RELIABILITY OF ELASTIC–PLASTIC STRUCTURES

C. GAVARINI
Istituto di Scienza delle Costruzioni, Università di Roma,
Via Eudossiana 18, I-00184 Roma, Italy

The plasticity theory is, in itself, deterministic, but in the physical context all the data of the problems are affected with uncertainty. So the problem arises of a probabilistic approach to the theory of elastic-plastic structures.

After a brief presentation of the principal probabilistic concepts the lecture will consider the problem of plastic limit analysis in the probabilistic context. The presentation starts with the definition of the plastic limit domain in the space of all the parameters considered as free. Then, giving to the parameters their probabilistic character, a probabilistic theory is developped, which holds both for the case of one loading condition and for the case of repeated loadings.

The important numerical difficulties involved in the above presented theory are briefly discussed and the consequent opportunity of looking for approximate approaches is thus presented.

1. Introduction

Structural theories are, in their fundamental formulation, deterministic theories, i.e., theories based on the hypothesis of a perfect knowledge of all the problem data: loads and other external actions, physical parameters defining the materials behaviour, and the geometry of the structure.

But in reality, at the design stage, all the data are affected by an amount of uncertainty or randomness. The input uncertainties of the problem produce, by self-combination, an output uncertainty which it is essential to evaluate in order to know, and thus also to impose by acting on the design variables, the risk level of the structure.

We will consider here the central problem of the evaluation of the output uncertainty on the basis of the knowledge of all the input uncertainties, i.e., the reliability theory. The two other important problems of the statistical definition of data and of the risk level to be adopted are not considered.

In particular, we will consider the limit plastic behaviour of structures, or plastic collapse, defined by the propagation of a failure mechanism. In the deterministic context the problem leads us, by introducing a discretization technique, to problems of Mathematical Programming and in particular, via linearization, to problems of Linear Programming. In the stochastic context we are concerned with problems of Parametric and Stochastic Programming.

A great interest may be attributed to the concept of Plastic Domain, in the space of all parameters with random character, as the locus of the points for which the structure resists. The boundary of such a domain, which is the locus of the points associated with failure mechanisms and which is, in the case of linearization, a hyperpolyhedron, is obtained by solving two coupled problems of Parametric Programming.

Since the plastic domain is a parametric concept it is not of itself necessarily associated with the concept of randomness; but if randomness is introduced it is possible to elaborate, on the domain, a very general probabilistic theory which has as its essential result the computation of the failure probability of the structure. We can consider loads which are fixed, repeated, or with the character of stochastic processes; however, in this last case the analytical difficulties are very great.

Unfortunately, the computational problems become serious as soon as the number of random variables reaches a few units. This circumstance requires in practice a search for approximate methods at two different levels, so called 2 and 1 (the general theory being level 3); such methods will be mentioned briefly.

2. Parametric limit analysis

Consider a generic elastic-plastic (or rigid-plastic) structure reduced to a discrete equivalent system by means of a finite element technique, subjected to a set of given loads and to other loads regulated by a vector of intensity parameters $\underline{t}_1$. With reference to a <u>reduced statically determined system</u>, the vector of the generalized stresses in the structure at collapse may be expressed as the sum of three terms which are the contribution of the given loads, of the loads which depend on $\underline{t}_1$, and of a field of residual stresses, which depend on a vector of hyperstatic unknows $\underline{x}$, i.e.:

$$\underline{s} = \overline{\underline{s}}_o + \underline{A}_1\underline{t}_1 + \underline{S}\,\underline{x} \tag{1}$$

In every section, or nodal point, or element, the limit plastic behaviour will be defined by a plastic domain in the space of the related generalized stresses; such a domain may in general have a non-linear frontier but we will here consider equivalent linear approximate contours so that the condition, for every stress point, to belong to the plastic domain will be imposed by the linear inequalities:

$$\underline{D}\ \underline{s} \leq \underline{b}_1 \tag{2}$$

At the present stage, characterized by a parametric analysis, we consider the resistance of the structural elements to depend, as the loads, on a vector of parameters $\underline{t}_2$, and so we adopt for $\underline{b}_1$ the expression:

$$\underline{b}_1 = \underline{b}_2 + \underline{A}_2\underline{t}_2 \quad . \tag{3}$$

By introducing (3) and (1) into (2) we obtain a linear system of inequalities having as variables the componentes of $\underline{x}$ and as parameters the components of $\underline{t}_1$ and $\underline{t}_2$. With the positions,

$$\underline{C} = \underline{D}\ \underline{S}\ , \quad \underline{b} = \underline{b}_2 - \underline{D}\ \underline{\bar{s}}_o\ , \quad \underline{A} = [-\underline{D}\ \underline{A}_1 \vdots \underline{A}_2]\ , \quad \underline{t} = \begin{Bmatrix} \underline{t}_1 \\ \cdots \\ \underline{t}_2 \end{Bmatrix}\ ,$$

this system may be written in the compact form:

$$\underline{C}\ \underline{x} \leq \underline{b} + \underline{A}\ \underline{t} \quad . \tag{4}$$

If $\underline{t}$ is fixed, any vector $\underline{x}$ which satisfies (4) corresponds to a <u>statically admissible solution</u> in the sense of plastic analysis. If $\underline{t}$ is free, the set of vectors $\underline{t}$ for which at least one statically admissible solution exists defines the <u>plastic domain</u> of the structure D_p in the space of the parameters R_t:

$$D_p = \{\underline{t} \mid \exists\ \underline{x} \in R_t \text{ so that } \underline{C}\ \underline{x} \leq \underline{b} + \underline{A}\ \underline{t}\} \quad . \tag{5}$$

Now the problem is to determine the boundary of D_p.

Let μ denote an arbitrary element of vector $\underline{t}$ and $\underline{\bar{t}}$ be the rearranged vector of the remaining components. For any fixed value of $\underline{\bar{t}}$ one of the following statements holds true: a) for no value of μ does any vector $\underline{x}$ satisfy (4), i.e. the straight line $\underline{\bar{t}}$ = const does not intersect D_p; b) there is an interval, $\mu_i \leq \mu \leq \mu_s$, corresponding to the intersection of the straight line $\underline{\bar{t}}$ = const with D_p.

If $\underline{\bar{t}}$ is fixed the values μ_s and μ_i are given by the solutions of <u>two linear programming problems</u>,

$$\underline{C}\ \underline{x} - \underline{\bar{a}}\ \mu \leq \underline{b}_p + \underline{\bar{A}}\ \underline{\bar{t}} \quad , \tag{6}$$

$$\mu = \begin{cases} \max & (7a) \\ \min & (7b) \end{cases}$$

where the matrix $\underline{\bar{A}}$ is obtained from $\underline{A}$ by simply removing the column $\underline{\bar{a}}$, corresponding to μ.

If $\underline{\bar{t}}$ is allowed to vary, programs (8) (9) become <u>parametric programming problems</u>:

$$\underline{C}\ \underline{x} - \underline{\bar{a}}\ \mu \leq \underline{b}_p + \underline{\bar{A}}\ \underline{\bar{t}} \quad ,$$

$$\underline{\bar{t}} \in R_{\bar{t}} \quad , \tag{8}$$

$$\mu = \begin{cases} \max \\ \min \end{cases} \quad .$$

The boundary of D_p, as defined by (8), is divided into two parts: the upper part, $\mu = \mu_s$, defined by the parametric problem with μ = max; the lower part; $\mu = \mu_i$, defined by the condition μ = min. The identification of the separation line (or surface) between the two parts leads to a coupling between the two problems (fig.1).

Each face of the boundary is associated with a distinct mode of failure (collapse mechanism).

Inactive failure modes are automatically disregarded by parametric analysis.

If $\underline{t}$ contains resistance parameters, the plastic domain is unlimited in the associated positive directions.

A very simple example, which illustrates the theory, is shown in fig.2.

In this case it is $\underline{t} \equiv \underline{t}_1$, $\underline{t}_2$ does not exist, R_t is the plane; the possible collapse mechanisms are indicated in the figure, each of them being active in two directions, so that the plastic domain has eight sides.

Figure 3 is related to the same structure but with t_2 = const = 3 and a new parameter belonging to $\underline{t}_2$, say $\underline{t}_3$ related to the resistance of the structure (the same in all the sections). In this case there are non-active mechanisms and the domain is unlimited in the t_3 - direction.

A more complicated example, automatically analysed by parametric programming, is illustrated in fig.4. Here the interaction between normal force and bending moment is considered in the sections.

We can observe that it is obviously possible, and easier than in the plastic field, to define in R_t the elastic domain D_p. For example in the case of fig.2 D_E is given in fig.5, while fig. 6 shows an interesting comparison between the two domains, D_E and D_p; this figure gives a more general picture of the so-called reserve of strength of an elastic-plastic structure than that which is usually considered in the case of proportional loading, i.e. of one parameter.

3. Probabilistic theory

If the quantities associated with the parameters have random character, the plastic domain may be used directly in computing the structural reliability. This is the case in which all the parameters are random variables (independent of time): the structural reliability, L, is the probability of a point $\underline{t} \in R_t$ being inside the plastic domain:

$$L = 1 - p_f = \int_{\mathbf{D}} f_{\underline{t}}(\underline{z})\, d\underline{z} \quad , \tag{9}$$

where $f_{\underline{t}}(\underline{z})$ is the joint probability density function of $\underline{t}$.

When only one component of $\underline{t}$ is time-dependent (random process) the computation of L, or p_f, is still possible provided that the distributions of the extremes of the process are known. If there is more than one process, drastic approximations must be introduced.

Although the computation of the failure probability is the primary goal of reliability analysis, some intermediate statistics may also be of interest. Consider the case in which the random quantities (components of $\underline{t}$) are not simultaneously active. At a given time some of them, subvector $\underline{t}_a$, have attained their final, random, values, and their joint probability density function, $f_{\underline{t}_a}(\underline{z}_a)$, is already known. These variables are called "active". The remaining variables, grouped in the subvector $\underline{t}_p$, are those which have not yet been specified and are called "passive".

For any value $\underline{z}_p$ of the vector $\underline{t}_p$, the intersection of the plastic domain with $\underline{t}_p = \underline{z}_p$ is a subdomain in R_{t_a}, denoted by $D_a(\underline{z}_p)$ (see fig.7, drawn for the case of two active-one passive variables). We define the failure probability function, $G(\underline{z}_p)$ as the probability of failure conditional to the passive vector having value $\underline{z}_p$:

$$G(\underline{z}_p) = \int_{R_{t_a}-D_a(\underline{z}_p)} f_{\underline{t}_a}(\underline{z}_a)\, d\underline{z}_a \tag{10}$$

It is evident from this definition, that $G(\underline{z}_p)$ accounts both for the randomness of the active (already specified) variables, and for the domain of the structure; on the contrary, the random character of the passive variables is extraneous to $G(\underline{z}_p)$ ($f_{\underline{t}_p}(\underline{z}_p)$ may in fact be unknown at the time $G(\underline{z}_p)$ is computed).

When the joint probability density function of the passive variables becomes known, the reliability analysis is completed by the integral:

$$p_f = \int_{R_{t_p}} G(\underline{z}_p) \cdot f_{\underline{t}_p}(\underline{z}_p) d\underline{z}_p \tag{11}$$

One might notice that the function G is itself the result of a probabilistic analysis: given the statistics of the active variables, $G(\underline{z}_p)$ quantifies the probabilistic characteristics of the structural resistance in terms of the passive variables.

The potential usefulness of this function will now be illustrated in a hypothetical situation.

Imagine a plant producing structures, or structural components, to be used by many "branches" under different environmental conditions. The final behaviour of each system is affected by common (random) factors due to the production plant, and by local, particular (random) factors. In this situation the common factors may be recognized as active variables from the viewpoint of the plant and thus contribute to the definition of the function $G(\underline{z}_p)$, identical for all the structures (and therefore computed only once, by the plant). In each branch, the statistical data of the local factors (passive variables) will enable the failure probability to be computed through eq. (11).

In other words, the "label" of the generic structure, travelling towards a branch, will not contain separately the safety domain and the function $f_{\underline{t}}(\underline{z})$, but only the function $G(\underline{z}_p)$.

It may be of interest to point out that, in the particular case of one active and one passive variable, say ρ and μ, the function $G \equiv G_\mu(z)$ becomes simply expressed by

$$G_\mu(z) = 1 - F_{\mu_i}(z) + F_{\mu_s}(z) \quad , \tag{12}$$

where $F_{\mu_i}(z)$ and $F_{\mu_s}(z)$ are the cumulative distribution functions of the max and the min of program (8) as they result from the Stochastic Programming theory applied to the program itself. And in the more particular case in which μ is essentially non-negative and $\mu_i(z)$ is negative for any value of z, so that it is $F_{\mu_i} \equiv 1$, equation (12) reduces to

$$G_\mu(z) \equiv F_{\mu_s}(z) \quad ; \tag{13}$$

the function G becomes a common cumulative distribution function.

Other classic results may be obtained on the basis of the approach outlined above, but as space is short let us go on to the problem of repeated loading.

4. Repeated loading

The general case of a vector $\underline{t}$ with a number of random processes among the components is very difficult to handle, as it is associated with the problem of computing the probability of the first exit from a domain.

Easier is the case of repeated loading which is, on the other hand, of no little practical interest. In this case vector $\underline{t}$ will be decomposed into a subvector $\underline{t}_c$ of random variables (no repetition) and a subvector $\underline{t}_r$ of variables related to the intensities of certain repeated loads; $f_{\underline{t}_c}(\underline{z}_c)$ and $f_{\underline{t}_r}(\underline{z}_r)$ will be the corresponding joint probability density functions, the latter being valid at every loading application. Under such conditions it is quite easy to establish the following general formula which expresses the reliability of the structure, i.e. the survival probability after loading applications:

$$L_n = \int_{R_{t_c}} H_n(\underline{z}_c)\ f_{\underline{t}_c}(\underline{z}_c)\ d\underline{z}_c \quad , \tag{14}$$

with

$$H_n(\underline{z}_c) = \left[\int_{D_r(\underline{z}_c)} f_{\underline{t}_r}(\underline{z}_r)\ d\underline{z}_r\right]^n \quad , \tag{15}$$

where $D_r(\underline{z}_c)$ is the intersection of the plastic domain D_p with the hyperplane $\underline{t}_c = \underline{z}_c$ (fig.8). Formula (15) may be generalized by considering several loading conditions, differing both in spatial and statistical distribution; the generalized formula, with self-explanatory symbols, is the following:

$$H_n(\underline{z}_c) = \prod_j \left[\int_{D_{r,j}(\underline{z}_c)} f_{\underline{t}_{r,j}}(\underline{z}_r)\ d\underline{z}_r\right]^{n_j} \quad , \qquad n = \sum n_j \quad . \tag{16}$$

In the particular case of one component only in vector $\underline{t}_r$, say λ, formulas (15) and (16) become

$$H_n(\underline{z}_c) = \{F_\lambda[\lambda_s(\underline{z}_c)] - F_\lambda[\lambda_i(\underline{z}_c)]\}^n \quad , \tag{17}$$

$$H_n(\underline{z}_c) = \prod_j \{F_{\lambda,j}[\lambda_{s,j}(\underline{z}_c)] - F_{\lambda,j}[\lambda_{i,j}(\underline{z}_c)]\}^{n_j} \tag{18}$$

where F ( ) is the cumulative distribution function associated to f( ) and $\lambda_s(\underline{z}_c)$, $\lambda_i(\underline{z}_c)$ are the extremal ordinates of the segment on the axis λ to which $D_r(\underline{z}_c)$ is reduced in the present case.

Finally, if $\lambda_i(\underline{z}_c) \equiv 0$ and $F_\lambda[\lambda_i(\underline{z}_c)] \equiv 0$ hold, (17) becomes (with $\lambda = \lambda_s$)

$$H_n(\underline{z}_c) = F_\lambda^n[\lambda(\underline{z}_c)] \quad , \tag{19}$$

and if $R_{\underline{t}_c}$ too reduces to a one-dimensional space, we obtain the known formula:

$$L_n = \int_0^\infty F_\lambda^n(z)\ f(z)\ dz \quad . \tag{20}$$

5. Numerical problems and approximate approaches

The numerical problems related to the practical application of the formulas introduced above may soon become heavy. The difficulty is not generally connected with the structural complexity in itself, but rather with the number of random variables involved, and it arises at two levels: in the determination of the frontier of D_p, and in the computation of the multiple integrals. In such a situation the theory outlined above is not generally practicable in common applications, and simpler approaches, necessarily approximate, must be considered. Such approaches have been subdivided into two classes, designated level 2 and level 1 methods.

The level 1 methods are based on the known concepts of characteristic values and partial safety factors and are presently adopted by many international (ISO, CEB, ECCS, ...) and national codes of practice.

The level 2 methods, which are at an intermediate level of approximation, are based on the computation of a safety index which is approximately associated with reliability, the approximation being related both to the type of statistical distributions involved and to the actual shape of the plastic domain D_p. This measure of reliability becomes exact if the distributions are normal and the frontier of D_p a hyperplane.

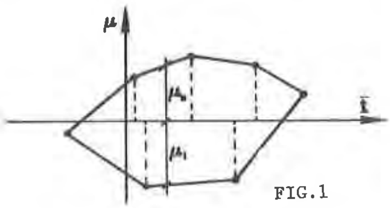


FIG.1

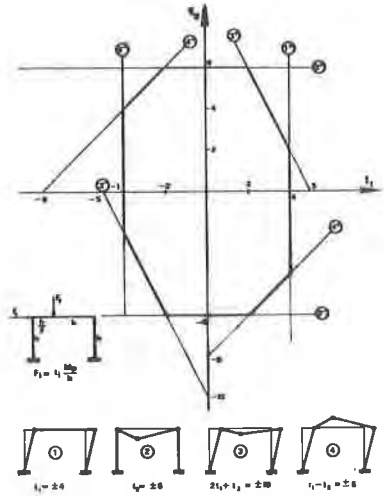
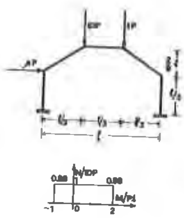


FIG.2



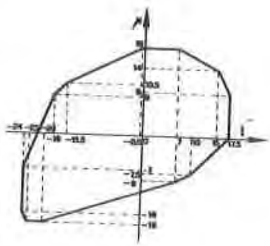


FIG.4

FIG.3

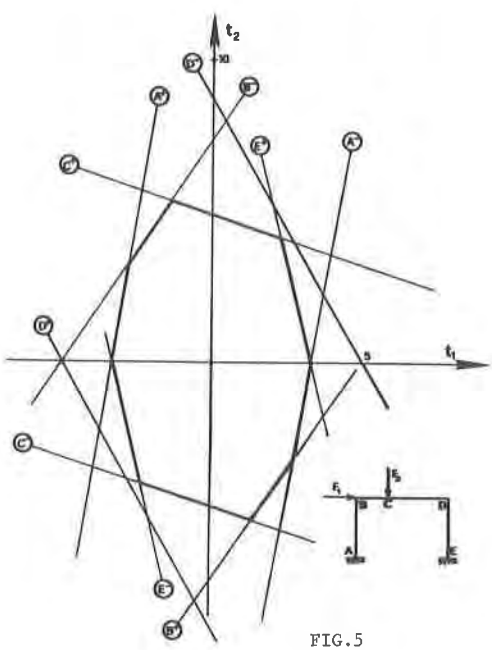
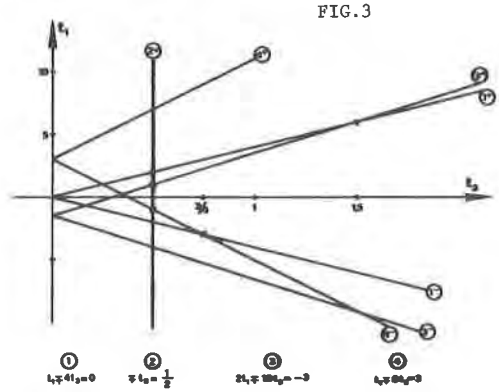


FIG.5



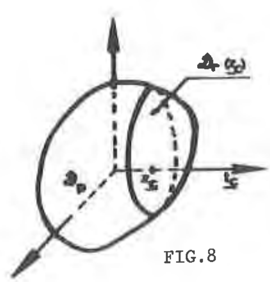


FIG.8

REFERENCES

(1) BOLOTIN V.V.: Statistical methods in structural mechanics - Holden - Day 1969.

(2) GAVARINI C.: Concezione probabilistica del calcolo a rottura, Giornale del Genio Civile, n.8, 1969.

(3) GAVARINI C., VENEZIANO D.: Sulla teoria probabilistica degli stati limite delle strutture - Giornale del Genio Civile, n.11-12, 1970.

(4) VENEZIANO D.: Calcolo a rottura probabilistico con più parametri casuali, Giornale del Genio Civile, n.2, 1971.

(5) GAVARINI C., VENEZIANO D.: On the safety domain of structures, I° Conv. Naz. di Mecc. Teorica ed Applicata, Udine, Juin 1971.

(6) GAVARINI C., VENEZIANO D.: Minimum weight limit design under uncertainty, Meccanica, n.2, 1972.

(7) GAVARINI C.: On the probabilistic limit theory of structures subjected to repeated loads, Annual Meeting of the CNR Research Group Advanced Problems of Structural Engineering (PAdIS), Catania, Dec. 1973.

(8) GAVARINI C.: A probabilistic theory of shake down for elastic plastic structures under repeated loads. II Congresso Nazionale AIMETA, Napoli, oct. 1974.

(9) BEOLCHINI G., CIOFANI F., GAVARINI C.: Sulla concezione probabilistica della sicurezza dei ponti, Convegno annuale PAdIS, Roma, nov. 1975.

(10) GAVARINI C., MERLANTI P.: Sulla concezione probabilistica della sicurezza a fatica dei ponti, Conv. Ann. PAdIS, Roma, nov. 1975.

(11) GAVARINI C., VENEZIANO D.: Probabilistic structural analysis and synthesis, 15th Polish Solid Mechanics Conference, Zakopane, sept. 1973.

(12) GAVARINI C.: L'analyse limite des structures au moyen de la programmation mathématique, Séminaire "Plasticité et Viscoplasticité", Ecole Polytechnique, Paris 1972; publié dans Sciences et Techniques de l'Armement, 2ème fasc. 1973.

(13) GAVARINI C.: Applications de la Programmation mathématique à l'analyse limite des structures, Revue française d'Automatique, d'Informatique et de Recherche Opérationelle, V-3, 1973.

(14) CEB: Bulletin d'information 112 - CEB, CECM, CIO, FIP, IABSE, Joint Committee on Structural Safety: First order reliability concepts for design codes, Juillet 1976.

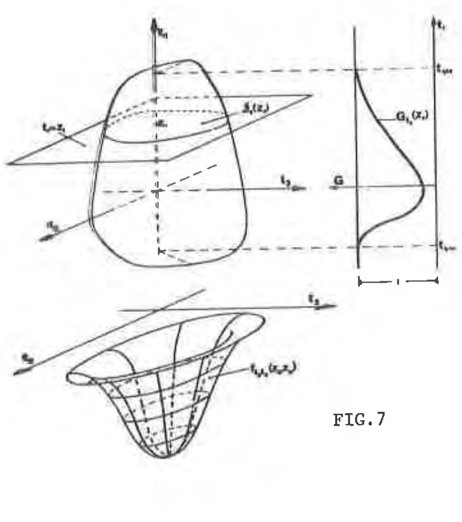


FIG.7

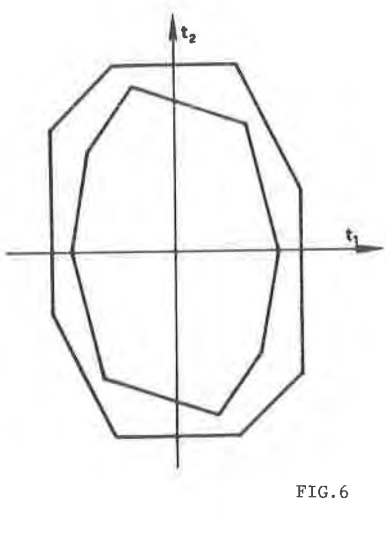


FIG.6

L 2/1*