# A Simplified Reliability Analysis for Pressurized Components at the Presence of Small Cracks

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Summary

The problem of crack depth probability densities which can be approximated by δ-functions at zero crack depth is discussed in the context of failure probability and failure rate assessment for pressurized components. An existing model is extended to the case of such δ-type densities; there, in addition to the crack depth and its critical value also the number of cracks is assumed to be a random variable. An expression is derived which involves only distributions as determined from measurements above a crack detection threshold, $a_D$, and which converges to the desired failure probability for $a_D \to 0$. General mathematical conditions are discussed for which non-trivial limiting values can exist. A special example demonstrates that, for δ-type densities of the crack depth, the rate of failures due to crack growth is nonzero even if cracks of small depth are assumed to be unable to grow.

## 1. Introduction

During the (first) Marshall Study [1] of PWR vessel integrity a theoretical procedure was proposed for the assessment of the failure probability of a pressurized structure. This led to a probabilistic analysis presented by Lidiard and Williams [2-5] and developed further by Harrop [6]. The analysis is based on the failure model of fast fracture described by linear elastic fracture mechanics. The data input to the analysis, in its simplest form, consists of the function Nf(a) where f(a) is the probability density of the crack depth, a, and N is the average number of cracks, and of the probability density of the critical crack depth determined by both the fracture toughness $K_{Ic}$ and the load.

The theory of ref. [2] implies that Nf(a) is a bounded function for all a. This condition appears natural as long as only distributions are considered which are truncated by the crack detection threshold, $a_D$, of the measuring device, i.e., the smallest crack depth which still can be measured. However, for $a_D \to 0$, the probability density $f(a;a_D)$ of the crack depth a, determined from measurements above $a_D$ should be expected, it appears, to converge to the singular $\delta$-function $\delta(a)$. Moreover the average number of cracks of depth larger than $a_D$ may diverge for $a_D \to 0$.

The fact that the observed shape of the probability density of the crack depth and the number of cracks depend, in principle, on the particular choice of the lower bound (threshold) value for the crack depth has been emphasized, it appears, by a number of authors [7-10].

This paper presents an extension of the theory of ref. [2] to the case of $\delta$-function crack depth densities. An important modification is the introduction of the number of cracks as a random variable.

We attack the problem by deriving a mathematical expression which is built up with the distributions of the number of cracks and that of the crack depth, only as far as they could be measured at crack depths larger than a treshold depth. Thus this expression depends on the threshold depth; it is however expected to converge to the desired failure probability if the threshold approaches zero. We will discuss the conditions under which the consequences of the earlier model can also be derived from our model. Finally an example should illustrate our considerations.

## 2. Description of the Probabilistic Model

We start with general assumptions similar to those of the existing model of refs. [2-5].

General assumptions:

The failure probability of a pressurized component can be formulated with the methods of linear elastic fracture mechanics.

Each crack is characterized only by its depth in the direction through the wall.

The component fails by fast fracture if the depth of at least one crack exceeds the critical depth which is uniquely determined by the fracture toughness $K_{Ic}$ and the load.

Each component may be subdivided in a number of partial domains of finite size with constant $K_{Ic}$ and homogeneous stress. All our calculations refer to single domains.

In a measurement only cracks of a depth bigger than a threshold $a_D$, characteristic for the measuring device can be detected. Thus the experimentally determined probabilities to observe a certain number of cracks and a certain crack depth depend on $a_D$.

Measured quantities and their probabilities:

We distinguish 4 randomly distributed quantities associated with a component domain: The number of cracks, the depth of an arbitrary crack, the depth of the biggest crack at a given number of cracks and the fracture toughness.

The conditional probability to detect n'=n cracks in a component domain is

$$P(n'=n \mid a' \geqq a_D) \equiv p_n(a_D) \; ; \quad n = \{0,1,2,\ldots\} \tag{1}$$

(We write a primed character for the random variable and an unprimed character for a special value.)

Given n' the depth of a detectable crack a' has the probability distribution

$$P(a' \leq a \mid (n'=n) \cap (a' \geqq a_D)) = F_n(a;a_D) \equiv \int_{a_D}^{a} f_n(\alpha;a_D)d\alpha \tag{2}$$

with $\quad F_n(h;a_D) = 1 \; ; \quad n \geq 1$

h is the physical upper bound of a'. Note that the distribution may depend on n.

Since the cracks are undistinguishable the depth, b', of the deepest crack in n detectable cracks has the probability distribution

$$P(b' \leq b \mid (n'=n) \cap (a' \geqq a_D)) = (F_n(b;a_D))^n \equiv \int_{a_D}^{b} w_n(b';a_D)db' \tag{3}$$

The probability density of the toughness $K_c'$ $(=K_{Ic}')$ is defined by

$$P(K_c \leq K_c' < K_c + dK_c) \equiv g(K_c)dK_c \; ; \qquad \int_0^\infty g(K)dK = 1 \tag{4}$$

We assume $K_c'$ to be statistically independent of n' and a'. For a description of possible correlations we refer to a proposal of Mariott and Churchill [11].

3. <u>The experimental failure probability</u>

With the notations of eqs. (1) to (4) we write down the probability to have $K_c'$ at $K_c$, to observe n detectable cracks and to have, for the deepest crack, b' at b

$$\begin{aligned} &P((K_c \leq K_c' < K_c + dK_c) \cap (n'=n) \cap (b \leq b' < b + db) \mid a' \geq a_D) \\ &= P(K_c \leq K_c' < K_c + dK_c)\, P(b \leq b' < b + db \mid (n'=n) \cap (a' \geq a_D)) \\ &\qquad \times P(n'=n \mid a' \geq a) \\ &= g(K_c)\, w_n(b;a_D)\, p_n(a_D)\, dK_c\, db \equiv u_n(K_c,b;a_D)dK_c\, db \end{aligned} \tag{5}$$

with the normalisation

$$p_o(a_D) + \sum_{n=1}^{\infty} \int_0^\infty dK_c \int_{a_D}^{h} db\, u_n(K_c,b;a_D) = 1 \tag{5a}$$

With the density eq.(5) we are able to formulate what we call the experimental failure probability. We make use of the assumption that for a given stress, $\sigma$, there is a one-to-one relationship between the stress intensity factor and the crack depth, a, $K_I(\sigma,a) \sim s\, a^{1/2}$. The condition for failure is then $K_I(\sigma,b) \geq K_c$ or

$$b' \geq b^{\text{critical}} = K_I^{-1}(\sigma,K_c) \equiv c(K_c) \sim (K_c/s)^2 \tag{6}$$

where b is the largest crack depth at a given number of cracks.

Using the definitions eqs. (1) to (5) and the failure criterion (6) the experimental failure probability for fixed stress is

$$E_F(a_D;c) = P(b' \geq c(K_c') \mid a' \geq a_D) = \sum_{n=1}^{\infty} p_n(a_D) \int_{b \geq c(K_c) \geq a_D} u_n(K_c,b;a_D)\, db\, dK_c \tag{7}$$

$$= \int_0^{K_D} g(K_c)dK_c + \int_{K_D}^{K_h} g(K_c) \sum_{n=1}^{\infty} p_n(a_D) \{1-F_n(c(K_c);a_D)^n\}dK_c$$

with

$$F_n(c(K_c);a_D) = \int_{a_D}^{c(K_c)} f_n(a;a_D)da \;;\quad K_D \equiv K_I(\sigma,a_D) \;;\; K_h \equiv K_I(\sigma,h)$$

The integration area is shown in Fig. 1.

An approximation formula:

Our eq. (7) can contain eq. (3) of ref. [2] as an approximation under the following conditions:

$$\int_{c(K_c)}^{h} f_n(a;a_D)da = 1 - F(c(K_c);a_D) \ll 1 \qquad \text{and independent of } n \tag{8}$$

$$\bar{n}_k(a_D) \equiv \sum_{n=k}^{\infty} p_n(a_D) n^k \qquad \text{is finite for all } a_D \geq 0 \text{ and all } k \geq 1. \tag{8a}$$

Then

$$E_F(a_D;c) \cong E_F^*(a_D;c) = \int_{K_D}^{K_h} g(K_c)\{\sum^{\infty} np_n(a_D) \int_{c(K_c)}^{h} f(a;a_D)da\}\, dK_c = \bar{n}(a_D) \int_{K_D}^{K_h} g(K_c)dK_c \int_{c(K_c)}^{h} f(a;a_D)da \tag{9}$$

This is essentially the expression for the failure probability as derived in ref. [2]. The product $\bar{n}(a_D)f(a;a_D)da$ is interpreted there as the average number of cracks of depth a to a + da. As is well known this product, as a function of a, is independent of $a_D$ in the following sense: Let be $a_{D2} < a_{D1}$. Then $\bar{n}(a_{D2})f(a;a_{D2}) = \bar{n}(a_{D1})f(a;a_{D1})$ for $a \geq a_{D1}$, and $E_F^*(a_{D2},c) \geq E_F^*(a_{D1},c)$ since $K_D = K_I(\sigma,a_D)$ is a decreasing function of $a_D$ (see fig.1).

## 4. The Failure Probability as a Limiting Value for $a_D \to 0$

The experimental failure probability, eq. (7), depends on the crack detection threshold $a_D$ since it is defined with the probability distributions of the crack depth and the number of cracks determined from measurements above threshold. No information on cracks of depth smaller than $a_D$ is required. It is assumed that the expression eq. (7) is an approximation of the (desired) failure probability, $P_F$, which would become arbitrarily close to it if $a_D$ could be lowered to zero. We thus postulate:

$$\lim_{a_D \to 0} E_F(a_D;c) = P_F(c) \qquad (= \text{failure probability}) \tag{10}$$

exists for all boundary curves $b' = c(K_c)$ in a $b$-$K_c$-plane as represented in fig. 1 and defined, at a fixed load, by eq. (6).

Sufficient conditions for the existence of the limit eq. (10) are the following:

- $\sum_{n}^{\infty} p_n(a_D)$ is uniformly convergent for all $a_D \geq 0$
- the functions $F_n(c;a_D)$ and $p_n(a_D)$ are continuous functions of $a_D$ in $0 \leq a_D < c$, for any fixed $c > 0$. (See eqs. (1-2)).
- $g(K_c)$ is continuous in $K_c$ (see eq. (4)).

Since, by definition, $\sum^{\infty} p_n(a_D)$ is convergent (=1) for all $a_D$, $E_F(a_D)$ of eq.(7) is continuous for $a_D \to +0$. If the sum over $p_n$ is not uniformly convergent, eq. (10), can no longer be proved in general with simple conditions. We can however, in this case draw an interesting conclusion in the context of discontinuous crack depth distributions.

Lemma: Let be $p_n(x) \geq 0$ and $\sum^{\infty} p_n(x)$ convergent but not uniformly convergent on an interval $\{I: x \in I, 0 \leq x \leq h\}$. Then $\bar{n}_k(x) = \sum_{n=L}^{\infty} p_n(x) n^k$, $k = \{1,2,\ldots\}$, $L \geq 0$, is a positive unbounded function of x on I.

Proof: Because of nonuniform convergence of the sum there exists an $x_M \in I$ and a fixed $\varepsilon > 0$ that for arbitrarily large $M > L$ we have $\sum_{n=M}^{\infty} p_n(x_M) \geq \varepsilon$. It follows

$$\bar{n}_k(x_M) \geq \sum_{n=M}^{\infty} n^k p_n(x_M) \geq M^k \sum^{\infty} p_n(x_M) \geq M^k \varepsilon \qquad \checkmark$$

According to the lemma the momenta $\bar{n}_k(a_D)$ of the numbers of cracks become infinite at least at one value of $a_D$ if $\sum p_n(a_D)$ doesn't converge uniformly. For physical reasons we must have $\bar{n}_k(a_D) \to \infty$ for $a \to +0$.

Types of crack depth densities:

Many authors assume that the probability density $f(a;0)$, eq. (2), is finite and smooth at $a = 0$, e.g., $f(a;0) = A \exp(-qa)$, refs. [12,13]. We will call these distribution densities continuous.

We call $f(a;a_D)$ delta-type distribution densities if $f(a;0)$ is the delta-function $\delta(a)$, e.g., $f(a;a_D) = \frac{1}{a} \frac{1}{\ln(h/a_D)}$

We are now able to discuss the conditions under which eq. (9) may be derived, as an approximation with the restrictions eq. (8), from the experimental failure probability, eq. (7), of our model.

1) Consider the case that the crack depth distribution density is of delta-type.

Case 1a): Assume $\sum^{\infty} p_n(a_D)$ is uniformly convergent in $0 \leq a_D \leq h$. Then it follows $P_F = 0$ (see eq. (10)), since in eq. (7) n-summation and the limiting process $a_D \to 0$ can be interchanged. On the other hand, if there are cracks of a detectable depth in a body, the expression, eq. (9), is unequal zero, even in the limit $a_D = 0$. Thus eq. (9) is not derivable from eq. (7) in case 1a).

Case 1b): Assume $\sum p_n(a_D)$ at $a_D = 0$ not uniformly convergent. Eq. (9) is not derivable from eq. (7) because, according to the lemma above the momenta $\bar{n}_k(a_D)$ diverge at $a_D = 0$.

2) Consider $f(a;0)$ is of the continuous type.

Case 2a): Assume $\sum p_n(a_D)$ is uniformly convergent for $0 \leq a_D \leq h$. In this case eq. (9) is consistent with eq. (7).

Case 2b): Assume $\sum p_n(a_D)$ is not uniformly convergent at $a_D = 0$. From the same arguments as in case 1b) eq. (9) is not derivable from eq. (7).

We conclude: For delta-type $f(a;a_D)$ the expression eq. (9) cannot be derived from eq. (7)

of our model. For continuous $f(a;0)$ mathematical consistency of eqs. (9) and (7) is possible depending on the uniformity of convergence of $\sum p_n(a_D)$.

## 5. An Example

We present a simple example with $\delta$-type crack depth density and nonuniform convergence of $\sum^{\infty} p_n(a_D)$. For the number probability, eq. (1), assume a poisson distribution with a parameter dependent on $a_D$.

$$p_n(a_D) = e^{-\mu(a_D)} \frac{1}{n!} \mu(a_D)^n \quad ; \qquad n = \{0,1,2,\ldots\} \quad ; \qquad 0 \leqq \mu(a_D) < \infty \quad \text{for} \quad a_D > 0 \quad ; \tag{11}$$

Let the depth distribution density be independent of n, $f_n(a;a_D) \equiv f(a;a_D)$. Then the sum in eq.(7) can be evaluated explicitly yielding the experimental failure probability

$$E_F(a_D;c) = \int_0^{K_D} g(K)dK + \int_{K_D}^{K_h} g(K) \; (1-e^{-\mu(1-F)})dK \tag{12}$$

with

$$F = F(c(K);a_D) = \int_{a_D}^{c(K)} f(a;a_D)da \; ; \qquad F(h;a_D) = 1$$

We specialize the distribution density to the $\delta$-type function

$$f(a;a_D) = \frac{1}{a} \; \frac{1}{\ln(h/a_D)} \tag{13}$$

For finite $\mu(a_D)$ at $a_D = 0$ the sum over $p_n(a_D)$ would be uniformly convergent and $P_F = 0$. (See eq.(10)). In fact, we have to choose $\mu(a_D)$ so that $\bar{n}(a_D)f(a;a_D) = \mu(a_D)f(a;a_D)$ is now independent of $a_D$ in order to be consistent with the requirements explained after eq. (9). Thus

$$\mu(a_D) = \ln(\frac{h}{a_D})^q \quad ; \qquad q > 0 \tag{14}$$

and we obtain from eqs. (7) and (10)

$$P_F = \lim_{a_D \to 0} E_F(a_D;c) = \int_0^{K_h} g(K) \; (1-(\frac{c(K)}{h})^q \;)dK \tag{15}$$

with $c(K)$ defined by eq. (6) and in fig. 1.

Expression (15) is obviously nonzero. From eq. (9) we would have expected

$$E_F^*(a_D;c) = \int_{K_D}^{K_h} g(K)q \; \ln(h/c(K)) \; dK \tag{16}$$

## 6. The Failure Rate

Another quantity, more important in practice than the failure probability is the failure rate of a component part. We demonstrate, under the special assumptions, eqs. (11,13,14) that also the failure rate has nontrivial values for $\delta$-type $f(a;a_D)$.

Let the failure rate be defined as the limiting value of an experimental failure rate which is defined for $a_D > 0$

$$\rho(t) = \lim_{a_D \to 0} r(t;a_D) = \lim_{a_D \to 0} \frac{\frac{d}{dt}(1-R(t;a_D))}{R(t;a_D)} \tag{17}$$

where t denotes time and $R(t;a_D)$ is the experimental reliability, i.e., an approximating expression, at $a_D > 0$, for the probability of no failure after $t = 0$. We again follow the simplifying model assumptions of ref. |2| with minor modifications. The non-reliability $1-R(t;a_D)$ is obtained by integration of the probability density, eq. (7), over an area in the $(b-K_c)$-plane as explained in fig. 2.

We assume that the component undergoes once a preoperational test with stresses much larger than those expected under operational conditions. The test removes all members of the population represented by points above the curve $b' = c_T(K_c)$, (see fig.1) representing the critical crack depth at fixed test load.

$$c_T(K_c) = K_I^{-1}(\sigma_{Test}, K_c) = (\frac{K_c}{s_T})^2 \tag{18}$$

The curve $b = c_{op}(K_c)$ represents the critical crack depth at average operational load

$$c_{op}(K_c) = (\frac{K_c}{s_{op}})^2 \quad ; \qquad s_{op} < s_T \tag{19}$$

Further we assume crack growth, accoridng to a Paris law, with a unique relation between the depth of a crack, $b_t$, at time t and its depth $b_o$ at $t = 0$.

$$b_t \equiv v(t,b_0) \qquad \text{with the inversion} \quad b_o = v^{-1}(b_t,t) \tag{20}$$

The line B in Fig. 2 indicates a possible bound of the crack depth below which cracks are unable to grow under operational cinditions.

The non-reliability at time $t_1 > 0$ is obtained by integration over an area (fig. 2) bounded by the curve, eq. (18), (critical crack depth at test) and by the curve

$$b' = b_0(K_c,t_1) \equiv v^{-1}(c_{op}(K_c),t_1) \tag{21}$$

which is the image of eq. (19) (critical crack depth at operational load) at $t = -t_1$.

From eqs. (17 - 21) we obtain an expression of the failure rate

$$r(t;a_D) = \frac{\int_{K_1(t)}^{K_2(t)} g(K)\ (-\frac{\partial b_0}{\partial t})\ \frac{\partial}{\partial b_0}\ (\sum_{n=1}^{\infty} p_n(a_D)F_n(b_0;a_D)^n)dK}{1-E_F(a_D;c_T) - \int_{K_1(t)}^{K_2(t)} g(K)\ \{\sum_{n=1}^{\infty} p_n(a_D)\ (F_n(c_T;a_D)^n - F_n(b_0;a_D)^n)\}dK} \tag{22}$$

with $\quad F_n(c_T(K);a_D) = 1 \quad$ at $\quad K > K_h$ ;

$F_n(b_0(K,t);a_D) = F_n(v^{-1}(c_{op}(K),t);a_D)$ ;

Using the special assumptions, eqs. (11, 13, 14), yields

$$r(t;a_D) = \frac{\int_{K_1(t)}^{K_2(t)} g(K)\left(-\frac{\partial b_o}{\partial t}\right)\frac{\partial}{\partial b_o}\left(\frac{b_o}{h}\right)^q dK}{1-\int_0^{K_2(t)} g(K)dK - \left(\frac{a_D}{h}\right)^q \int_{K_D}^{K_2(t)} g(K)dK + \int_{K_D}^{K_1(t)} g(K)\left(\frac{c_T}{h}\right)^q dK + \int_{K_1(t)}^{K_2(t)} g(K)\left(\frac{b_o}{h}\right)^q dK} \tag{23}$$

with

$$\lim_{a_D \to 0} K_D(a_D) = 0$$

which, for $a_D \to 0$, is easily seen to be nonzero if $K_1 < K_h$.

## 7. Conclusion

We have extended an earlier model for the assessment of the failure probability using linear fracture mechanics. The number of cracks is introduced as a random variable. The extended model is applicable to the case that the probability density of the crack depth in the delta-function at zero crack depth.

## 8. Acknowledgement

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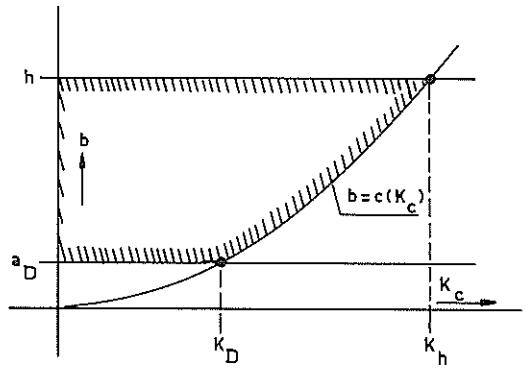


Fig. 1: Integration area of the experimental failure probability (shadowed). $K_c$: fracture toughness $K_{Ic}$; b : depth of the biggest crack on a given number of cracks; function $c(K_c)$ : critical crack depth at fixed load; $a_D$: threshold of crack detection; h:upper bound of crack depth.

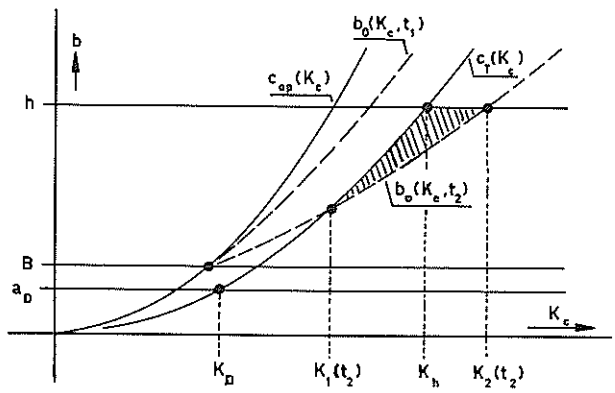


Fig. 2: Integration area of the non-reliability, $(1-R(t;a_D))$, after preoperational test (shadowed). $b=c_T(K_c)$ and $b=c_{op}(K_c)$: critical crack depths at test and operational load conditions respectively; $b=b_0(K_c,t)$ : image curves of $c_{op}(K_c)$ for the times $0<t_1<t_2$ ; B : threshold of crack growth; $a_D$ and h: see Fig. 1.