PROBABILISTIC METHODS IN PLASTIC STRUCTURAL ANALYSIS

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SUMMARY

This paper reviews some applications of probabilistic methods to plastic structures: only a brief account is however possible in this "compact" text, which is derived from a recent Lecture of wider scope /1/.

The distinction between elastic (brittle or weakest link) behaviour, and plastic (fail safe or alternate path) response is underlined first: the redistribution of stresses, although beneficial from the safety viewpoint, introduces another source of uncertainty in random strength structures, which makes rigorous probabilistic calculations of elastic-plastic structures pratically impossible. In fact, all codified methods for limit-state safety calculations allow only "local" checks, and leave undefined the actual overall reliability of the structure.

Practicable procedures for structural reliability calculations have however been developed for structures that satisfy the classical assumptions of plastic limit analysis. In particular, two theorems that allow to find rigorous upper and lower bounds on the probability of full plastic collapse under given loads, are presented. Other methods for probabilistic limit analysis are also indicated, including in particular a specifically developed parametric simulation procedure, which was recently extended to find the probability distribution function of the permament displacement at incipient plastic collapse.

The last part of the paper is devoted to the reliability analysis of plastic structures subject to loads varying (slowly) in time.

It is recalled first that probabilistic limit analysis can be easily extended to the shakedown/incremental collapse problem, provided the loads vary within a finite domain (as in the "classical" shakedown problem).

However, the significance of such an approach for stochastically varying loads is questioned. In fact, as time increases, the probability also increases that the loads cross any given threshold. Therefore, it is more appropriate to speak of "plastic adaptation" rather than "shakedown", and to focus the attention on the probability of reaching, in any given time interval, a certain permanent deformation. Again, only approximate solutions (in the form of upper and lower bounds) can be found to this question, but this appears to be a more rational and promising approach to the problem.

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1. Introduction

In the assessment of safety and reliability of a given structure with respect to any limit state, both the applied loads and the structural properties (yield stresses, geometrical imperfections, etc.) should always be regarded as affected by random uncertainties. If the structural strength does not deteriorate in time, the probability of failure can be in general expressed by

$$P_{fail} = \int_L f(\underline{W})\ P(\underline{W})\ d\underline{W} \qquad (1)$$

where $f(\underline{W})$ is the joint probability density function (PDF) of $\underline{W}$, the vector of the applied load parameters (or of their peak values, appropriately defined), L the region of definition of the loads, and $P(\underline{W})$ the conditional probability of failure (in the considered limit state) under loads $\underline{W}$ /2/. In this context, structural analysis is identified with the determination of the CPF function $P(\underline{W})$, which - in case of deterministic strength - is a 0-1 function whose discontinuity surface defines the failure locus or, when the loading system depends on only one scalar parameter, the failure load parameter.

2. Preliminary considerations on inelastic structural response

Let us refer explicitly to structures composed by (or discretized into) many elements.

If structural failure is identified with the first violation of a strength inequality in any point or element of the structure, the structural resistance is governed by the weakest link (such a structure is sometimes defined fully brittle with respect to the limit state under consideration): the probability of failure of the whole structure is given by the union of the probabilities of failure of each constitutive element. Typical example of this behaviour is the chain (Fig. 1a), but is worth noting that most (if not all) established methods for reliability-based design prescribe only local resistance verifications, thus implicitly assuming weakest link behaviour, although sometime stress redistribution is indirectly taken into account through the introduction of semi-empirical correction factors.

However, most often this way of operating leaves great uncertainties with regard to the actual overall reliability of redundant structures, especially when material and structural detailing are such that significant plastic deformations can take place. Indeed, the stress redistributions consequent to plastic deformations are in most cases beneficial to the overall structural strength, and may greatly reduce the effects of unexpected local weaknesses (which could be disastrous in case of weakest link behaviour) because the stresses can find alternate paths to resist the applied loads: this type of behaviour is also referred

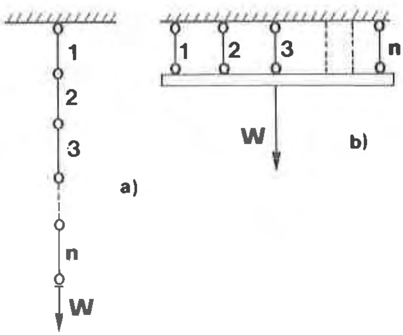

FIG. 1:
Limit cases of ultimate structural behaviour;
a) weakest link (brittle chain);
b) fail-safe (ductile bundle).

to as ductile or fail-safe. A typical, althoug limit, example of this behaviour is the bundle of elastic-perfectly plastic elements diagrammatically shown in Fig. 1b: it is easy to show that, if the yield loads of the n elements are statistically independent, with common average $\bar{R}_1$ and coefficient of variation V_1, the average and coefficient of variation of the limit load of the bundle are respectively

$$\bar{R} = n\,\bar{R}_1 \qquad V = \frac{V_1}{\sqrt{n}} \qquad (2)$$

i.e. the random variability of the overall strength decreases indefinitely (in other words, the effects of localized low or high strengths tend to vanish) as the number of independent redundant elements in parallel increases.

Unfortunately, even if all the relevant quantities are deterministic and known a priori, in most cases the complete response of a structure that partly yields under increasing loads can only be followed by numerical step-by-step integration. Therefore, if some parameters are random, it becomes necessary to resort to MonteCarlo procedures; i.e. solve numerically a sufficient number of deterministic problems ("simulated experiments") and thereafter evaluate statistics of the response quantity of interest.

To avoid lengthy step-by-step calculations, and to obtain a synthetic insight of the ultimate resistance of redundant ductile structure, the now classical procedure of plastic limit analysis was developed some years ago: its extension to structures with random strengths is the main subject of this paper. The basic hypotheses and theorems of classical limit analysis will be assumed known to the reader. Note that the behaviour of ductile structures is intermediate between the chain and the bundle of Fig. 1: in fact, at the ultimate collapse stage, more than one but not all the structural elements are in the limit (yield) condition.

3. CPF of plastic structures by combination of mechanisms.

Consider a structure satisfying all assumptions of limit analysis, subject to a set of forces wq_i whose intensity is defined by one positive factor w: the collapse load factor, R, is defined as the smallest value of w that causes formation of a complete collapse mechanism. Probabilistic limit analysis consists in the determination of the CPF function P(w) or, alternatively, of the PDF p(R):

$$P(w) = \text{Prob}\,\{R \leq w\} = \int_0^w p(R)\, dR \qquad (3)$$

For any plastic mechanism, which implies (virtual) rotations Θ_{kj} in the plastic hinges and (virtual) displacements v_{ki} of the points of application of the load (Fig. 2), collapse coincides with the occurrence of the event (•)

$$\{E_k\} \equiv \{wq_i\, v_{ki} \geq M_j\, \Theta_{kj}\} \qquad (4)$$

If the limit moments M_j are random variables, the probability of (4) is a nondecreasing function of w

$$\text{Prob}\,\{E_k\} = P_k(w) = P_k \qquad (5)$$

If n mechanisms are possible, collapse occurs when the inequality in (4) is verified for at least one mechanism. Denote this event by {E}: hence

$$P(w) = \text{Prob}\,\{E\} = \text{Prob}\,\{E_1 \cup E_2 \cup \ldots \cup E_n\} =$$

$$= P_1 + P_2 + \ldots + P_k - \text{Prob}\,\{E_1 \cap E_2\} - \text{Prob}\,\{E_1 \cap E_3\} - \ldots + \qquad (6)$$

$$+ \text{Prob}\,\{E_1 \cap E_2 \cap E_3\} + \ldots$$

If the n mechanisms are statistically independent, eq.(6) simplifies in

$$P(w) = P_1 + P_2 + \ldots + P_K - P_1P_2 - P_1P_3 - \ldots + P_1P_2P_3 + \ldots \qquad (7)$$

However, the mechanisms are usually correlated (in fact, they often share one or more hinges: cf. Fig. 2), and the calculation of the cross-terms of eq.(6) becomes rather cumbersome. Therefore, many symplified methods have been (and are being) proposed to evaluate eq.(6) approximately, in such a way to bound P(w) on both sides (see e.g. /3/ /4/ /5/ /6/): the most simple (and obvious) such bounds are

$$\max_k P_k \leq P(w) \leq P_1 + P_2 + \ldots + P_n \qquad (8)$$

Note that, in the range of very small probabilities of failure, the upper bound in eq.(8)

(•) Summation over repeated indices is implied.

is in many cases a gooa estimate of the true P(w).

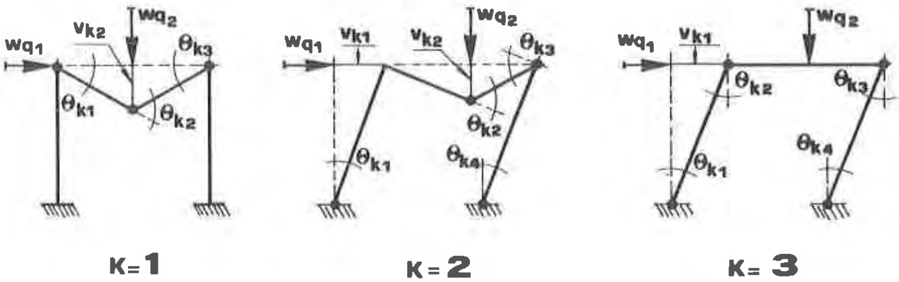

FIG. 2: Plastic collapse mechanisms of simple frame.

The previous treatment, which corresponds to the simplest case of classical limit analysis, can be similarly extended to loads defined by two (or more) parameters and to structures in which the limit moments are influenced by the axial forces (or other generalized stress components).

4. CPF-bounding theorems.

If not all the possible mechanisms are known, or also if their number is very large, eqs.(6-8) become inapplicable. This is not only a speculative possibility: in fact, the limit moment M(x) is a random process along the structural members, and, unless the bending moment diagram presents sharp and well localized vertices, many alternative locations are possible for the plastic hinges. Consequently, it may become unsafe to accept an estimate of the sum on the r.h.s. of eq.(8), necessarily limited to a finite number of terms, as a true upper bound to the CPF P(w): as a matter of fact, this upper bound (like analogous ones proposed in recent years) is obtained from a mathematical simplification of eq.(6), i.e. presupposes the knowledge of all relevant collapse mechanisms. These observations have motivated the development of two theorems, which allow to obtain rigorous and physically evident upper and lower bounds to the CPF (/7/ /8/), as it will be now briefly illustrated.

Suppose that a number of collapse mechanisms are taken into consideration, but that this number does not generally comprise all the possible mechanisms. Denote by {F} the event {one of the investigated mechanisms would allow the structure to fail, when subjected to the load W}: it is evident that

$$\{F\} \Rightarrow \{E\} \qquad (9)$$

whence

$$\text{Prob}\ \{F\} \leq \text{Prob}\ \{E\} \qquad (10)$$

Taking account of (6) and with the further position

$$P_\gamma(w) = \text{Prob}\ \{F\} \qquad (11)$$

the first bounding theorems (kinematic theorem) is written in the form

$$P_\gamma(w) \leq P(w) \qquad (12)$$

It has already been noted that this theorem remains valid if in its definition other failure modes (e.g. buckling modes) are included besides the plastic collapse mechanisms /9/.

The kinematic theorem does not add much the procedures summarized in Section 3, but just evidences the inherent unsafety of the kinematic approach in limit analysis. However, in classical limit analysis a dual approach, the static approach, is also well known: it is based on the statement that the structure does not fail if there exists a statically admissible stress field (i.e. a stress field by which the structure can withstand the applied loads, without the strength limit condition being violated in any point). Viceversa, if the structure fails, no statically admissible stress field can exist.

Then, suppose that a number (not necessarily all) of the stress fields in equilibrium with the load w are examined; denote by {D} the event {none of the investigated fields is statically admissible}. Because of the just stated theorem,

$$\{E\} \Rightarrow \{D\} \tag{13}$$

With the position

$$P_\psi(w) = \text{Prob}\ \{D\} \tag{14}$$

the second bounding theorem (static theorem) can be written

$$P(w) \le P_\psi(w) \tag{15}$$

When the loads are defined by more parameters, ineqs. (12) and (15) imply that the contour $P_\psi(\underline{w}) = \text{const} = P_o$ is internal (or better, not external) to the contour $P(\underline{w}) = P_o$, which in turn is internal to $P_\gamma(\underline{w}) = P_o$, for any P_o : $(0 < P_o < 1)$, provided that (as physically likely) all are closed surfaces enclosing the origin $\underline{w} = 0$.

The static approach, first formulated by Augusti and Baratta /7/ /8/, has found further improvements and applications in recent papers /10/ /11/ /12/.

5. Probabilistic limit analysis via numerical methods.

It is well known that the determination of the plastic collapse load factor can be formulated by means of linear programming or, if the limit moments are random variables, stochastic programming: this approach will be the object of another Invited Lecture at this Conference /13/.

MonteCarlo methods can also be applied to find directly the statistics of the collapse load starting from those of the limit moments and performing in each "simulated experiment" a complete limit analysis in the classical, deterministic sense. However, the amount of computations required can be reduced drastically making use of the so-called "parametric MonteCarlo technique" /14/. In fact, in the space of the (random) structural properties t_i, t_j (Fig. 3), "decision regions" can be individuated, which correspond to each plastic collapse mechanism, and therefore to the same constants A_{ki} in the relationship

$$R = R_K = A_{ki}\ t_i \tag{16}$$

that yields the collapse load factor R. Therefore, only one complete limit analysis is required. In fact, the first experiment carried out allows (e.g. by using linear programming) to determine R and the collapse kinematism for the simulated structure. Moreover a parameter study allows to define in the t_i space a decision region Π whose points have the same collapse mechanism of the first simulated structure. For determining all regions Π_k in the t_i space, it would be necessary to solve a stochastic programming problem, with a computational effort which increases very rapidly with the number of relevant random parameters. However, once the decision region Π corresponding to the first simulated structure is known, the points, that in t_i space represent the other simulated structures, may be classified according to their belonging or not-belonging to Π. For the first ones the collapse load R is a known linear combination of the t_i, eq. (16), whereas for the others the new coefficients in the expression of R may be determined by performing a further pivotal-step of the linear programming algorithm. In this way the repetition of the structural analysis for each simulated experiment is avoided.

The difference between the usual and the parametric MonteCarlo technique is shown diagrammatically in Fig. 3.

6. Permanent displacement.

The computation of residual deformations is also of interest in the analysis of plastic structures. In fact, a structure can be put out of effective service, before ultimate plastic collapse, by the accumulation of plastic displacements. Therefore, in a complete structural reliability analysis, it may be necessary to evaluate the probability that the inelastic deformations δ_r attains a given permissible limit Δ_L during the structural lifetime T

$$\text{Prob}\ \{\delta_r > \Delta_L | T\} = \text{Prob}\ \{w_m > \xi^{-1}(\Delta_L) | T\} \tag{17}$$

where w_m is the largest value of load in the time interval (0,T), and $\delta_r = \xi(w_m)$ is the relationship between the residual deformation and the maximum load.

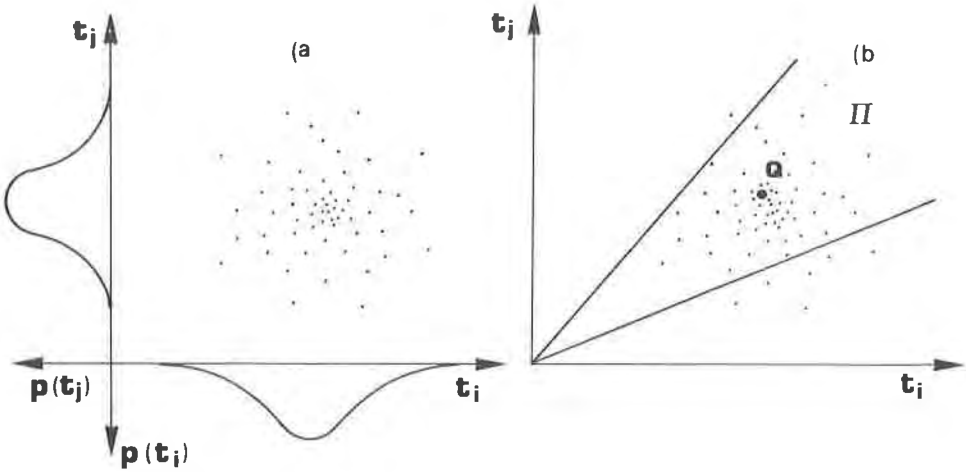

FIG. 3: Usual (a) and parametric (b) MonteCarlo procedures:

a) the structural analysis is performed in every point (t_i, t_j) yielded by a numerical experiment;

b) the structural analysis is developed completely only in Q, the first simulated structure; the same structural solution holds for all points in the decision region Π.

If the randomness of the structural properties is not negligible, a probabilistic definition of the function ξ(.) is required. In order to avoid step-by-step techniques the following upper-bound to the probability (17) can be introduced /15/. With reference to Fig. 4a, the ξ(.) function is idealized with the straight line ξ'(w) joining the points representing the elastic limit and the plastic collapse. Since $\xi^{-1}(\Delta_L) > \xi'^{-1}(\Delta_L)$, it is possible to write:

$$\text{Prob}\ \{\delta_r \geq \Delta_L | T\} \leq \text{Prob}\ \{w_m \geq w^+ | T\} \qquad (18)$$

The PDF of w^+ may be easily computed after the distribution laws of the parameters defi-

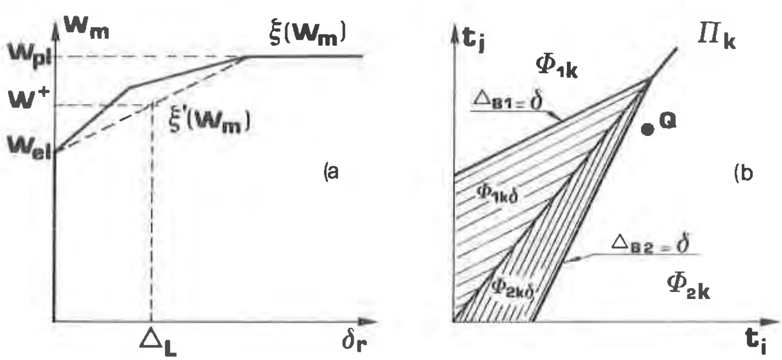

FIG. 4: Determination of residual displacement:

a) Approximate load-permanent displacement relation;
b) Decision regions for collapse mechanism (Π_k) and last hinge (Φ_{jk}).

ning the elastic limit and the plastic collapse are known.

A related problem, important in view of a more complete consideration of the plastic collapse limit state, is that of the deflections at incipient collapse: these, as well known, can be evaluated for an elastic-perfectly plastic structure, without recurring to step-by-step techniques, provided some additional assumptions are introduced, under which, when the collapse load is reached, all plastic hinges except one (two or more in exceptional cases) have formed and undergone rotation without ever unloading /16/. The hinge rotations can be computed by elastic analysis, if the last formed hinge is known: thence the deflections are immediate. A probabilistic approach to the above problem, taking into account the random nature of the mechanical characteristics of the material, is generally possible by using the MonteCarlo techniques. In this case, for each simulated structure, it is necessary to determine the last formed plastic hinge and then to obtain the parameters of the deformation state as linear combinations of the independent random design parameters. In order to make more efficient the procedure, a parametric study is possible /15/: in fact, in every decision region Π_k of the design parameter space (i.e. for each of the possible collapse mechanisms), the points corresponding to the same last plastic hinge define a cone Φ_{jk} with the vertex in the origin: these cones Φ_{jk} form new decision regions that allow to apply a parametric MonteCarlo method, as discussed in Section 5, for solving the considered problem.

A full description of deflections at collapse would involve to specify all possible regions Π_k and Φ_{jk}, whose knowledge allows to determine the true distribution function of any deflection parameter Δ, given by (Fig. 4b):

$$F_\Delta(\delta) = \sum_k \sum_j \int_{\Phi_{jk\delta}} p(t_i)\, dt_1 \ldots dt_n \qquad (19)$$

where $p(t_i)$ is the JPDF of the design parameters t_i; $k = 1, \ldots, h$, with h number of the regions Π in the t_i space; $j = 1, \ldots, n_k$, with n_k number of the regions Φ in the k-th region Π and $\Phi_{jk\delta} = (\Phi_{jk} \cap \Omega)$, being Ω the half-plane $\Delta(t_i) - \delta \leq 0$. However, the use of eq. (19) involves great computational difficulties and the development of approximate techniques seems more promising /17/.

7. Repeated loads: shakedown and plastic adaptation.

Classical limit analysis theory includes the treatment of (statically) variable repeated loads: in the load-parameters space, there exist a shakedown domain S (internal to the static collapse domain) such that, if the loads are always comprised in S, displacements and dissipated plastic work remain limited, irrespective of the number and details of the load applications, and the structure tends to shake down to purely elastic behaviour with increasing number of loading cycles.

The shakedown problem can be extended to structures with random strengths, in a similar way as illustrated for static plastic collapse: if the loads vary within a well-defined domain S, there is a probability $(1-P_s)$ that the structure shakes down, and a probability P_s that it does not, as the number of load cycles tends to infinity /18/.

The concept of shakedown, as a matter of fact, is a specific asymptotic view of the more general phenomenon of plastic adaptation of structures, which by plastic deformations and stress redistributions try to adapt themselves in the best way to resist the applied loads. However, the very starting point of shakedown theory loses logical significance, if the stochastic character of the loading history is taken into account: rather, the correct way of formulating the problem of plastic adaptation to variable repeated loads in a probabilistic context is analogous to that of permanent displacements (Sec. 6): namely, to start from probabilistic data on the loading process (in particular, the probability of excursion out of any limited domain in a given time interval), then try and derive from this data the probability of reaching given values of displacements in the prescribed time interval.

This approach, including also dynamic load effects, has been followed in a recent paper by Baratta /19/. From the available bounding techniques for displacements of plastic structures under dynamic loads, and for the threshold violation probability of stochastic processes, a procedure for bounding the probability of exceeding a given displacement in function of time was developed and illustrated by an example.

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