MATHEMATICAL PROGRAMMING METHODS IN ENGINEERING PLASTICITY

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Summary

Since the early fifties when problems of plastic limit analysis were first recognized as linear programs, the applications of mathematical programming methods to structural mechanics have grown continuously.

This lecture will briefly outline some of the main concepts of MP which are believed to be essential for such applications and will illustrate them through a set of selected problems and specific examples which have mostly appeared recently in the technical literature and are likely to be of interest in nuclear technology- oriented structural mechanics.

Engineering plasticity (EP) is concerned with the applications of the mathematical theory of plasticity to structural analysis and design. Mathematical programming (MP) is that branch of optimization theory which is concerned with inequality constraints in the algebraic context of finite-dimensional spaces.

The present conspectus is an attempt to summarize some of the significant advances which have been achieved in applying MP methods to EP problems, in particular to plastic analysis of structures.

Because of the present time and space limitations, several interesting developments and important contributions could not be mentioned here. The reference list is quite uncomplete with respect to the abundant literature now available; reference was deliberately limited to mostly recent sources which were believed to provide up-to-date information directly connected with the specific aspects to be considered in this lecture.

Perhaps the most obvious applications of MP to engineering are to problems of structural design where the function to minimize encodes the design objective and the constraints express the codified requirements imposed on the design. Thus if a reinforced concrete frame whose geometry and concrete dimensions are fixed a priori, is to be designed by fixing the reinforcement details, then a possible objective function might be the total volume of reinforcement and the optimality criterion would be the minimisation of that function. The constraints to be imposed would be selected to ensure satisfactory structural behaviour of the frame. Other more complex design problems can be envisaged where the increased complexity may stem from the adoption of an objective function such as total cost or where the increased complexity may arise from the design variables also fixing the geometry and topology of the frame.

The MP formulation of analysis problems arises from the inequality constraints imposed, for example, by the plastic constitutive relations or from unilateral contact. The objective function may be a load factor or an energy functional. Various surveys on structural sythesis, including plastic design and MP methods are available (see e.g. |19, 54|). Therefore, only analysis problems via MP will be considered here, in order to supplement and to update, rather than to replace, previous surveys such as |11, 32, 33| .

The increasing application of MP to structural problems has been facilitated by the use of discretized models of structural systems and the subsequent reduction to finite dimensions. The introduction of graph-theoretical concepts, particularly with respect to structural frames, has permitted the adoption of alternative "mesh" or "nodal" descriptions of of the underlaying static and kinematic relations which themselves are linked by a fundamental static-kinematic duality (SKD) |32, 38, 39| .

The necessary conditions for the solution of an MP are known as Kuhn-Tucker (KT) conditions and, under certain conditions of convexity (concavity), they become sufficient. The KT conditions for linear programming (LP) and quadratic programming (QP) problems can be cast in a linear complementarity (LC) form.

LP and QP problems are concerned with the minimization (maximization) of linear and quadratic functions, respectively, both under linear inequality and equality constraints. An LC problem concerns two sign-constrained variable vectors related to each other by a linear (matrix) equation and by an orthogonality (or complementarity) condition. The vectorial relations (equilibrium and compatibility) for many problems, when linked with the appropriate constitutive relations, occur naturally in LC format. If these fundamental structural relations are arranged in the form of KT conditions then the corresponding primal-dual programs may be inferred directly. These programs encode the extremum principles appropriate to the considered class of problems.Thus the KT theory of MP can be seen to be a vehicle through which the appropriate primal-dual extremum principles may be generated for discretized structural mechanics |6, 11, 32, 37, 55 |.

The simplest form of MP is the LP whose KT conditions are not only in LC form but also display a form of uncoupling. The fundamental relations for plastic limit analysis and shakedown analysis of discretized systems for a single generalized stress-resultant occur naturally in this form |9, 11, 38, 39 |. Such problems can therefore be transformed directly to a pair of primal-dual LPs for each of the two graph descriptions, giving four possible LPs for each problem |39 |. Some attention has been given to the most convenient formulation with respect to the standard simplex algorithm and to the simplification of the necessary data preparation. Mesh description have been shown to lead to LPs which require less computational effort (at least, with respct to simplex-based computer codes) than the corresponding nodal LPs. However the nodal programs generally require less data preparation. The advantages of both systems may be achieved by using node-mesh transformations. Modifications to the simplex algorithm to accomodate the special features of such structural problems have also been explored.

The problems of plastic limit design of dicretized systems for a single generalized stress-resultant occur naturally in the form of LPs. The KT conditions for such LPs are of interest and the generalized Foulkes' conditions for optimal design are obtained in this way. The design LPs can be dualised and once again alternative graph descriptions may be used. The analysis and design problems may also be generalized readily to multi-component cases if a piece-wise linearisation of the yield criterion is adopted.

Some problems which, at first sight, appear to be a different type can be arranged in LP form. Thus the limit analysis of voussoir structures |31| and the limit analysis with respect to overall locking under imposed straining effects in elastic-perfectly locking systems, can also be formulated in terms of linear programming problems.

The mesh and nodal static-kinematic descriptions along with the alternative flexibility and stiffness relations of elasticity lead to four equivalent formulations for the elastoplastic analysis of frames. They each have a parametric LC form in terms of a single parameter controlling a proportional load path. A procedure based on the Wolfe and Markowitz algorithms for quadratic programming can be employed in the solution of the parametric LC problem. This procedure can be regarded as a variant of the traditional "simplex" algorithm for LP, supplemented by an additional rule which enforces complementarity in each pivotal transformation ("restricted basis" LP) |8, 13, 16, 20, 28, 29, 34, 49| . It automatically traces the development of plasticity in the structure as the load path is followed and accounts for the unstressing and reactivation of elements which have previously been plastified. This procedure also identifies the collapse mode (or modes) for structures with a limit load. Elastoplastic solutions are not necessarily unique and any multiplicity, including that induced by the presence of a pseudomechanism, is automatically identified. This technique can also be used to study the behaviour of individual plastic sections for multicomponent cases and the effects of the stress-resultant path and the presence of residual stresses can be considered.

The methods which have proved successful with respect to planar frames have been extended to three-dimensional frames. The effects of strain-softening have been included and this has required to deal with possible branching of the equilibrium configuration. Applications of LP and QP methods have also been made, through finite differences |9| and finite element (FE) models, to axisymmetric and twodimensional continua and to plate structures (both steel and re. concrete slabs) with respect to plastic limit analysis and to elastoplastic analysis. The enforcement of SKD enables a wide range of FE models for surface structures and continua to be developed statically and kinematically |27, 56, 57| . Particularly simple specializations have been the nodal and mesh descriptions of the yield line method for reinforced concrete slabs. These have been employed in deriving programs for the analysis and synthesis of such slabs and the comparative computational efforts of differing approaches and methods have been studied |40| .

Elastic-plastic analysis of discretized structures has been recognized to be ameneable to the mathematical model of QP in various ways, both for the incremental formulations (in terms of rates) and step-by-step evolutive solutions along any general loading path, and for holonomic solutions in finite terms in the presence of proportional loading and piecewise linearized yield surfaces and linear hardening (kinematic, isotropic or according to Koiter hypothesis of non-interacting yielding modes) |13, 15, 17, 30, 34, 37| . The computational aspects of QM in this context have been fairly extensively but not yet exaustively investigated |1, 17, 18| ; to this aim the elastoplastic torsion seems to provide a sort of benchmark problem |24, 41| .
Unilateral contact problems even for linear elastic structures imply quite obviously complementarity relationships,bear a formal analogy with incremental or holonomic piecewiselinear elastoplastic problems and, hence, lead to QP problems or related mathematical formulations and solution procedures |35, 36, 43, 44| .

Rigid-plastic constitutive laws, also allowing for rate-sensitivity (rigid-viscoplastic idealization) is known to provide a convenient and realistic description of the local deformability for analysing ductile structures subjected to impulsive loading when permanent deformations prevail on the reversible ones in their response. The search for the acceleration field when the velocities are known is a central problem in rigid-plastic dynamics and becomes a problem in QP as long as discrete structural models are concerned |4, 5, 6, 55| . It has been shown recently |51| that rigid plastic dynamic analysis via QP, even in the presence of large displacements, compares well with elastoplastic transient analysis from the computational point of view.

Some very recent work has been concerned with deriving mesh and nodal static-kinematic descriptions for frames undergoing large displacements.These descriptions are given linear forms by enforcing SKD and using it to define uniquely appropriate additional forces and deformations dependent upon the current displaced configuration. The use of perturbation methods has allowed the elastoplastic behaviour of such structures to be given an exat parametric LC form which does not require iterative solutions |50| .

The foregoing has largely been concerned with EP problems which can be formulated as QP or LP problems or in some other similarly well-structured algebraic form. A wide class of nonlinear programming (NLP) techniques are available, i.e. algorithms capable to lead to a minimum of a general, possibly nonlinear function over a feasible domain defined by linear or nonlinear equations and inequalities. Some but still unsufficient attention has been devoted to classification of the NLP methods and to the identification of categories of structural problems which may be tackled by each class.

Pretensioned cable structures and similar systems (e.g. a pipeline laid to rest freely on a rough sea-bottom) can be analyzed fairly effectively in the range of large displacements allowing for loosening and/or plastic yielding of cables and structural elements,by minimizing a nonlinear, nonquadratic potential energy function under sign constraints only on the variables |11, 12, 36| .

Rigid-plastic limit analysis with nonlinear yield conditions in more than one (true or generalized) stress components, leads to NLP problems which have been solved, (notably by the sequential unconstrained minimization technique, SUMT, resting on the use of penalty functions) since the earliest applications of the finite element methods to this area |3, 47| . Remarkable computational improvements have been achieved more recently |7, 25, 26, 48| also with reference to more complicate structures such as shells. However, it is not generally clear yet whether this approach is advantageous with respect to the more familiar one resting on picewiselinearization of the yield loci and consequent use of LP techniques |21, 22, 33, 40, 45, 56, 57 |.

In elastoplastic and rigid plastic dynamics (and particularly in dynamic shakedown analysis) deformation bounding techniques may provide technically useful information without lengthy time integration, if the upper bounds are improved by some minimization procedure. Therefore, this interesting, still growing topic gives rise to various kinds of MP problems (LP and QP, NLP), which generally are the more laborious the closer the bound should be to the real deformation parameter. A fairly comprehensive and systematic discussion of the several alternatives can be found in |14| .

A peculiar class of nonconvex NLP problems is the minimization of a linear or of a convex-quadratic form under linear inequalities and equations and complementarity constraints (a complementarity constraint, as in the LC problems, consists of an orthogonality condition concerning sign-constrained vectors, i.e. it requires that in each pair of corresponding components of the two vectors at least one vanishes). This type of MP problem occurs in the optimal design of elastic-plastic structures under behaviour constraints concerning displacements and/or local deformations (such as plastic hinge rotations in critical sections of a beam or frame model). The same kind of problems arises when the yield limits in a discrete structural system with piecewiselinear yield loci have to be identified on the basis of the minimization of the discrepancy between measured and theoretical displacements, i.e. through a least-square approach to this "indirect identification" of local properties |11, Chapter 24| .

This class of nonconvex NLP problems has been found (with reference to an unusual kind of design optimization |36|) to be solvable by decomposition into a convergent sequence of LP or QP problems (depending on the linear or quadratic nature of the objective function, respectively), in the conceptual framework of combinatorial opimization involving binary variables.

Indirect identification or "inverse" problems in the above sense, though concerning excitation characteristiscs of dynamic systems, have been recently solved by dynamic programming a MP technique which has been successfully applied earlier in rigid-plastic limit analysis |42, 46| .

MP methods and their applications to EP have been exposed to a period of fairly intense research development and have now reached the stage of successful application to practical problems. Commercially available computer software packages for MP have been developed and it would appear that the subject is entering a phase of increased practical utilisation |10, 11, 18| .

A discussion of some specific examples of engineering relevance, necessary to substantiate and illustrate the preceding conspectus are not compatible with the limitations of this compact and will form the main object of the lecture.

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