ON HISTORY DEPENDENCE OF STRESS–STRAIN DIAGRAMS AND CREEP CURVES UNDER VARIABLE REPEATED LOADING

D. A. GOKHFELD, O. S. SADAKOV, M. E. MARTYENENKO
Chelyabinsk Polytechnical Institute, Lenin Avenue 76, Chelyabinsk 454044, U.S.S.R.

ABSTRACT

The ability of structural alloys to "keep in memory" the loading prehistory becomes of special importance when inelastic variable repeated loading is considered. There are two main approaches to the development of the mathematical description of this phenomenon: the inclusion of hidden state variables in the incremental theory constitutive equations (\(\varphi\)) and construction of proper hereditary functionals (\(\beta\)). In this respect the assumption that the "memory" regarding the previous deformation history is due to structural nonhomogeneity of actual materials proves to be fruitful.

A version of structural model reflecting formally the nonhomogeneity is considered below with emphasis on the description of history-dependent effects. As it is assumed an infinitely small element of medium acts like an assemblage of subelements having equal values of total strain, temperature and elastic moduli. The inelastic strain rate of any subelement depends on its stress and temperature exclusively. The corresponding functions of all the subelements considered to be similar to the rheological function of the material. The latter function represents the dependence of secondary creep rate upon stress and temperature. Nonhomogeneity of medium is reflected by the similarity coefficients distribution within the element of a body. Both determinative functions of the model (the rheological function and the distribution density) can be obtained for a certain material employing the corresponding test data.

The experimental data show that within the operating temperature range the rheological functions obtained for different structural alloys have a specific form: strain rate is negligibly small under moderate stresses and grows sharply with the stress approach to some limit value. Then the subelement stress–strain diagram turns to be similar to that of elastic perfectly plastic body. This peculiarity allows to simplify essentially the corresponding analysis. As a result the similarity principle has been formulated to determine the material deformation behaviour under proportional variable nonisothermal loading programs including hold-time periods.
1. Introduction

The history-dependent effects play a leading role in structural alloy deformation behavior under arbitrary variable repeated loading and temperature conditions. But these phenomena are completely ignored in the theories of steady creep and perfect plasticity. To improve this Rabotnov and other investigators have suggested to introduce some hidden state variables into the corresponding constitutive equations. The same objective was pursued by the authors of the theories of hereditary type. The advantages and limitations of the mentioned two approaches are discussed in ref. [1].

Experiments show that the material "memory" is not of continuous but of discrete character; there exist certain turning-points in loading history whose influence on the subsequent deformation behavior is of prime importance. The well known Masing principle is the most simple example of such "memory" but it does not embrace various situations occuring under variable repeated non-isothermal loading (see, for example, [2]).

In the present paper the results of structural model behavior analysis are discussed. The model is intended to reflect time-dependent properties of actual structural alloys. For the case of proportional loading the mentioned analysis permits to formulate the corresponding state equations and memory rules in the form simple enough to employ in engineering applications.

2. A structural model of medium

According to the structural model [3-5] the behavior of a body element under proportional loading is described by the following equations:

\[ \varepsilon = r + p = \varepsilon^K = r^K + p^K \quad (1) \]

\[ r = \sigma/E; \quad r^K = \sigma^K/E \quad (2) \]

\[ p^K = \Phi(r^K, z^K, T) \quad (3) \]

\[ \sigma = \langle \sigma^K \rangle = \int_0^{z^K} \sigma(z^k) dz^k \quad (4) \]

Here \( \varepsilon, r, p \) are total, elastic and plastic strain respectively, \( \sigma \) -stress, \( E \) - Young's modulus, \( T \) - temperature, \( z^K \) - similarity parameters having constant values for each subelement, \( y(z^K) \) - distribution function, \( \Phi \) - rheologic function. The corresponding parameters of subelements are marked by index \( k \).

As it was shown [3,4] the function \( \Phi \) coincides with the dependence of the steady creep rate on stress and temperature

\[ \dot{\varepsilon}_{st} = \Phi(\sigma/E, T) \quad (5) \]

Experiments revealed that the isothermal sections of the surface (5) may be schematically represented by a curve in Fig.1. Then the \( r^k-\varepsilon \) curve for a sub-element (at a certain strain rate \( \dot{\varepsilon} \) and temperature \( T \)) is close to the perfectly plastic stress-strain diagram (Fig.2). The ultimate value of elastic strain is

\[ r_{u}^k = z^K \Phi(\dot{\varepsilon}, T) \quad (6) \]
where \( \Phi \) is the inverse function \( \Phi(\tilde{\Phi}(x,T),T) = x \). With the variation of the rate \( \dot{\varepsilon} \) and/or temperature \( T \) the values of \( r_u^k \) for all the subelements change in equal proportion. It follows that if the curve \( r_\varepsilon \) of the element is described by the function \( F \) related with some base values of \( \dot{\varepsilon}_0 \) and \( T_0 \)

\[
\dot{\varepsilon} = \dot{\varepsilon}_0; \quad T = T_0; \quad r = F(\varepsilon)
\] (7)

then the diagram corresponding to some other values of \( \dot{\varepsilon} \) and \( T \) is centrally similar (homothetic) to the function \( F \) with similarity coefficient \( \eta \)

\[
\dot{\varepsilon} = \text{const}; \quad T = \text{const}; \quad r = \eta F(\varepsilon/\eta); \quad \eta(\dot{\varepsilon},T) = \frac{\tilde{\Phi}(\dot{\varepsilon},T)}{\tilde{\Phi}(\dot{\varepsilon}_0, T_0)} = \frac{r_u}{r_u^0} \] (8)

where \( r_u = r_u^k \) and \( r_u^0 = r_u^k \) are ultimate elastic strains of an element.

3. The state equations

The calculation analysis of model behavior under loading cycles with hold-time periods has revealed an important regularity. The latter can be illustrated by a particular example that follows (Fig.3). Regardless whatever way the state point reaches position \( A \) either as a result of creep under constant stress \( OBA \), stress relaxation \( ODA \) or an intermediate trajectory \( OCA \) the stress distribution among the subelements proves to be practically identical. This distribution in all the mentioned cases is the same as after a monotonous deformation process at a certain strain rate (the dashed line, Fig.3). Hence the creep strain rate \( \dot{\varepsilon} \) at this very instant is defined by point \( A \) coordinates \( (r,\varepsilon) \) and temperature \( T \) exclusively, and it is independent of the loading history. Thus the creep rate can be determined by the strain rate \( \dot{\varepsilon} = \nu \) and the tangential modulus at point \( A \), such as if it were the loading case \( OA \):

\[
K = \frac{\partial r}{\partial \dot{\varepsilon}} = r/\dot{\varepsilon} = (\dot{\varepsilon} - \dot{r})/\dot{\varepsilon}; \quad \rho = \dot{\varepsilon}(1-K) = \nu F'(C)
\] (9)

It has been taken into account that the tangential modulus \( K \) for the whole family of centrally-similar curves \( r = \eta F(\varepsilon/\eta) \) is defined by the secant modulus \( C = r/\dot{\varepsilon} \) only.

The function \( F(C) = 1-K(C) \) can be easily determined by the base curve \( F \) (line \( OA' \), Fig.3). On the other hand each value of the secant modulus \( C \) is associated with the single point on the base curve (point \( A' \), Fig.3) with the coordinates \( \varepsilon_b = F_b(C); r_b = F(\varepsilon_b) \). This point allows to obtain the similarity coefficient \( \theta = OA/OA' \) for the diagram \( OA \) with respect to the base curve \( F \). Then the strain rate can be found from Eq.(8).

Thus

\[
\dot{\rho} = \Phi(r_u^0 \theta, T) F'(C)
\] (10)

where

\[
C = r/\varepsilon; \quad \theta = \varepsilon_b/\varepsilon = r/F(\varepsilon_b); \quad \varepsilon_b = F_b(C)
\] (11)

The creep rate at a certain constant temperature \( T \) can be defined as the field on the plane \( \{r,\varepsilon\} \). The latter can be considered as product of two fields: the field \( \Phi(r_u \theta, T) \) having the stress-strain curves (8) as
the level lines and the field \( F_1(C) \) the level lines of which are the radials spreading from the coordinate centre (Fig. 4). Numbers on the first family lines denote the values of \( \theta \) while on the second - the values of the secant modulus \( C \).

The obtained state equation (10) contains two determining functions of material: the rheological function (Eq. (5)) and the stress-strain curve or, to be more exact, the functions \( F_1 \) and \( F_2 \) which are formed by \( F \).

Under initial loading with constant strain rate and temperature the solution of Eqs. (1), (10), (11) reduces to \( \theta = \eta(\epsilon, T) \) and the stress-strain curve (8) results.

4. The "turning-points" of the loading trajectory

Assume that the coefficient \( \eta(\epsilon, T) \) increases or changes its sign starting with a certain time instant (let us call it the turning-point).

Then Eqs. (10), (11) cease to reflect the actual model behavior. The analysis analogous to the previous one shows that Eq. (11) is to be replaced by the following

\[
C = r^* \epsilon^*; r^* = r - r_m; \epsilon^* = \epsilon - \epsilon_m; \eta^* = \eta - \eta_m; \eta^* = \epsilon^*/\epsilon_m^*; \epsilon^*_o = F_2(C). \tag{12}
\]

Here \( \eta_m, \epsilon_m, r_m \) denote the values of the parameters \( \eta, \epsilon, r \) at the turning-point. Fig. 5 illustrates the level lines of the multipliers \( \phi \) and \( \epsilon \), corresponding to a certain triple of numbers \( \{r_m; \epsilon_m; \eta_m\} \). These lines are similar to those of the initial state. In the coordinate system \( \{r^*, \epsilon^*\} \) the second family of lines \( F_1 \) coincides with the initial one while the former \( \phi \) proves to be centrally-similar to the corresponding initial family. For example, line \( \eta^*_m = 0, \epsilon^*_o \) stretching from the turning-point has a similarity coefficient equal to \( 0, \epsilon^*_m - \epsilon_m \) with respect to the initial line having the same value of \( \theta = 0, \epsilon \). The creep rates at the same temperature, for example, at points \( B \), Fig. 5 are equal in absolute values.

Eq. (12) is true not only after the first turning-point but also after the second, the third etc. turning-points. In this case the numbers \( \{r_m; \epsilon_m; \eta_m\} \) correspond to the last turning-point while all the preceding ones (and the associated values of the secant modulus \( C \)) have also to be kept in the material "memory". Moreover Eq. (12) is also true for the description of the initial loading if the four memory numbers \( \{m\} = \{r_m; \epsilon_m; \theta_m; \eta_m\} \) are to be considered equal to \( \{m_0\} = \{0; 0; 0; 0\} \).

The turning-points emerge and have to be kept in memory each time when the following conditions are violated:

a) \( \epsilon^* d\epsilon^* \geq \epsilon^* d\theta^* \);

b) \( \epsilon^* d\epsilon^* \geq 0 \). \tag{13}

Certain situations when the turning-points are to be excluded from memory are possible. This takes place when the running value of \( C \) reaches \( C_m \); then the last turning-point is forgotten and it is replaced by the preceding one. If the inequality \( |\theta| < \theta_{m-1} \) is fulfilled along with the equality \( C = C_m \) (where \( \theta_{m-1} \) is related with the last but one turning-point)
the last two turning-points are "forgotten" simultaneously. For example Eqs.(10), (12) with a steady set \([M]\) are true as long as the point \([\mathcal{C}_n, \mathcal{C}_m]\) is between the radials 1 and 0. At an instant when \(C = \mathcal{C}_m\) the set \([M]\) is excluded from the memory. The set \([m_j]\) remains and Eq.(12) reduces to (11). The corresponding level lines are illustrated in Fig.4.

5. Nonisothermal loading

When loading with the prescribed constant values of \(\dot{\varepsilon}\) and \(T\) is considered the coefficient \(\eta\) is to be determined from Eq.(8).

Then the stress-strain diagram corresponding to a certain level line of the first family results:

\[
\mathcal{R}^* = \eta^*(\varepsilon^*/\eta^*) ; \eta^* = \eta(\dot{\varepsilon}, T) - \eta_m \tag{14}
\]

If under variable-repeated loading \(|\dot{\varepsilon}|\) and \(T\) are constant Eq.(14) reduces to

\[
\eta_m = \pm \eta(|\dot{\varepsilon}|, T), \quad \eta^* = \pm 2\eta(|\dot{\varepsilon}|, T) - \text{ the Masing principle. The turning-points coincide with loading reverse points. At time-instants when } C = \mathcal{C}_m \text{ a pair of the last loading reversals are excluded from the memory - except the case when there is only one set in } [m_0] = [0; 0; 0] \text{ in the memory.}
\]

6. Example: determination of stress-strain diagram due to a complex loading history

Consider the following loading history (Fig.6a): the initial loading at temperature \(T_1\) continues until the strain \(\varepsilon_f\) is reached; then unloading and reverse loading with simultaneous change of temperature \(T_1 \rightarrow T_2\) is performed; after stress \(\sigma_2 = E(T_2)\sigma_2\) is reached a hold-time period takes place; the stress is constant till the moment the creep strain becomes \(\varepsilon = \varepsilon_2\) : the last stage includes one more loading reversal with a simultaneous temperature change \(T_2 \rightarrow T_1\).

The initial loading is characterized by the diagram \(OA\) (Fig.6b) which is centrally-similar to the base curve \(F\), the similarity coefficient being \(\theta(T_i)\). With the reverse at point \(A\) the values \([\mathcal{R}_A; \varepsilon_A; \theta_A = \theta(T_1)); \mathcal{C}_A = \varepsilon_2/\varepsilon_1\) are remembered. Section \(AB\) of the stress-strain curve \(r = f(\varepsilon)\) in the coordinate system \((r^*; e^*)\) is described by the following equation

\[
r^* = \theta^*(\varepsilon^*/\theta^*) \tag{15}
\]

where \(r^* = r - \mathcal{R}_A; \varepsilon^* = \varepsilon - \varepsilon_1; \theta^* = \theta(T) - \theta_A\)

When \(r = \mathcal{R}_B, \varepsilon = \varepsilon_2\) (point \(B\)) there occurs a creep stage. This point is not to be considered as a turning-point as the conditions (13) are not violated (\(\theta^*, \varepsilon^*, d\varepsilon^*/d\theta^* < 0)\). The more creep strain accumulates the farther the point \((r, \varepsilon)\) displaces along line \(r = \mathcal{R}_B\). Then the secant modulus \(C = \frac{r - \mathcal{R}_A}{\varepsilon - \varepsilon_1}\) decreases as well as the value \(\theta^* = \frac{\varepsilon - \varepsilon_A}{\varepsilon - \varepsilon_1}\) does up to the moment these values become \(C^* = (\mathcal{R}_B - \mathcal{R}_A)/\varepsilon_2 - \varepsilon_1), \theta^* = (\varepsilon_2 - \varepsilon_1)/\varepsilon_2 - \varepsilon_1\) \(\theta^* = \theta(T) - \theta_A\) at point \(D\). The latter is a turning-point. It is meant that the second turning-point parameters \([m_2] = [T_2; \varepsilon_2; \theta_2; \varepsilon_2]\) are to be fixed in memory.

The subsequent loading \(\varepsilon_2 \rightarrow \varepsilon_1; T_2 \rightarrow T_1\) follows the law \(\theta^* = \theta(T) - \theta_2\).
At the end of this stage $\theta = \theta(T_f) - \theta(T_i) + \frac{\Delta \varepsilon}{\varepsilon_b(C_b)} = (\varepsilon_f - \varepsilon_2)/\varepsilon_b(C_b) = -\theta_D$. Thus a point corresponding to the loading stage end is on the same curve with respect to point $D$ as point $D'$ with respect to point $A$. Hence regardless the value $\varepsilon_2$ the loading cycle will be closed at point $A$.

However if the creep stage ended at point $D'$ which is to the left of line $OA$ the secant modulus would reach the value of $\varepsilon_A$ on crossing line $OA$. Then the turning-point $A$ would be excluded from the memory. After line $OA$ being crossed creep proceeds in such a way as it were preceded by the initial loading up to the value $r = \tau_{D}$ at $T = T_2$. With the loading reverse at point $D'$ the deformation curve will come above point $A$ (see plotted line in Fig. 6b).

7. Conclusion

On the basis of the structural model behavior analysis in the case of proportional cyclic loading programs which can comprise temperature changes and hold-time periods it becomes possible to formulate a rather simple principle defining the corresponding stress-strain diagrams and creep curves. By this principle the rules of "keeping in memory" and "forgetting" the turning-points of the loading trajectory have been established. As the central similarity of stress-strain curves with respect to the turning-points is of prime importance the mentioned set of rules can be referred to as similarity principle.

The experimental verification shows that all the main deformation peculiarities of cyclically steady (or stabilized) structural alloys are described by the model with satisfactory adequacy. The deformation processes are meant to take place at temperatures within the operation temperature range for the corresponding alloys.

In a more general case of non-proportional loading the corresponding response established for each particular program only by means of calculations employing a properly generalized structural model. Probably there are reasons to count on the adequacy with the experimental data in this case as well [7].

References


[3] Sadakov O.S. "The stress-strain analysis of structural elements under cyclic nor-isothermal loading on the basis of the structural model of medium" In: Trans. of the All-Union Simpos. on Low-Cycle Fatigue at


