INELASTIC ANALYSIS OF STRUCTURES WITH APPLICATIONS TO CYCLIC LOADINGS

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Summary :

The paper is concerned with the derivation of cyclic constitutive relations of materials during inelastic regime and the development of the eventual periodical state in a cyclically loaded structure.

To describe inelastic constitutive relations for metals (we limit ourself to this special material), usually two different approaches are possible : phenomenological relations, valid for the results of only a few experiments, can be guessed and developed in the macroscópic approach ; the different local processes leading to plastic deformation can be analysed and the behaviours can be deduced after some computations in the microscopic approach.

An intermediate approach will be proposed in the regime where hardening and creep have to be considered. It is also based on the fact that the fundamental volume element we have to represent is indeed a polycristal, i.e. an aggregate of crystals and thus similar to any structure.

It is moreover known that when a sample is subjected to cyclic uniaxial stress, three regimes can be reached after some cycles :

elastic shakedown, where the material will sustain a large number of cycles before failure ; plastic shakedown (periodic plastic strain), and ratcheting (progressive plastic strain) where low cycle fatigue will generally occur. The modeling of material has to take into account these facts.

With some special constitutive laws, one has then to look at what will happen within the structure during a cyclic thermomechanical loading. Many experimental, numerical and theoretical studies have been performed to give solutions, or at least partial ones, to this problem. One of the most important activity was to define a method in order to find straightforwardly the limiting state, or some bounds, without following, incrementally step by step, the structure.

We shall describe our own general framework to solve these two problems. Its essential feature consists in the introduction of a family of transformed parameters $\hat{\underline{X}}$ which are linked to the viscoelastic, plastic and viscoplastic strains field $\underline{X}$, through a symmetric non-negative matrix $\mathfrak{B}$ (if the structure is discrete) which may be singular. The position of the field $\hat{\underline{X}}$ may be known from only the simple purely elastic (or viscoelastic) analysis of the structure. The uses of this property are evident. When the matrix $\mathfrak{B}$ is regular (case of the kinematical hardening material), a local study can be performed;on the contrary,for a singular matrix $\mathfrak{B}$ (case of perfectly plastic material), a global study is necessary.Bounds can be easily found.

1. Introduction

The effects of cyclic loads and temperature upon the performance and time-life of engineering components are of fundamental interest to nuclear designers. Two main problems have to be considered :

i) the derivation of cyclic constitutive relations of metals during inelastic regime (local constitutive equations),

ii) the development of the eventual periodical state in the structure (global evolution boundary value problem).

The paper is concerned with these problems. It will show how simple they will appear within a particular framework which was already presented in a previous report [12].

2. The global constitutive equations

2.1 Experimental facts and principle of the description

Under applied cyclic stress in the homogeneous uniaxial condition, some important facts are classically underlined :

i) the eventual existence of a steady cyclic state for which the stress and strain change periodically, there is elastic shakedown or plastic shakedown ;

ii) some asymmetrical stress cycles may induce progressive ratcheting even if during some lower mean stresses there is shakedown ;

iii) some cycles are necessary to reach the eventual limiting state ;

iv) the eventual ratcheting or cyclic strain may be the result of both plastic and viscous effects during cyclic tests with varying frequency or with rest-periods.

On the other hand, the metal is at our macroscopical level a polycrystal i.e. the fundamental volume element is an assembly of single crystals which contain themselves some various components and some structural defects. Its global behaviour will be associated with the local behaviour of its different constituents.

A complete review of classical approaches, particularly during cyclic loadings, was given in [6]. Here, we shall just say, when considering the volume element symbolized by a discrete assembly of local subelements of simple mechanical properties, that a reasonable description of the cyclic behaviour may be obtained.

Four types of subelements may be distinguished ; they are similar to :

1. slider
2. dashpot
3. in parallel slider-dashpot and linked between them by
4. linear springs [9].

By taking all the subelements as perfect, workhardening and worksoftening will only be the results of creation and redistribution of residual stresses within the volume element [4].

2.2 General formulae

2.2.1 Transformed parameters

At the level of any inelastic subelement of type 1, 2, 3, the local stresses are given by :

$$\begin{pmatrix} \underline{\sigma}^{(1)} \\ \underline{\sigma}^{(2)} \\ \underline{\sigma}^{(3)} \end{pmatrix} = \begin{pmatrix} \underline{A}^{(1)} \\ \underline{A}^{(2)} \\ \underline{A}^{(3)} \end{pmatrix} \sigma - \begin{pmatrix} \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \\ \hat{\underline{\gamma}} \end{pmatrix} \equiv \underline{A}\sigma - \hat{\underline{X}} \qquad (1)$$

where $\underline{\sigma}^{(1)}$ is the vector made of the local stress tensors for all subelements 1, so for $\underline{\sigma}^{(2)}$ and $\underline{\sigma}^{(3)}$; $\underline{A}^{(1)}$ is the vector made of the localisation tensors of elastic stresses for all subelements 1, so for $\underline{A}^{(2)}$ and $\underline{A}^{(3)}$;

$\hat{\underline{X}} = \begin{pmatrix} \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \\ \hat{\underline{\gamma}} \end{pmatrix}$ are the transformed parameters which are here indeed respectively the opposite of the residual stresses in all subelements 1, 2, 3. They are linked to the local inelastic strains at the level of subelements 1, 2, 3, $\underline{X} = \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \\ \underline{\gamma} \end{pmatrix}$, through a linear application :

$$\hat{\underline{X}} = \begin{pmatrix} \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \\ \hat{\underline{\gamma}} \end{pmatrix} = \mathfrak{B}\underline{X} \equiv \begin{pmatrix} B^{(1)} & C & D \\ C^T & B^{(2)} & E \\ D^T & E^T & B^{(3)} \end{pmatrix} \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \\ \underline{\gamma} \end{pmatrix} \qquad (2)$$

where $\mathfrak{B}$ is a non negative symmetrical matrix.

When $\mathfrak{B}$ is regular, there is a one to one application between $\hat{\underline{X}}$ and $\underline{X}$. If $\mathfrak{B}$ is singular, $\hat{\underline{X}}$ must belong to a linear subspace $\hat{\mathcal{L}}$, whose dimension is related to the rank of $\mathfrak{B}$, there will thus exist an infinite number of $\underline{X}$ such that eq. (2) is satisfied.

2.2.2 Evolution laws for the inelastic subelements

1 : They symbolize the perfect plasticity i.e. instantaneous inelastic strains and a threshold defined by a fixed convex set C_o. Locally, for a particular subelement, $\sigma^{(1)}$ must belong to C_o and the associated strain rate $\dot{\alpha}$ is an external normal or included in the case of external normals to C_o in $\sigma^{(1)}$.

In the $\hat{\underline{\alpha}}$ space, these conditions are globally expressed by [5], [12];

$\dot{\underline{\alpha}} \in - \partial\Psi_{\hat{\mathcal{C}}(\sigma)}(\hat{\underline{\alpha}})$ (3) .

The equation (3) implies that $\hat{\underline{\alpha}}$ belongs to the mobile convex set $\hat{\mathcal{C}}(\sigma)$, and $\dot{\underline{\alpha}}$ is an internal normal, or included in the cone of internal normals, to $\hat{\mathcal{C}}(\sigma)$; it is evident that $\hat{\mathcal{C}}(\sigma)$ is only subjected from $\hat{\mathcal{C}}_o$ to the translation $\underline{A}^{(1)}\sigma$ and may be built locally very easily.

2 : They symbolize the pure viscosity i.e. delayed inelastic strain without a threshold. Their evolution laws can be written :

$$\left.\begin{array}{l} \dot{\underline{\beta}} = \dfrac{\underline{\sigma}^{(2)}}{\underline{\eta}} = \dfrac{\underline{A}^{(2)}\sigma - \hat{\underline{\beta}}}{\underline{\eta}} \quad \text{or} \quad \dot{\beta} = - \dfrac{\partial\omega}{\partial\hat{\underline{\beta}}}(\underline{A}^{(2)}\sigma - \hat{\underline{\beta}}) \\ \text{with when } \underline{\sigma}^{(2)} \equiv 0 \Rightarrow \dot{\underline{\beta}} \equiv 0, \end{array}\right\} \qquad (4)$$

and where, wether linear or nonlinear viscoelasticity is implied

$\underline{\eta}$, representing the set of damping factors,

$\underline{\omega}$, the set of viscoelastic convex potentials.

3 : They symbolize the perfect viscoplasticity, i.e. delayed inelastic strain with a a threshold. One can locally take $\dot{\gamma} = \dfrac{\partial\Omega}{\partial\sigma^{(3)}}$ with when $\sigma^{(3)} \in D_o \Rightarrow \dot{\gamma} = 0$, D_o being a fixed convex set associated to the particular subelement in the $\sigma^{(3)}$ space, and Ω is any viscoplastic potential, convex function of $\sigma^{(3)}$. Globally, one can write :

$\dot{\underline{\gamma}} = - \dfrac{\partial\Omega}{\partial\hat{\underline{\gamma}}}(\underline{A}^{(3)}\sigma - \hat{\underline{\gamma}})$ with $\dot{\underline{\gamma}} \equiv 0$ when $\hat{\underline{\gamma}} \in \hat{\mathfrak{D}}(\sigma)$ (5)

where $\hat{\mathfrak{D}}(\sigma)$ is, in the $\hat{\underline{\gamma}}$ space, a mobile convex set subjected from $\hat{\mathfrak{D}}_o$ to the translation $\underline{A}^{(3)}\sigma$ and may be built locally easily (contrary to subelements 1, for subelements 3, $\hat{\underline{\gamma}}$ do not necessarily belong to $\hat{\mathfrak{D}}(\sigma)$).

2.2.3 Evolution laws for the transformed parameters and global strains

Thanks to (2), we have :

$\dot{\hat{\underline{X}}} = \mathfrak{B}\,\dot{\underline{X}}$ (6)

It may then be easily proved that the global inelastic strain rate for the volume element is given by :

$\dot{\varepsilon}^p = \dot{\underline{\alpha}}\,\underline{A}^{(1)} + \dot{\underline{\beta}}\,\underline{A}^{(2)} + \dot{\underline{\gamma}}\,\underline{A}^{(3)} = \underline{A}^T\,\dot{\underline{X}}$ (7) ,

and the global elastic strain rate is equal to :

$\dot{\varepsilon}^e = M\,\dot{\sigma}$ (8)

where M is the global elastic compliances matrix for the volume element.

So, in order to define the global behaviour of the volume element, we need to know :

$\underline{A}$, , M (a pureley elastic analysis),
all $\underline{C}_o$, $\underline{D}_o$, $\underline{\omega}$, $\underline{\Omega}$ (local inelastic constitutive laws),
and the initial values of $\underline{X}(0)$.

2.3 Cyclic global inelastic behaviour

The volume element is subjected to a cyclic homogeneous stress state $\sigma(t)$.

2.3.1 Eventual purely viscous response

For sake of simplicity, we shall assume here that if there may be some viscous subelements in the volume element, their evolution laws are given by the first equation of (4) i.e. we have a linear viscosity.

If $\underline{\alpha}$ and $\underline{\gamma}$ are all fixed to a zero value, we must solve :

$$\dot{\underline{\beta}}(t) = \frac{\underline{A}^{(2)}\sigma(t) - B^{(2)}\underline{\beta}(t)}{\eta} \qquad (9)$$

where $B^{(2)}$ is a symmetrical non negative matrix taken from $\mathfrak{B}$ and which has real non negative eigen values λ_k. The general solution of this system will be given by :

$$\underline{\beta}(t) = \underline{\beta}_v(t) + \sum_k \exp\{-\lambda_k t/\eta_k\}\, \underline{P}_k(t) \qquad (10)$$

where $\underline{\beta}_v(t)$ is a periodical bounded particular solution, and $\underline{P}_k(t)$ polynomial functions of t whose degree is equal to the order of multiplicity of $\lambda_k - 1$. When $B^{(2)}$ is a strictly definite matrix, what we shall assume from now, for t going towards ∞, $\underline{\beta}(t)$ will always go towards $\underline{\beta}_v(t)$.

2.3.2 General response

For the volume elements, let us set :

$$\underline{\beta}(t) = \underline{\beta}_v(t) + \underline{\beta}_1(t) \qquad (11)$$

($\underline{\beta}_v(t)$ is $\underline{0}$ if there is no viscous subelement), and define the new transformed parameters $\hat{\underline{X}}_1$ such that :

$$\hat{\underline{X}}_1 = \mathfrak{B}\begin{pmatrix}\underline{\alpha}\\ \underline{\beta}_1\\ \underline{\gamma}\end{pmatrix} \quad \text{and} \quad \hat{\underline{X}} \equiv \hat{\underline{X}}_1 + \begin{pmatrix} C\,\underline{\beta}_v \\ B^{(2)}\,\underline{\beta}_v \\ E^T\,\underline{\beta}_v\end{pmatrix} \qquad (12)$$

The evolution laws for the subelements will be transformed into :

$$\text{i)}\ \dot{\underline{\alpha}} \in -\partial\Psi_{\mathcal{C}_1(\sigma,\underline{\beta}_v)}(\hat{\underline{\alpha}}_1) \quad \text{with} \quad \hat{\underline{\alpha}}_1 \in \hat{\mathcal{C}}_1(\sigma,\underline{\beta}_v) = \hat{\mathcal{C}}_o + \underline{A}^{(1)}\sigma(t) - C\underline{\beta}_v(t) \qquad (13)$$

$$\text{ii)}\ \dot{\underline{\beta}}_1 = -\hat{\underline{\beta}}_1/\underline{\eta} \Rightarrow \dot{\underline{\beta}} = \dot{\underline{\beta}}_v + \dot{\underline{\beta}}_1 \qquad (14)$$

$$\left.\begin{aligned} &\text{iii)}\ \dot{\underline{\gamma}} = -\frac{\partial\Omega}{\partial\underline{\gamma}_1}(\underline{A}^{(3)}\sigma - C^T\underline{\beta}_v - \hat{\underline{\gamma}}_1) \\ &\quad \text{with when } \hat{\underline{\gamma}}_1 \in \hat{\mathfrak{G}}_1(\sigma,\underline{\beta}_v) = \hat{\mathfrak{G}}_o + \underline{A}^{(3)}\sigma(t) - E^T\underline{\beta}_v(t) \Rightarrow \dot{\underline{\gamma}} \equiv 0 \end{aligned}\right\} \qquad (15)$$

It is evident that two particular cases of loading, which will give direct bounds for the real one, may be considered :

i) a "static" loading such that $|\dot{\varepsilon}^p|$ is very near from 0 at any time ; this will imply that $\hat{\underline{\beta}}$ will always be near from 0 and that $\hat{\underline{\gamma}}$ will always approximately belong to $\mathfrak{G}(\sigma)$; in other words, we must only consider subelements of type 1 and 3 in the volume element ;

ii) a "dynamic" loading such that $|\dot{\varepsilon}^p|$ is almost infinite at any time ; this will produce $\underline{\beta} = \underline{\gamma} \equiv 0$; only the subelements 1 will take part in the deformation.

2.3.3 Eventual elastic shakedown [2], [1]

i) When $\mathfrak{B}$ is regular, the elastic shakedown condition is expressed very easily : there is a non void intersection $\hat{\mathcal{C}}_\ell$ of convex sets $\hat{\mathcal{C}}_1(\sigma(t),\underline{\beta}_v(t))$ in the $\hat{\underline{\alpha}}_1$ space and also a non void intersection $\hat{\mathcal{D}}_\ell$ of convex sets $\hat{\mathcal{D}}_1(\sigma(t),\underline{\beta}_v(t))$ in the $\hat{\underline{\gamma}}_1$ space when $t \in [0,\tau]$, period of the loading ; this is a local condition.

ii) When $\mathcal{B}$ is singular, the condition is more difficult and corresponds to a generalized Melan's theorem : there is a fixed $\hat{\underline{X}}_1^*$ such that $\hat{\underline{X}}_1^*$ belongs to $\hat{\mathcal{L}}$, $\hat{\underline{\alpha}}_1^*$ and $\hat{\underline{\gamma}}_1^*$ are respectively strictly included in the non void intersections $\hat{\mathcal{C}}_\ell$ and $\hat{\mathcal{D}}_\ell$ and at last $\hat{\underline{\beta}}_1^* = 0$.

iii) The number of cycles necessary to reach this limiting state may be infinite.

2.3.4 Eventual plastic shakedown [1]

i) For a regular $\mathcal{B}$ matrix, there will always be plastic shakedown (without failure or ratcheting) when the previous condition is not satisfied.

ii) For a singular $\mathcal{B}$ matrix, an a-priori knowledge of the limiting state is not yet possible when the previous condition is not verified. However, when the intersection between $\hat{\mathcal{L}}$ and $\hat{\mathcal{C}}_1(\underline{\sigma}(t),\underline{\beta}_v(t))$ is non void at any time t (necessary condition for the existence of $\hat{\underline{\alpha}}_1(t)$ or no failure) there will be a periodic solution $\hat{\underline{X}}_1(t)$ while $\underline{X}_1(t)$ may be or not be periodic ; so plastic shakedown or ratcheting may be obtained.

iii) The number of cycles necessary to reach the eventual limiting state cannot generally be known and may be any finite or infinite number.

2.4 An elementary example

Let us assume, we want to use only one yield criterion when the loading is uniaxial and there is no time effect.

For a one-dimensional criterion, within this framework, we obtain :

i) $\sigma_1^{(1)} = A_1^{(1)}\sigma - \hat{\alpha}_1$

ii) $\hat{\alpha}_1 = B_1^{(1)}\alpha_1$ with $B_1^{(1)} \geq 0$

iii) $C_o : |\sigma_1^{(1)}| < S_1 \Rightarrow \hat{\mathcal{C}}(\sigma) = \hat{\mathcal{C}}_o + A_1^{(1)}\sigma$

$\hat{\mathcal{C}}_o : |\hat{\alpha}_1| \leq S_1$

and $\dot{\hat{\alpha}}_1 \in -\partial\Psi_{\hat{\mathcal{C}}(\sigma)}(\hat{\alpha}_1) = \begin{cases} \dot{\hat{\alpha}}_1 \geq 0 & \text{if } A_1^{(1)}\sigma - \hat{\alpha}_1 = S_1 \\ \dot{\hat{\alpha}}_1 = 0 & \text{if } |A_1^{(1)}\sigma - \hat{\alpha}_1| < S_1 \\ \dot{\hat{\alpha}}_1 \leq 0 & \text{if } A_1^{(1)}\sigma - \hat{\alpha}_1 = -S_1 . \end{cases}$

iv) $\dot{\varepsilon}^P = A_1^{(1)}\dot{\alpha}_1$.

Two cases are possible. If $B_1^{(1)} = 0$ (singular matrix), we find the perfectly plastic material with the compatibility condition $\hat{\alpha}_1 = 0$. If $B_1^{(1)} > 0$ (regular matrix), we find the kinematic hardening material.

For a bi-dimensional criterion, always within this framework, we shall write :

i) $\begin{cases} \sigma_1^{(1)} = A_1^{(1)}\sigma - \hat{\alpha}_1 \\ \sigma_2^{(1)} = A_2^{(1)}\sigma - \hat{\alpha}_2 \end{cases}$

ii) $\hat{\underline{X}} = \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = \begin{pmatrix} B_{11}^{(1)} & B_{12}^{(1)} \\ B_{12}^{(2)} & B_{22}^{(1)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \mathcal{B}\,\underline{X}$ with $\Delta = B_{11}^{(1)}B_{22}^{(1)} - (B_{12}^{(1)})^2 \geq 0$

iii) $C_o : (\sigma_1^{(1)})^2 + (\sigma_2^{(2)})^2 < S_1^2 \Rightarrow \hat{\mathcal{C}}(\sigma) = \hat{\mathcal{C}}_o + \begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix}\sigma, \quad \hat{\mathcal{C}}_o : (\hat{\alpha}_1)^2 + (\hat{\alpha}_2)^2 < S_1^2,$

$\dot{\hat{\underline{X}}} = \begin{pmatrix} \dot{\hat{\alpha}}_1 \\ \dot{\hat{\alpha}}_2 \end{pmatrix} \in -\partial\Psi_{\hat{\mathcal{C}}(\sigma)}(\hat{\underline{X}})$

iv) $\dot{\varepsilon}^P = A_1^{(1)}\dot{\alpha}_1 + A_2^{(1)}\dot{\alpha}_2 = \underline{A}^T\,\dot{\underline{X}}$

The two possible cases correspond to $\Delta > 0$ and $\Delta = 0$ with when $B_{12}^{(1)} = \delta\sqrt{B_{11}^{(1)}B_{22}^{(2)}}$, the compatibility condition $\hat{\mathcal{L}}$: $\delta\sqrt{B_{22}^{(2)}/B_{11}^{(1)}}\;\hat{\alpha}_1 - \hat{\alpha}_2 = 0$.

Some simulated curves are plotted in fig. 1. We can easily find again that when $\Delta > 0$, there is always elastic or plastic shakedown and on contrary when $\Delta = 0$, there may be ratcheting or failure.

Several other examples which are more related to experimental results are given in another paper [13] .

3. The evolution of the structures

3.1 Problem

Let us now consider the problem of a structure V with boundary ∂V subjected to the cyclic thermomechanical loading : $\underline{\underline{\vec{X}}}^d(t)$ (*) body forces per unit mass in V, $\underline{\underline{\theta}}(t)$ temperature change $\underline{\underline{\theta}}^d$ in V, $\underline{\underline{F}}_i^d(t)$ surface forces on a part $\partial_{F_i}V$ of ∂V and displacements $\underline{\underline{u}}_j^d(t)$ on the complementary part $\partial u_j V$.

We shall assume here that the constitutive laws are obtained from section 2 and that they are temperature independent (indeed some special temperature dependences may be easily assumed too). One cumbersome method to analyze the structures would have been to follow it step by step and to look at the eventual final limiting state. Such an analysis is very expensive indeed ; so only a small number of cycles and empirical extrapolations are used.

The following special procedure is more efficient and practical ; it is only available when $\mathcal{B}$ at the scale of the volume element (constitutive relations) is regular. (It was already applied for structure with kinematic hardening material [10] , [11] , and it is, here, generalized to more complex viscoplastic behaviours).

It is just necessary to use a purely elastic analysis of the structure with only some minor modifications.

The principle will be however similar as in section 2. We shall introduce again new transformed parameters $\underline{\underline{\hat{X}}}'$ which will allow to treat easily the yield criteria for subelements 1 and 3 and which are linked to $\underline{\underline{X}}$ by a one-to-one relation.

3.2 Global evolution of the structure

3.2.1 Purely elastic response of the structure

Had the structure been made of a purely elastic material, with the same elastic matrix $\underline{\underline{M}}$ as the real one, we should have obtained the fields :

$\underline{\underline{\sigma}}^{el}(t)$ which is Statically Admissible (S.A) with $\underline{\underline{\vec{X}}}^d(t)$ in V and $\underline{\underline{F}}_i^d(t)$ on $\partial_{F_i}V$ and :

$\underline{\underline{\varepsilon}}^{el}(t) = \underline{\underline{M}}\underline{\underline{\sigma}}^{el}(t) + \underline{\underline{\varepsilon}}^{\theta}(t)$ which is Kinematically Admissible (K.A) with $\underline{\underline{u}}_j^d(t)$ on $\partial_{u_j}V$

where $\underline{\underline{\varepsilon}}^{\theta}(t)$ is the global dilatation strain field associated to $\underline{\underline{\theta}}(t)$.

3.2.2 Evolution of the structure

Since the actual material is not purely elastic, we shall have for the global total strain field within the structure :

$$\underline{\underline{\varepsilon}}(t) = \underline{\underline{\varepsilon}}^e(t) + \underline{\underline{\varepsilon}}^{\theta}(t) + \underline{\underline{\varepsilon}}^p(t) \qquad (16)$$

with

$$\underline{\underline{\varepsilon}}^e(t) = \underline{\underline{M}}\,\underline{\underline{\sigma}}(t) . \qquad (17)$$

By putting

$$\underline{\underline{\sigma}}(t) = \underline{\underline{\sigma}}^{el}(t) + \underline{\underline{\rho}}(t) , \qquad (18)$$

it is known that the residual stress field $\underline{\underline{\rho}}(t)$ is such that :

$$\left.\begin{array}{l} \underline{\underline{\rho}}(t) \text{ is S.A with } \vec{0} \text{ in V and 0 on } \partial_{F_i}V \\ \underline{\underline{M}}\,\underline{\underline{\rho}}(t) + \underline{\underline{\varepsilon}}^p(t) \text{ is K.A with 0 on } \partial_{u_j}V \end{array}\right\} \qquad (19)$$

and thus that $\underline{\underline{\rho}}(t)$ is associated to the global plastic strain field $\underline{\underline{\varepsilon}}^p(t)$ by :

$$\underline{\underline{\rho}}(t) = \mathbf{B}_o\,\underline{\underline{\varepsilon}}^p(t) \qquad (20)$$

where $\mathbf{B}$ is in fact a matrix when the structure is discrete or discretized as we shall assume it and which corresponds to the solution of an elastic problem with $\underline{\underline{M}}$ as the elastic matrix field, $\underline{\underline{\varepsilon}}^p(t)$ as initial strains field and the homogeneous boundary conditions

(*) Two bars $=$ under a quantity means a field of this quantity in the whole structure ; on the contrary, one bar _ means, as previously in § 2, the field in the volume element.

($\mathcal{O}$ in V, 0 on $\partial_{F_i}V$ and 0 on $\partial_{u_j}V$).

By introducing (18) in (1), we find locally :

$$\begin{pmatrix} \underline{\sigma}^{(1)} \\ \underline{\sigma}^{(2)} \\ \underline{\sigma}^{(3)} \end{pmatrix} = \underline{A}\,\sigma^{el}(t) - \{\hat{\underline{X}} - \underline{A}\,\rho\} \qquad (21)$$

which implies that the new natural transformed parameters family has to be introduced :

$$\hat{\underline{X}}' = \hat{\underline{X}} - \underline{A}\,\rho \qquad (22)$$

As we have (7), (20), (2), we deduce that :

$$\hat{\underline{\underline{X}}}' = \mathcal{B}'\,\underline{\underline{X}} \qquad (23)$$

where $\mathcal{B}'$ is now a (symmetrical non-negative) matrix with a much bigger dimension than $\mathcal{B}$.

The evolution laws for the structure will be then similar to those given in section 2. The new convex sets $\mathcal{E}'(\underline{\underline{\sigma}}^{el}(t))$ and $\hat{\mathcal{S}}'(\underline{\underline{\sigma}}^{el}(t))$ are still locally built. During cyclic loadings, the same conditions will be available, as in 2.3.3 and 2.3.4. When $\mathcal{B}$ is regular, it is straightforward to find $\mathcal{B}'^{-1}$ from a simple elastic calculus.

Indeed, we can write :

$$\varepsilon^p = \underline{A}^T\underline{X} = \underline{A}^T\mathcal{B}^{-1}\hat{\underline{X}} = \underline{A}^T\mathcal{B}^{-1}(\hat{\underline{X}}' + \underline{A}\rho) = \underline{A}^T\mathcal{B}^{-1}\hat{\underline{X}}' + \underline{A}^T\mathcal{B}^{-1}\underline{A}\rho \qquad (24)$$

which gives, by introduction in (19) :

$$(\underline{\underline{M}} + \underline{\underline{A^T\mathcal{B}^{-1}A}})\underline{\underline{\rho}} + \underline{\underline{A}}^T\mathcal{B}^{-1}\hat{\underline{\underline{X}}}' \quad \text{K.A. with } 0 \text{ on } \partial_{u_j}V \qquad (25)\ .$$

This means, we must solve another homogeneous elastic problem with the new elastic matrix field $\underline{M + A^T\mathcal{B}^{-1}A^T}$ and the new initial strains field $\underline{A^T\mathcal{B}^{-1}\hat{X}'}$. Symbolically, (25) implies :

$$\underline{\underline{\rho}} = \mathcal{B}'_o\,\hat{\underline{\underline{X}}}' \qquad (26)$$

and we deduce : $\underline{\underline{X}} = \mathcal{B}'^{-1}\,\hat{\underline{X}}'$ (27)

This proves that a regular $\mathcal{B}$ matrix (constitutive relations) will produce a regular $\mathcal{B}'$ matrix (structure). (It is only a sufficient condition to reach a regular $\mathcal{B}'$ matrix). There will then always be shakedown.

By considering "static" and "dynamic" loadings, direct bounds can be produced.

4. The classical three bars system

This classical elementary structure is only given here in order to show how to use our framework. A general structure will be treated similarly.

4.1 Description

It consists of 3 bars fixed to two rigid supports to which we apply a constant axial force F. Two identical bars (index 1) are placed symmetrically with respect to the third one (index 2). The two bars 1 are subjected to the change of temperature $\theta_1(t)$ assigned to be $\theta_1(t) = \Delta T \sin 2\pi\omega t$ while the bar 2 is kept at a constant temperature.

i) The rigid supports impose at any time the Kinematically Admissible condition : $\varepsilon_1 = \varepsilon_2$ which we can write as : $\sigma_1/E_1 + \alpha_1\theta_1 + \varepsilon_1^p = \sigma_2/E_2 + \varepsilon_2^p$

where ε_1, ε_2, and ε_1^p, ε_2^p are the total and plastic axial strains,

σ_1, σ_2, the axial stresses and E_1, E_2, the elastic Young's modulus in the indicated bars, and α_1 the coefficient of thermal expansion in 1.

ii) The global equilibrium requires the Statically Admissible condition with F :

$$2\,S_1\,\sigma_1 + S_2\,\sigma_2 = F$$

where we shall assume $2S_1 = S_2$; we can thus write : $\sigma_1 + \sigma_2 = F/2S_2 \equiv 2\sigma_A$.

This cyclic thermomechanical loading will produce the elastic stress field :

$\sigma_1^{el} = 2\sigma_A\,E_1/(E_1+E_2) - \sigma_T \sin 2\pi\omega t$, $\sigma_2^{el} = 2\sigma_A\,E_2/(E_1+E_2) + \sigma_T \sin 2\pi\omega t$,

where $\sigma_T = E_1E_2/(E_1+E_2)\alpha_1\Delta T$;

4.2 Constitutive hypothesis

We shall assume for bars 1 :

i) $\underline{X}_1 = \begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$

ii) $\sigma_1^{(1)} = \sigma_1 - h_1\alpha_1$

$\sigma_1^{(2)} = \sigma_1 - k_1\beta_1$

$$\hat{\underline{X}}_1 = \begin{pmatrix}\hat{\alpha}_1\\ \hat{\beta}_1\end{pmatrix} = \begin{pmatrix}h_1 & 0\\ 0 & k_1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \mathfrak{B}_1\,\underline{X}_1$$

iii) $\varepsilon_1^p = \alpha_1 + \beta_1 = \underline{A}_1^T\,\underline{X}_1$,

iv) and the evolution laws : $\begin{cases}\dot{\alpha}_1 - \partial\,\mathscr{C}_{1}(\sigma_1)(\alpha_1)\\ \dot{\beta}_1 = \sigma_1^{(2)}/\eta_1\end{cases}$

with similar expressions for $\underline{X}_2$, $\hat{\underline{X}}_2$, $\mathfrak{B}_2$, $\underline{A}_2$, ε_2^p in bar 2 .

Indeed for each bar we have taken an inserial assembly of one slider-spring, one dashpot-spring and one spring.

4.3 Systematical treatment of the structure

4.3.1 Residual stress field (20)

The matrix $\mathbf{B}$ is here obtained by writing (19) :

$\rho_1 + \rho_2 = 0$ S.A with 0

$\rho_1/E_1 + \varepsilon_1^p = \rho_2/E_2 + \varepsilon_2^p$ K.A with 0 .

$$\underline{\underline{\rho}} = \begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = \begin{pmatrix}-G & G\\ G & -G\end{pmatrix}\begin{pmatrix}\varepsilon_1^p\\ \varepsilon_2^p\end{pmatrix} \equiv \underline{\underline{\varepsilon}}^p \quad \text{with} \quad G = E_1E_2/(E_1+E_2)$$

4.3.2 New transformed parameters (22)

$$\hat{\underline{X}}'_1 = \hat{\underline{X}}_1 - \underline{A}_1\rho_1$$
$$\hat{\underline{X}}'_2 = \hat{\underline{X}}_2 - \underline{A}_2\rho_2$$
$$\Rightarrow \quad \hat{\underline{\underline{X}}}' = \begin{pmatrix}\hat{\underline{X}}'_1\\ \hat{\underline{X}}'_2\end{pmatrix} = \begin{pmatrix}\hat{\alpha}'_1\\ \hat{\beta}'_1\\ \hat{\alpha}'_2\\ \hat{\beta}'_2\end{pmatrix} = \mathfrak{B}'\begin{pmatrix}\alpha_1\\ \beta_1\\ \alpha_2\\ \beta_2\end{pmatrix} = \mathfrak{B}'\underline{\underline{X}}$$

$$\mathfrak{B}' = \begin{pmatrix}G+h_1 & G & -G & -G\\ G & G+k_1 & -G & -G\\ -G & -G & G+h_2 & G\\ -G & -G & G & G+k_2\end{pmatrix}$$

When $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are regular ($h_1k_1 > 0$ and $h_2k_2 > 0$), $\mathfrak{B}'$ will also be regular and its inverse will be obtained through (24) :

$$(1/E_1 + 1/h_1 + 1/k_1)\rho_1 + (\hat{\alpha}'_1/h_1 + \hat{\beta}'_1/k_1) = (1/E_2 + 1/h_2 + 1/k_2)\rho_2 + (\hat{\alpha}'_2/h_2 + \hat{\beta}'_2/k_2) \ ;$$

by putting respectively for bars 1 and 2 :

$1/E' = (1/E + 1/h + 1/k)$, $\varepsilon^{p\prime} = \hat{\alpha}'/h + \hat{\beta}'/k$, we deduce :

$$\underline{\underline{\rho}} = \begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = \begin{pmatrix}-G' & G'\\ G' & -G'\end{pmatrix}\begin{pmatrix}\varepsilon_1^{p\prime}\\ \varepsilon_2^{p\prime}\end{pmatrix} \quad \text{with} \quad G' = E'_1E'_2\,/\,(E'_1 + E'_2)$$

At last, we find for $\mathfrak{B}'^{-1}$:

$$\underline{\underline{X}}' = \begin{pmatrix}\alpha_1\\ \beta_1\\ \alpha_2\\ \beta_2\end{pmatrix} = \begin{pmatrix}(1-G'/h_1)/h_1 & -G'/k_1h_1 & G'/h_2k_1 & G'/k_2h_1\\ -G'/h_1k_1 & (1-G'/k_1)/k_1 & G'/h_2k_1 & G'/k_2k_1\\ G'/h_1h_2 & G'/k_1h_2 & (1-G'/h_2)/h_2 & -G'/k_2h_2\\ G'/h_1k_2 & G'/k_1k_2 & -G'/h_2k_2 & (1-G'/k_2)/k_2\end{pmatrix}\begin{pmatrix}\hat{\alpha}'_1\\ \hat{\beta}'_1\\ \hat{\alpha}'_2\\ \hat{\beta}'_2\end{pmatrix}$$

(It is however evident that when only one among h_1, k_1, h_2, k_2, is null, the matrix $\mathfrak{B}'$ is still regular even if one local $\mathfrak{B}_1$ or $\mathfrak{B}_2$ is no more regular.)

4.3.3 Direct bounds

i) "static" loading : Since $\hat{\beta}_1 = \hat{\beta}_2 \equiv 0$, we just find a material with h as the kinematic hardening modulus, and a new elastic Young's modulus $E'' = Ek/(E + k)$; thus the elastic stresses induced by the loading are :

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$\sigma_1''^{el} = 2\sigma_A\ E_1''/(E_1''+E_2'') \ - \ E_1''E_2''/(E_1''+E_2'')\ \alpha_1\ \Delta T \sin 2\pi\omega t$

$\sigma_2''^{el} = 2\sigma_A\ E_2''/(E_1''+E_2'') \ + \ E_1''E_2''/(E_1''+E_2'')\ \alpha_1\ \Delta T \sin 2\pi\omega t$.

ii) "dynamic" loading : Since $\beta_1 = \beta_2 = 0$, we find again a material with h as the kinematic hardening modulus and E as the Young's modulus ; the elastic stresses are σ_1^{el} and σ_2^{el} with "instantaneous" changes between their extremal values.

iii) J. ZARKA and J. CASIER [10] gave explicitely for such cases the limiting states.

4.3.4 General response

It will be a function of the frequency of the loading and obtained as in § 2.3. Some trajectories in the $(\hat{\alpha}_1, \hat{\alpha}_2)$ plane and the corresponding strains are given in fig. 2.

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Complete references could be found in detail in [6] .

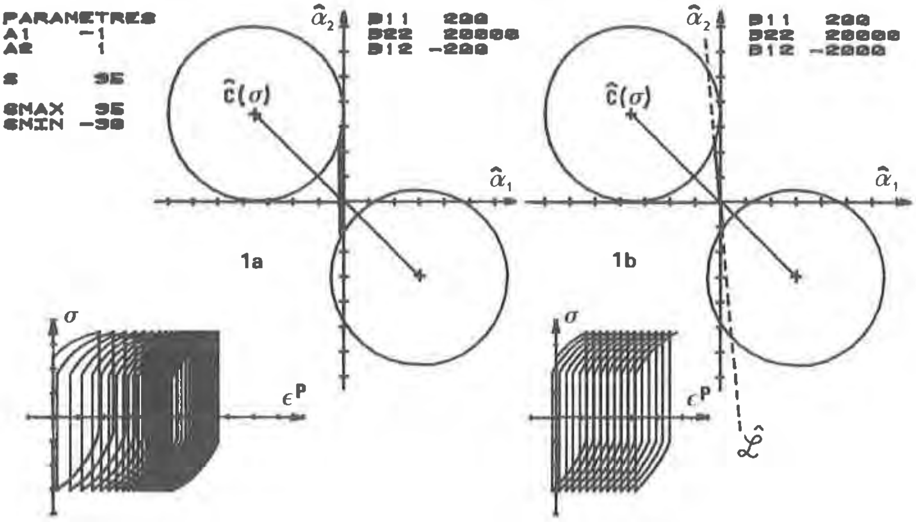

Figure 1 : A bidimensional criterion during uniaxial loading

1_a) Regular matrix $\mathcal{B}$
Plastic shakedown after some cycles

1_b) Singular matrix $\mathcal{B}$
Ratcheting after the first cycle

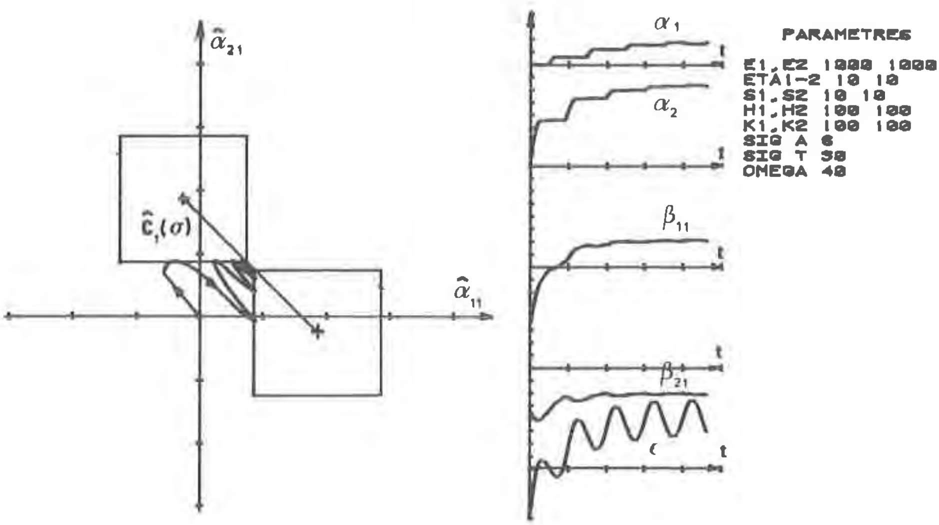

Figure 2 : An example of plastic shakedown in the three bar system