A Bounding Technique for Dynamic Plastic Deformations

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Summary

The paper considers elastic-perfectly plastic discrete structures subjected to dynamic excitations, whose exact time sequence is supposed to be unknown. A general bounding principle is formulated, which provides bounds on several kinds of dynamic plastic deformations, such as plastic strains, residual displacements and plastic work produced within a single or more structural elements

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1. Introduction

After that Ceradini [1] extended Melan's shakedown theorem to the domain of dynamic loadings in 1969, a considerable research effort has been made through all the Seventies in order to better clarify and improve the theorem itself as related to concepts of dynamic adaptation and inadaptation. This research effort, in majority due to the Italian school, has also been concerned with providing appropriate bounding techniques for the assessment of dynamic plastic deformations produced during the adaptation process. A few recent survey papers give a quite complete account of the most significant contributions in the field of dynamic shakedown and related bounding techniques [2-4].

There are many instances of engineering practice and modern technology (such as seismic and ocean engineering, chemical and metallurgical industries, nuclear reactor technology and the like) in which conditions for structures to escape failure due to progressive accumulated plastic deformations are of vital importance and for this an estimation of deformations by means of a bounding technique is quite frequently adequate, at least for preliminary evaluations. Within these applications, rarely the loading history is completely known at all times; more frequently, the loading process consists of a sequence of short-duration excitations with intervals during which the structure remains unloaded.

The primary purpose of the present paper is the formulation of a bounding technique which can be applied not only when the loading history is fully specified (as usually assumed within the known bounding techniques) but also when the loading process is a more general one. To this aim the following hypotheses are made:

a) The loading history is unspecified but identifies with any sequence of short-duration excitations belonging to a given convex set. Every excitation is supposed to be separated from the subsequent one by a sufficiently long period during which the unloaded structure rapidly ceases to oscillate due to internal damping. If the aforementioned set contains a single excitation and futherly its duration is supposed to be unlimited, the typical loading of classical shakedown theory is recovered (i.e. the loading history is fully specified).

b) The given set of excitations is a hyperpolyhedron of the function space, whose vertices constitute what will be called the "basic" excitations. Since all the excitations can be always thought of as having the same (finite) duration, every excitation of the given set can therefore be expressed as a convex linear combination of the basic ones only.

Other hypotheses of the present paper are:

c) The structure is supposed to be a discrete model constituted by elastic-perfectly plastic elements and subjected to dynamic actions in the form of nodal forces and imposed strains, the latter being caused by temperature variations. The displacements of the resulting motion are treated as infinitesimal.

d) A piecewise linear description of the plastic behavior is adopted (see, e.g., [2,3,5-9]) according to which the elastic domain of every structural element is a hyperpolyhedron of the stress space, specified through the (fixed) unit external normals of the polyhedral facets together with the distances of these facets from the origin. Since, for the sake of simplicity, no workhardening is considered, these distances are constant (plastic resistances).

e) The dynamic characteristics of the structure considered as being purely elastic, such as the natural frequencies and the associated displacement and stress mode shapes, are supposed to have been supplied through a modal analysis [10,11].

The above hypotheses enable one to obtain bounds expressed in terms of the (dynamic) elastic stress response of the structure to the basic excitations, plus some additional terms which can be suitably adjusted in order to improve the stringentness of the bounds. The formulation of these bounds will prove to be a unified one, for these bounds are all incorporated in a single general bound inequality and they can be deduced as particular cases when some free perturbation terms are suitably specified. The deformation parameters which can in practice be bounded are the plastic strain components, the residual displacements and the plastic work relative to a single or more elements [5-7, 12].

2. Derivation of a general bounding principle

Let $\underset{\sim}{F}_j(\tau)$, $\underset{\sim}{\theta}_j(\tau)$, $0 \leq \tau \leq T$, denote the nodal forces and the imposed strains associated with the j-th basic excitation ($j = 1,2,..,m$) and let $\underset{\sim}{\sigma}_j^E(\tau)$ indicate the corresponding elastic stress response. A typical excitation can be represented in the form

$$\underset{\sim}{F}(\tau) = \sum_{j=1}^{m} \alpha_j \underset{\sim}{F}_j(\tau), \qquad \underset{\sim}{\theta}(\tau) = \sum_{j=1}^{m} \alpha_j \underset{\sim}{\theta}_j(\tau) \tag{1}$$

where the nonnegative coefficients α_j must satisfy

$$\sum_{j=1}^{m} \alpha_j = 1. \tag{2}$$

Denoting by Int(t) the greatest integer contained in the time variable $t \geq 0$, and setting

$$\tau = t - (r-1)T, \qquad r = 1 + \mathrm{Int}(t) \tag{3}$$

enables us to construct an Admissible Loading History (ALH) $\underset{\sim}{F}(t)$, $\underset{\sim}{\theta}(t)$, i.e.

$$\underset{\sim}{F}(t) = \sum_{j=1}^{m} \alpha_j(r) \underset{\sim}{F}_j(\tau), \qquad \underset{\sim}{\theta}(t) = \sum_{j=1}^{m} \alpha_j(r) \underset{\sim}{\theta}_j(\tau) \tag{4}$$

where the α's coefficients are to be considered as functions of r and eqs. (3) are to be substituted. The elastic stress response to the loadings in eq. (4) is

$$\underset{\sim}{\sigma}^E(t) = \sum_{j=1}^{m} \alpha_j(r)\, \underset{\sim}{\sigma}_j^E(\tau) \tag{5}$$

where $\underset{\sim}{\sigma}_j^E(\tau)$ is the analogous response to the j-th basic excitation.

Let us suppose that there exist a time-independent self-stress distribution, $\underset{\sim}{\rho}$, and free-motion stress distribution, $\underset{\sim}{\sigma}^F(\tau)$, such that the total stresses

$$\underset{\sim}{s}_j^* = \underset{\sim}{\sigma}_j^E(\tau) + \underset{\sim}{\sigma}^F(\tau) + \underset{\sim}{\rho}, \quad (j = 1,2,..,m), \quad (0 \le \tau \le T) \tag{6}$$

satisfy the following plasticity conditions:

$$\underset{\sim}{N}^T \underset{\sim}{s}_j(\tau) - \underset{\sim}{k} + \beta \underset{\sim}{d} \le \underset{\sim}{0} \quad (j = 1,2,..,m), \quad \forall\tau \in (0, T) \tag{7}$$

Here $\underset{\sim}{N}$ is the matrix of the unit external normals of the yield facets, $\underset{\sim}{k} > \underset{\sim}{0}$ is the vector of plastic resistances, while $\underset{\sim}{d}$ is a time-independent vector, called perturbation vector, and $\beta > 0$ is a scalar (perturbation multiplier) |5|. If ω_i, $(i = 1,2,..,N_f)$, indicate the natural frequencies of the N_f-degree-of-freedom elastic system and further $\underset{\sim}{\Phi}_i$, ξ_i are the associated displacement mode shapes and damping ratios, respectively, then we can write [10,11]

$$\underset{\sim}{\sigma}^F(\tau) = \sum_{i=1}^{N_f} \underset{\sim}{S}_i \left[a_i \cos \omega_i^D \tau + b_i \sin \omega_i^D \tau\right] \exp(-\xi_i \omega_i \tau) \tag{8}$$

where $\underset{\sim}{S}_i$ is the stress mode shape associated with $\underset{\sim}{\Phi}_i$, i.e.

$$\underset{\sim}{S}_i = \underset{\sim}{D}\,\underset{\sim}{C}\,\underset{\sim}{\Phi}, \tag{9}$$

ω_i^D is the damped i-th frequancy, i.e.

$$\omega_i^D = \omega_i \sqrt{(1 - \xi_i^2)} \tag{10}$$

and finally a_i, b_i are arbitrary constants. Since

$$\underset{\sim}{u}^F(\tau) = \sum_{i=1}^{N_f} \underset{\sim}{\Phi}_i \left[a_i \cos \omega_i^D \tau + b_i \sin \omega_i^D \tau\right] \exp(-\xi_i \omega_i \tau) \tag{11}$$

the constants a_i, b_i are related to the initial conditions of the free motion as

$$\underset{\sim}{u}^F(0) = \underset{\sim}{c} = \sum_{i=1}^{N_f} \underset{\sim}{\Phi}_i a_i \tag{12a}$$

$$\dot{\underset{\sim}{u}}^F(0) = \dot{\underset{\sim}{c}} = \sum_{i=1}^{N_f} \underset{\sim}{\Phi}_i \omega_i^D \left[b_i - \frac{\xi_i a_i}{\sqrt{(1 - \xi_i^2)}}\right] \tag{12b}$$

Multiplying eq. (7) by $\alpha_j(r)$, summing with respect to j and taking into account eqs.(2) and (5) yields

$$N^T s(t) - k + \beta d \le 0\,, \qquad \forall t \ge 0 \tag{13}$$

where, in virtue of the first eq. (3),

$$s(t) = \sigma^E(\tau(t)) + \sigma^F(\tau(t)) + \rho\,. \tag{14}$$

Eq. (13) says that the plasticity conditions as in eq. (7) are also satisfied by any ALH. On the other hand, the real elastic plastic response to this ALH satisfies the relationships

$$f = N^T \sigma - k \le 0, \quad \dot{\lambda} \ge 0, \quad f^T \dot{\lambda} = 0, \qquad \forall t \ge 0, \tag{15}$$

so that through a suitable comparison between eqs. (13) and (15) we obtain the inequality

$$d^T \lambda(t_1) \le \beta^{-1} Q(\rho, c, \dot{c}) \tag{16}$$

where

$$Q = \tfrac{1}{2} \rho^T D^{-1} \rho + \tfrac{1}{2} c^T K c + \tfrac{1}{2} \dot{c}^T M \dot{c}. \tag{17}$$

We can always set $t_1 = nT$ with n being an integer, so eq. (16) is a bound on plastic deformation produced in the structure after any ALH, i.e. after any number of applications of excitations, no matter what the actual sequence of the latter may be, nor the number of repetitions of an individual excitation. Note that eq. (16) holds for any choice of the perturbation vector d, of the self-stress ρ and of the free motion $u^F(\tau)$, provided eqs. (7) are satisfied.

3. Specializations of the perturbation vector

The bound inequality eq. (16) can generate several particular cases as soon as the perturbation vector d is suitably specified. The following cases can be distinguished:

3.1. Bound on plastic strains

Let d be chosen as $d = N^T \bar{\sigma}$, where $\bar{\sigma}$ is an arbitrary stress vector; then $d^T \lambda = \bar{\sigma}^T N \lambda = \bar{\sigma}^T p$ and eq. (16) transforms into

$$\bar{\sigma}^T p(t_1) \le \beta^{-1} Q(\rho, c, \dot{c}) \tag{18}$$

which is a bound on the plastic strains $p(t_1) = N \lambda(t_1)$.

3.2. Bound on residual displacements

If the stress $\bar{\sigma}$ in eq. (18) is assumed to be the statical elastic stress response to some arbitrary nodal force $\bar{F}$, we can write

$$\bar{\sigma}^T p = \bar{\sigma}^T (C u^R - D^{-1} \sigma^R) = \bar{F}^T u^R, \tag{19}$$

where $(\cdots)^R$ stands for "residual"; thus eq. (16) becomes

$$\bar{F}^T u^R(t_1) \le \beta^{-1} Q(\rho, c, \dot{c}), \tag{20}$$

which is a bound on the residual displacement $u^R(t_1)$.

3.3. Bound on plastic work

Let us observe that the product $d^T \lambda(t_1)$ in eq. (16) can also be written as

$$d^T \lambda(t_1) = \sum_{h=1}^{N_e} d_h^T \lambda_h(t_1) \tag{21}$$

where N_e is the number of structural elements and the product $d_h^T \lambda_h$ represents the contribution of the h-th element to the total deformation measure. Let d_h be chosen in the form

$$d_h = \gamma_h k_h \qquad (h = 1,2,..,N_e) \tag{22}$$

where the γ_h's are arbitrary nonnegative constants; then, from eq. (16) we obtain

$$\sum_{h=1}^{N_e} \gamma_h W_h(t_1) \le \beta^{-1} Q(\rho, c, \dot{c}) \tag{23}$$

where $W_h(t_1) = k_h^T \lambda(t_1) = \int_o^{t_1} \sigma_h^T \dot{p}_h \, dt$ is the plastic work dissipated within the h-th structural element in the interval $(0, t_1)$. Thus, eq. (23) is a bound on the weighted plastic work.

4. Bound optimization

In the bound equation (16), the vectors $\rho, c, \dot{c}$ are completely free, provided eqs. (7) are satisfied. Thus, the bounding quantity in the right-hand member of eq. (16) can be minimized under the constraints of eqs. (7). The resulting problem is the following:

$$\min \Phi_0 = \beta^{-1} Q(\rho, c, \dot{c}) \tag{24a}$$

subject to:

$$N^T[\sigma_j^E(\tau) + \sigma^E(\tau) + \rho] - k + \beta d \le 0 \tag{24b}$$

$$C^T \rho = 0, \quad \beta > 0, \tag{24c}$$

$$(j = 1,2,..,m), \quad (0 \le \tau \le T) \tag{24d}$$

where $\sigma^F(\tau)$ is as given in eq. (8), while c and $\dot{c}$ are as given in eqs. (12a,b). This problem is strictly convex in the (ρ, a_i, b_i, β)-space and possesses linear constraints. Applying the Lagrangian multiplier method gives the appropriate optimality conditions which characterize the most stringent bound.

5. Conclusion

The bounding technique presented possesses a quite unified character in the sense that it is formulated in terms of a general perturbation vector which represents several kinds of deformation parameters. In this way, the solution methods and the computational procedure devised can be applied independently of the specific bound involved.

The practical numerical computation of the bounds require a discretization also in the time space. It is also required the modal analysis of the structure, but such an analysis is often necessary in other analysis stages.

The methods shown are mainly valid for loads below the shakedown limit. For loads above this limit, the procedure should be somewhat modified.

References

1. G. Ceradini, "On the adaptation of elastic-plastic bodies subjected to dynamic actions" (in Italian), Giornale del Genio Civile 107, (4-5), 1969, pp. 239-250.
2. M. Capurso, L. Corradi and G. Maier, "Bounds on deformations and displacements in shakedown theory", in: Matériaux et structures sous chargement cyclique, (Association Amical des Ingenieurs Anciens Elèves de l'E.N.P.C., Paris, 1979) pp. 231-244.
3. J.A. König and G. Maier, "Shakedown analysis of elasto-plastic structures: a review of recent developments, Nucl. Engrg. Des. 66, 1981, pp. 81-95.
4. G. Ceradini, "Dynamic shakedown in elastic-plastic bodies", J. Engrg. Mech. Div., Proc. ASCE, 106, (3), June 1980, pp. 481-498.
5. C. Polizzotto, "A unified treatment of shakedown theory and related bounding techniques", SM Archives 7, (1), 1982, pp. 19-75.
6. C. Polizzotto, "On workhardening adaptation of discrete structures under dynamic loadings", Archives of Mechanics, 32, (1), 1980, pp. 81-99.
7. C. Mazzarella, T. Panzeca and C. Polizzotto, "A discrete approach to bound techniques for dynamic plastic deformation", in: Proc. 5-th Nat. Congress of the Italian Association of Theoretical and Applied Mechanics, AIMETA, Palermo, 1980, Vol. 2, pp. 249-259.
8. G. Maier, "A matrix structural theory of piecewise linear elastoplasticity with interacting yield planes", Meccanica, V, (1), 1970, pp. 54-66.
9. G. Maier, "Piecewise linearization of yield criteria in structural plasticity", SM Archives, 1, (2/3), 1976, pp. 239-281.
10. R.W. Clough and R.W. Penzien, Dynamic of Structures, (McGraw-Hill, 1975).
11. C. Polizzotto, "Dynamic shakedown by modal analysis", Meccanica, (submitted to).
12. T. Panzeca, "Bounds on displacement and on velocity for elastic-plastic solids under dynamic actions", in: Proc. 6th Nat. Congress of the Italian Association of Theoretical and Applied Mechanics, AIMETA, Genova 1982, Vol. 2, pp. 329-339.