STRUCTURAL COLLAPSE DUE TO PLASTIC INSTABILITY

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In a number of circumstances the collapse and failure of structures is due to bifurcation of the load-deflection-path, as in the buckling failure of thin walled plate and shell structures subject to compressive loading. The aim of this paper is to give an overview of what is known theoretically about bifurcation induced structural collapse in which plastic deformation plays a significant role. This includes not only situations in which bifurcation takes place in the plastic range, but also situations in which plastic deformation limits the maximum support load of a structure that bifurcates in the elastic range. Particular emphasis is given to recent progress that has been made, in both analytical and numerical studies, in elucidating the combined effects of material nonlinearity and structural imperfections.

When bifurcation of a perfect structure takes place in the plastic range, assessments of imperfection sensitivity have been obtained based on an asymptotic analysis employing a material model that neglects the effect of elastic unloading. Examples are discussed which illustrate that, at least in certain circumstances, this analysis is capable of revealing the effect of imperfections on the overall buckling behaviour of structures, including indicating the circumstances in which a nominally imperfection insensitive structure, such as a flat plate, exhibits imperfection sensitive behaviour due to the destabilizing effect of plastic deformation. The utility of this type of analysis is assessed by comparing its predictions with those of numerical solutions which take full account of elastic unloading. Numerical results are also discussed which illustrate the role played by the combined effects of plastic deformation and imperfections in determining the ultimate load carrying capacity of a structure, when bifurcation of the corresponding perfect structure takes place in the elastic range.

At high temperatures, the effect of time dependent deformations plays an important role in determining the buckling behaviour of a structure. Here, the buckling process is initiated by small imperfections that grow due to creep, even at constant loads. In several cases, it has been possible to directly relate the creep buckling behaviour to the type of postbuckling behaviour exhibited by the structure in the absence of creep deformations.

In addition to buckling type failures, bifurcation induced structural collapse can occur under tensile loading, for example in internally pressurized spheres and cylinders and in rotating disks. A summary is given of the state of knowledge concerning tensile instability induced structural collapse, including the role of imperfections.
1. Introduction

Although the analysis of plastic instabilities dates back to the nineteenth century, it was not until the late 1940's that the fundamental distinction between loss of uniqueness and loss of stability in the plastic range was elucidated by Shanley [1] and von Karman [2]. Also, during the late 1940's and early 1950's it was noted that calculations based on the simplest deformation theory of plasticity gave much better agreement with test results than did similar calculations based on the simplest flow theory, leading to what was termed the "plastic buckling paradox". Attempts to resolve this paradox took essentially two distinct lines of attack; one approach pursued the justification of the deformation theory calculations in terms of a sophisticated flow theory involving the development of a vertex on the yield surface; the other approach sought the resolution of the paradox by ascribing the discrepancy between the simplest flow theory predictions and experimental observations to the presence of unavoidable imperfections. A comprehensive discussion of the history of this controversy, which has been revived by recent work on tensile plastic bifurcations, and the fundamental issues raised, is given by Hutchinson [3].

Despite this impasse, considerable progress has been made. Hill [4] developed a general framework for analyzing questions of uniqueness and bifurcation in the plastic range, for solids with cornered as well as smooth yield surfaces. A foundation has also been laid for a general theory of post-bifurcation phenomena in the plastic range, Hutchinson [3]. Furthermore, a number of recent studies have contributed to obtaining an understanding of imperfection sensitivity in plastic bifurcation problems (see e.g. Hutchinson [3], Tvergaard [5]). Until very recently, the role of bifurcation in precipitating structural collapse under tensile loading conditions was not much appreciated, although Hill's theory encompasses these phenomena, as well as the more familiar plastic buckling applications. With the exception of a brief review of some recent work on creep buckling, we confine attention to rate and time independent materials, for which Hill's [4] theory of bifurcation is applicable.

2. Plastic buckling

Analyses of plastic buckling for thin walled plate or shell structures under compressive loading are usually based on small strain elastic-plastic constitutive relations and on a standard nonlinear shell theory such as Donnell-Mushtari-Vlasov shell theory. Thus, a linear variation of strains through the shell thickness is assumed, and the stress state in the shell is taken to be approximately plane. In practically all the results to be discussed here $J_2$ flow theory is used to describe elastic-plastic material behaviour.

The classical buckling concept hinges on determining the critical bifurcation point, at which the fundamental state of deformation loses uniqueness. The actual behaviour observed at bifurcation varies, however, from gradual growth of buckles with further loading in some cases to sudden explosive collapse in other cases. For buckling in the elastic range these differences are very well explained by Koiter's general theory of elastic stability [6].

When bifurcation occurs in the plastic range the strongly nonlinear effect of elastic unloading enters as a considerable extra complication. The initial post-bifurcation behaviour is given by the following asymptotic expression for the load parameter $\lambda$ in terms of the buckling mode amplitude $\xi$ [3]
\[ \lambda = \lambda_c + \lambda_1 \xi + \lambda_2 \xi^{1+\beta} + \ldots, \quad \text{for} \quad \xi \geq 0 \quad (1) \]

where \( \lambda_c \) is the critical bifurcation load, \( \lambda_1 \) is positive, \( \lambda_2 \) is negative, and \( 0 < \beta < 1 \). This expression accounts for important effects of elastic unloading, but does not distinguish between structures with a stable post-buckling behaviour and structures, for which the collapse load is imperfection-sensitive.

Some assessments of the imperfection-sensitivity corresponding to bifurcation in the plastic range have been obtained on the basis of an asymptotic analysis employing a material model that neglects the effect of elastic unloading [7,8]. For such a hypoelastic material the post-bifurcation expansion (1) for the perfect structure is replaced by an expansion of the form

\[ \lambda = \lambda_c + \lambda_1 \xi + \lambda_2 \xi^2 + \ldots \quad (2) \]

The influence of small initial imperfections in the shape of the critical bifurcation mode has been studied for some cases, where symmetry results in \( \lambda_1^{he} = 0 \). Then, according to an asymptotic analysis for the hypoelastic structure, no imperfection-sensitivity is found for a structure with stable initial post-bifurcation behaviour (\( \lambda_2^{he} > 0 \)), whereas for negative \( \lambda_2^{he} \) the reduction of the snap buckling load \( \lambda_s \) due to an imperfection of amplitude \( \xi \) is given by the asymptotic estimate

\[ \frac{\lambda_s}{\lambda_c} = 1 - \mu \xi^2/(2\psi+1) \quad (3) \]

where \( \mu > 0 \) and \( \psi > 1 \). Thus, according to the hypoelastic approximation of elastic-plastic material behaviour, the imperfection-sensitive structure is distinguished from the insensitive structure by the sign of \( \lambda_2^{he} \). This is analogous to elastic buckling results, and in fact for an elastic material eqns. (2) and (3) reduce to well-known equations of the elastic post-buckling theory [6].

An attempt to obtain a different asymptotic estimate of the maximum support load with imperfections has been made for a spring model of an elastic-plastic column [9]. This approach employs an expansion of the locus of maxima around the reduced modulus load and is found quite useful for the model studied; but generalization of the approach to real structures has not yet been carried out.

Predictions of the hypoelastic expansion (2) have been compared with numerical computations taking full account of elastic-plastic material behaviour. For simply supported square plates under axial compression, either with the edges constrained to remain straight or without in-plane constraints, the stable elastic post-buckling behaviour is also found in the plastic range for high hardening materials as shown in Fig. 1. Here the yield stress \( \sigma_y \), Young's modulus \( E \) and the strain hardening exponent \( N \) refer to a uniaxial stress-strain behaviour represented by the special power hardening law

\[ \sigma = \begin{cases} E \epsilon, & \text{for} \quad \sigma < \sigma_y \\ \sigma_y \left( \frac{E}{Ng'_y} \epsilon + 1 - \frac{1}{N} \right)^N, & \text{for} \quad \sigma > \sigma_y \end{cases} \quad (4) \]

in which the tangent modulus is continuous. For low hardening materials the plates are

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imperfection-sensitive [7]. This is also predicted by the hypoelastic expansion, which agrees nicely with the general trends of the elastic-plastic post-bifurcation behaviour at deflections up to one or two plate thicknesses, although the details near the bifurcation point described by (1) are not accounted for in (2). In particular for the case in Fig. 1 the hypoelastic expansion shows the stable post-bifurcation behaviour for constrained edges and the slight imperfection-sensitivity without constraints. A similar good agreement has been found for axially compressed cylindrical panels occurring between stiffeners in a longitudinally stiffened circular cylindrical shell [8]. Thus, although the hypoelastic material model is certainly not in general valid for elastic-plastic structures, it does give a very useful indication of the actual elastic-plastic buckling behaviour.

A more complex buckling behaviour is found for an oval cylindrical shell under axial compression [10]. Here, the asymptotic estimate of the imperfection-sensitivity is not even valid in the elastic range, for sufficiently eccentric cross-sections, as the initially unstable behaviour is followed by a stable advanced post-buckling behaviour with a maximum support load above the primary bifurcation load. For elastic-plastic oval shells the initial drop in post-bifurcation load is still found, but the high subsequent load carrying capacity has vanished due to the effect of the material nonlinearity, so that even for a rather high hardening material the shells are moderately imperfection-sensitive.

A different example of a structure for which the plastic buckling behaviour has been investigated in detail is a wide eccentrically stiffened panel under axial compression [11]. In the elastic range this structure is imperfection-sensitive due to nonlinear interaction between column buckling and local buckling of the plate between the stiffeners, and the material nonlinearity adds to this effect. Furthermore, plasticity completely changes the asymmetry of the post-bifurcation behaviour, in addition to increasing the sensitivity to imperfections.

When bifurcation of the perfect structure takes place in the elastic range just below the yield stress, asymptotic estimates of post-bifurcation behaviour and of imperfection-sensitivity are entirely based on elastic theory. However, in such cases the actual buckling
behaviour often follows the general trends of plastic buckling, and even results in a relatively high imperfection-sensitivity due to rapid imperfection growth prior to initial yielding [10,11,12]. Results for an eccentrically stiffened panel with simultaneous buckling in the column mode and the local plate mode a little below the yield stress are shown in Fig. 2. In the range of negative $\xi_w$ elastic panels are insensitive to imperfections, but in Fig. 2 the largest sensitivity is found in this range, just like when bifurcation occurs in the plastic range. On the other hand, for very highly imperfection-sensitive structures such as the axially compressed circular cylindrical shell, the maximum support load may be attained before plastic yielding sets in [3].

3. Creep buckling

At high temperature and complex deformations have an important influence on the buckling behaviour of a structure. Here the buckling process is initiated by small imperfections that grow due to creep, even at constant loads, until structural failure occurs at a certain critical time. Thus, creep buckling is not a stability problem in the classical sense, but rather a question of determining how long time it takes, before failure occurs.

In the computations of creep buckling to be discussed here the total strain $\eta_{ij}$ is taken to be the sum of the elastic, plastic and creep strains

$$\eta_{ij} = \eta^E_{ij} + \eta^P_{ij} + \eta^C_{ij}$$

so that creep and plasticity are considered as independent processes. Only secondary creep is considered, as described by the generalized Norton's law, and the simplest flow theory of plasticity is employed.

The behaviour observed in creep buckling is directly related to the type of post-buckling behaviour exhibited by the structure in the absence of creep. After the load has been applied, creep adds to the permanent deformations of the structure, and simultaneously the elastic and plastic strains are redistributed. Now, if the structure is insensitive to imperfections in the absence of creep, as for elastic plates [13,14], the growth of the
permanent deformations does not lead to sudden collapse. In such cases the creep buckling process is merely characterized by deformations growing continuously, until the structure ceases to be useful for engineering purposes. If, however, the structure is imperfection-sensitive in the absence of creep, the permanent deflections may grow to a level, at which collapse occurs instantaneously under the given loads.

The axially compressed circular cylindrical shell is one example of an imperfection-sensitive structure, for which the creep buckling behaviour has been investigated. When axisymmetric imperfections are considered, bifurcation into a nonaxisymmetric mode will usually occur after some creep, and collapse occurs instantaneously, if the corresponding post-bifurcation behaviour is unstable [15]. Also for an axially compressed rectangular plate made of a sufficiently low hardening elastic-plastic material the imperfection-sensitivity in the absence of creep [7] results in sudden collapse after some creep [14]. It should be noted that the load levels corresponding to sudden creep buckling collapse are in the range of maximum support loads obtained for various imperfection levels in the absence of creep. For lower load levels the creep buckling behaviour is similar to that of insensitive structures.

4. Plastic collapse under tensile loading

In recent years, Hill’s bifurcation theory [4] has been increasingly utilized in the solution of tensile bifurcation problems. Imperfection-sensitivity has been less widely explored. Most of what is known concerning imperfection-sensitivity under tensile loading conditions is very recent and has focussed on applications to metal forming, material testing and questions related to the ultimate limits set on ductility (see e.g. Needleman and Rice [16] and references cited therein).

Here, we shall focus on a few selected problems, which are of potential significance in structural applications. Our discussion will be confined to consideration of results and the implications for structural analysis of these results. It should be noted at this point that a fundamental unresolved issue that arises in these tensile bifurcation problems concerns the appropriate description of the multi-axial constitutive behaviour of metals after a finite pre-strain. The specific results discussed here are unusual in that they are relatively insensitive to the details of the plastic constitutive model employed.

First we consider a thin walled spherical shell subject to internal pressure loading. One available state of deformation corresponds to a spherically symmetric expansion with the pressure, \( p \), given by

\[
p = 2 \left( \frac{t}{R} \right) \sigma \tag{6}
\]

Here, \( t/R \) is the current thickness to radius ratio and \( \sigma \) is the true stress (i.e., the principal membrane stresses are \( \sigma_1 = \sigma_2 = \sigma \)). Neglecting the small effect of elastic compressibility, and assuming that the material exhibits pure power law hardening in equal biaxial tension so that,

\[
\sigma = \epsilon^N \tag{7}
\]

where \( \epsilon \) is the natural strain (i.e., the principal membrane strains are \( \epsilon_1 = \epsilon_2 = \epsilon \)), eqs. (6) and (7) imply that the maximum pressure is reached when the radius of the spherical
shell has increased to the value $R_{\text{max}}$ given, in terms of the initial radius, $R_0$, by

$$R_{\text{max}} = R_0 e^{N/3}$$  \hspace{1cm} (8)

Conventionally, the attainment of this pressure maximum is associated with instability. This is appropriate only if the pressure is actually the controlled quantity.

The possibility of bifurcation from the spherically symmetric deformation state has been investigated, employing various plastic constitutive models and in varying degrees of generality, by Miles [17], Strifors and Storåkers [18] and Needleman [19]. For sufficiently thin shells, with the material characterized by eq. (7), these analyses all give the result that bifurcation is possible when $R = R_b$, where

$$R_b = R_0 e^{N/2}$$  \hspace{1cm} (9)

The bifurcation mode deflections correspond to local thinning at one pole with a corresponding thickening at the opposite pole.

From eq. (8) and eq. (9) for a power law hardening material, the radius of the spherical shell at bifurcation, $R_b$, is related to that at the maximum pressure, $R_{\text{max}}$, by

$$\left(\frac{R_b}{R_0}\right) = \left(\frac{R_{\text{max}}}{R_0}\right)^{3/2}$$  \hspace{1cm} (10)

Obviously, from eq. (10) bifurcation takes place at a later stage of the deformation history than does the maximum pressure. However, as discussed by Needleman [19], the bifurcation result, eq. (9), applies when the internal volume of fluid is regarded as the controlled quantity, as well as to the case when the pressure is taken to be the controlled quantity. In this sense, eq. (9) provides an inherent limit to the persistence of the spherically symmetric deformation state in a spherical shell subject to internal pressure loading.

However, even when the pressure is controlled, significant growth of an initial imperfection in the shape of the preferred bifurcation mode can take place prior to the bifurcation point. Numerical computations for imperfect shells in [20] were restricted to axisymmetric behaviour (the preferred bifurcation mode is an axisymmetric one), but full account was taken of variations through the shell thickness. The material was characterized by the simplest flow theory of plasticity with isotropic hardening and with the stress-strain law (4). Fig. 3 depicts results for a thin shell with the geometric and material parameters given in the figure caption. In Fig. 3a, curves of pressure versus a measure of the increase of average internal radius are shown for various initial imperfection magnitudes, $\epsilon^*_{\text{m}}$. The initial imperfection, $\epsilon^*_{\text{m}}$, is given by $(1 - t_{\text{min}}/t_{\text{nom}})$ where $t_{\text{min}}$ is the minimum initial thickness, $t_{\text{nom}}$ is the nominal initial thickness, and imperfection shape is that of the preferred bifurcation mode. The curves in Fig. 3a show that for an imperfect shell, the maximum pressure is only slightly reduced, but that this maximum pressure occurs at a possibly significantly reduced value of the current radius. The lower curves in Fig. 3b, show noticeable thinning prior to the maximum pressure point. We are not aware of any experimental evidence against which to compare the predictions of this analysis. However, for the closely related problem of an internally pressurized long cylindrical tube, the experiments of Larsson [21] reveal that collapse involves a mode analogous to that found in the spherical.
Fig. 3. (a) Pressure versus increase of average internal radius for spherical shell with initial thickness to radius ratio $0.002$ ($\sigma_y/E = 0.005$, $N = 0.125$). (b) Variation of current thickness at the two poles.

shell bifurcation problem. Chu [22] has recently carried out a bifurcation analysis for long cylindrical elastic-plastic shells. For thin cylindrical shells, the relation analogous to eq. (10) that emerges from Chu's [22] analysis is

$$\frac{R_b}{R_0} = \left( \frac{R_m}{R_0} \right)^2$$  \hspace{1cm} (1)$$

For moderately thick cylindrical shells, which are expected to be less imperfection-sensitive than thin shells, Storakers [23] has noted at least qualitative agreement between experimental observation and the bifurcation predictions of Chu's analysis [22].

A somewhat different example is provided by Tvergaard's analysis [24] of the influence of necking behaviour on the burst strength of rotating annular disks. Experiments, in which disks have been stopped just prior to bursting, show that necking has initiated at the bore, Percy, Ball and Mellor [25]. Conventionally, theoretical analyses of the burst behaviour of rotating disks have been carried out presuming an axisymmetric deformation state in the disk. However, the bifurcation analysis in [24] showed that a point of bifurcation from the axisymmetric state could be attained before the point of maximum angular velocity. The predicted critical bifurcation mode, apart from the critical speed of the rotating system that is usually passed in the elastic range, involves local thinning at two diametrically opposite points of the bore. This agrees with the experimental observations of Percy et al. [25].

Initial non-axisymmetric imperfections were also considered within the framework of the plane stress approximation. Fig. 4 shows results for imperfect disks, employing material and geometric parameters chosen to model an experimental configuration studied by Percy et al. [25]. The imperfection consists of a circumferential variation of the yield stress $\sigma_y$ in a
wave length corresponding to the preferred bifurcation mode, and $\Delta \sigma_y/\sigma_{y0}$ denotes the imperfection amplitude relative to the nominal yield stress. The curves in Fig. 4 plot the square of angular velocity, $\omega^2$, normalized by the square of a conveniently chosen reference angular velocity, $\omega_0^2$, ($\omega_0$ depends only on the disk geometry and material properties) against the average expansion of the bore, $U/R_1$, normalized by the inner edge radius, $R_1$. For comparison purposes the axisymmetric solution for a perfect disk is also shown in this figure. The points marked x indicate the maximum angular velocity, while the points marked o indicate the point of stationary angular momentum. For the imperfect disks the point of stationary angular momentum occurs near the maximum of angular velocity. However, for the axisymmetric state no point of stationary angular momentum was found. Thus, ductile bursting does not occur in the axisymmetric mode. Consequently, bifurcation is essential for bursting, even though relatively little growth of the bifurcation mode amplitude takes place prior to bursting.

References


