the relations between the contact width and the maximum contact pressure and the load factor α is illustrated. At four contact widths the contact pressure distributions are depicted in Fig 5.3. Fig 5.4 illustrates the plastic zone at three intensities of the load factor α.

The residual indentation depths between the bodies at the origin are shown in Fig 5.5.

5.2 A smooth circular disc in an infinite plate with hole loaded under unaxial tension

In the last example a problem where the contact surfaces will separate as the load is applied; it concerns a smooth circular disc in a circular hole in an infinite plate under a uniform tension σ_x^∞ at the infinity of x. The diameter of the disc and that of the hole are identical as well as the material properties, see Fig 5.6. Exact solutions have been given in [10] and [11] and finite element calculations including friction and with the disc shrinked into the plate is presented in [12]. The result reviewed here are from [13]. The exact solutions give that the contact surface is constant then the structure is loaded which was shown in the yield load calculation reviewed in Table II.

After the yield load calculation the load was increased up to α = 2 and then decreased until α = 0 where

$$\alpha = \frac{\sigma_x^\infty}{\sigma_x^*} \tag{5.3}$$

and

σ_x^* uniform tension initiating plastic flow

In Fig 5.7 the maximum value of the contact pressure is plotted against the load factor. Fig 5.8 shows the contact pressure distribution at three specific values of α during the loading and unloading respectively. At last the extension of the plastic zone is depicted in Fig 5.9 and as we can observe the disc never will reach the yield stress in this load interval.

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TABLE I

$\frac{P^*_{FEM}}{P^*[8]}$	$\frac{p^*_{FEM}}{p^*[8]}$	$\frac{b^*_{FEM}}{b^*[8]}$	$\frac{z^*_{FEM}}{z^*[8]}$
0.87	0.94	0.98	0.94

TABLE II

θ_{FEM}	$\theta[11]$	$\frac{\alpha^*_{FEM}}{\alpha^*[11]}$
20.°	20.°	1.002

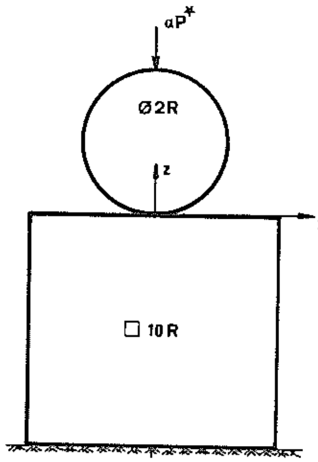

Fig 5.1 A circular cylinder in contact with a half plane, hardening coefficient equal to 0.6

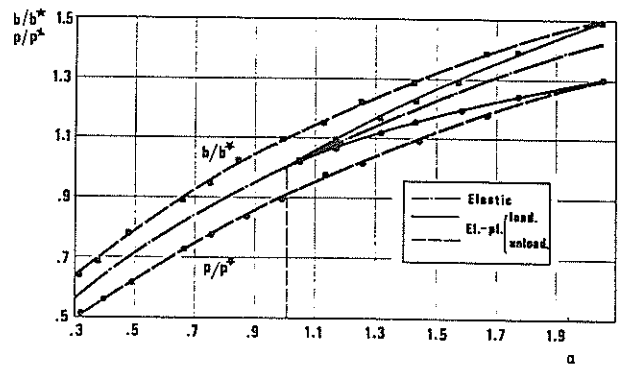

Fig 5.2 Maximum contact pressure and contact width against the load factor

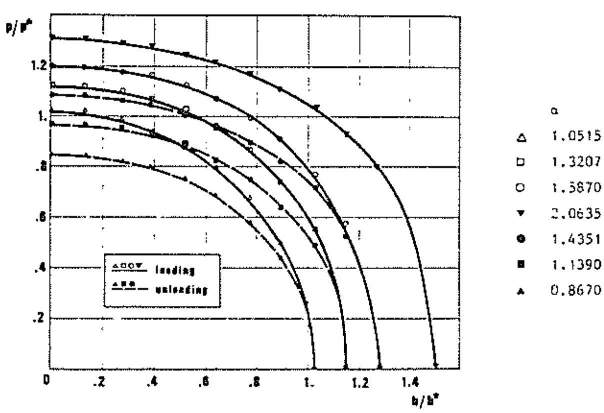

Fig 5.3 Contact pressure distribution during loading and unloading at four contact widths

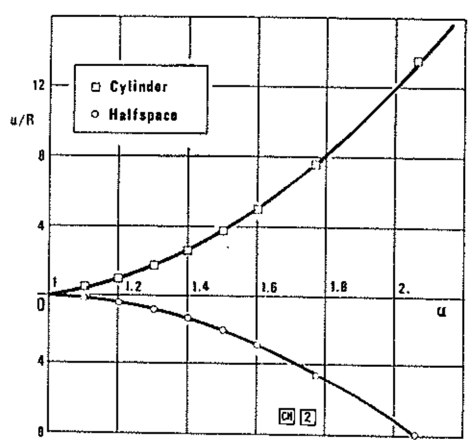

Fig 5.4 Residual indentation depths at (x,z) = (0,0) against the load factor

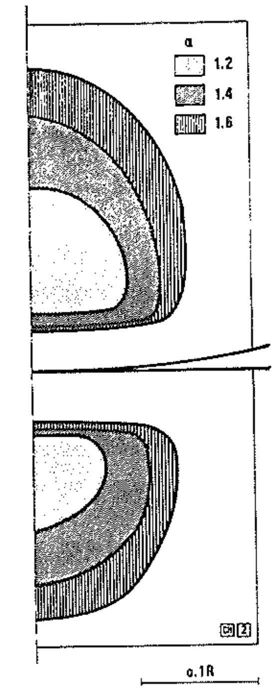

Fig 5.5 Schematic representation at the plastic zones for various intensities of α

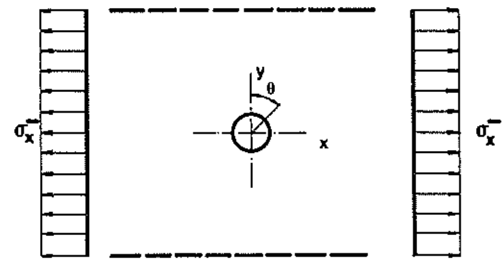

Fig. 5.6 A smooth circular disc in an infinite plate with hole loaded under unaxial tensile

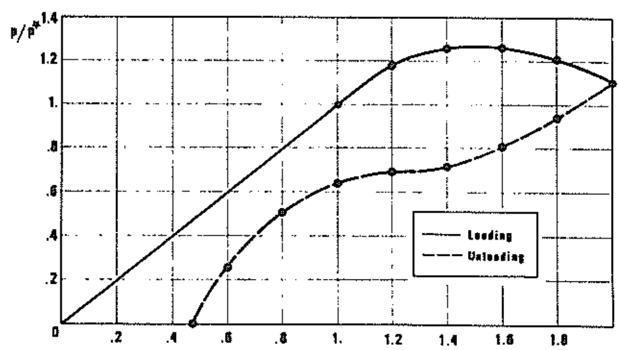

Fig 5.7 Maximum contact pressure against the load factor .

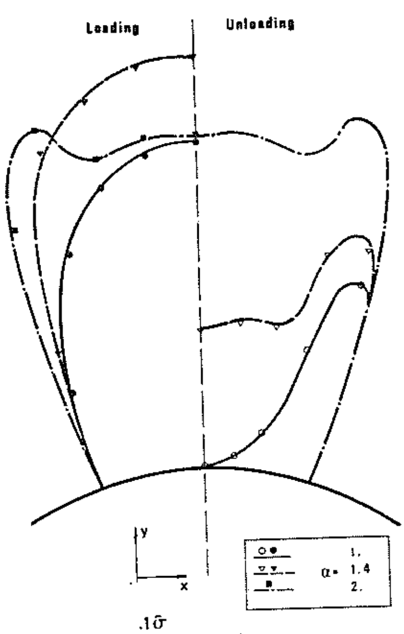

Fig 5.8 The contact pressure distribution at three load levels during loading and unloading

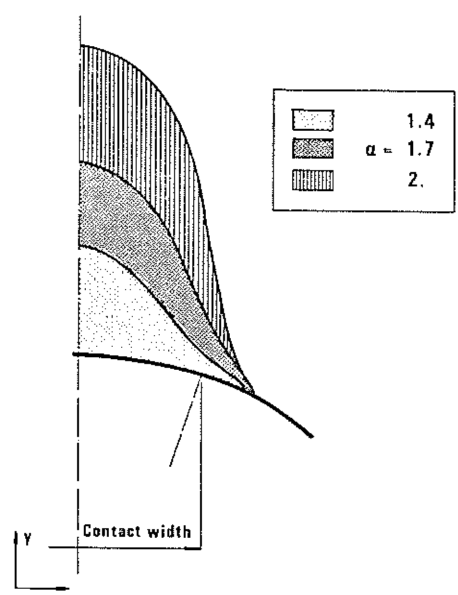

Fig 5.9 Schematic representation of the plastic zones for various intensities of α