AN INTERIOR COLLOCATION METHOD FOR VIBRATION OF A RECTANGULAR PLATE CARRYING ATTACHED MASS

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SUMMARY
In the analysis of equipment foundations of many Category I equipment, an upper bound analysis is performed to determine the loads on the foundation during a Safe Shutdown Earthquake (SSE) condition. Often the analysis model reduces to the determination of the fundamental frequency of a rectangular plate (foundation) with attached mass (equipment). The objective being to determine if the system is rigid (fundamental frequency of the system \( \geq 33 \) cycles/sec.) or flexible (fundamental frequency of the system \(< 33 \) cycles/sec.). This determination is an important part of the seismic qualification analysis of all Category I equipment. In cases when the fundamental frequency is greater than 33 cycles/sec. (or in some specific combined loading cases - such as Safety Relief Valve (SRV) loading + SSE - greater than any specified threshold frequency), the seismic qualification method simplifies to an equivalent static analysis. This latter analysis is generally based on the ZPA of the applicable response spectra and is very well adaptive to standardization of the equipment qualification for various nuclear plants.

A new, simplified method is developed for computing the fundamental frequency of this coupled plate/mass system. This approach is based on the interior collocation method for vibration analysis. The differential equation for small free oscillations of the homogeneous flat plate of uniform thickness with no constraints other than at the boundaries is given by:

\[
D \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] + \rho \frac{\partial^2 w}{\partial t^2} = 0
\]

where
- \( D \) = structural rigidity of the plate.
- \( x, y \) = Cartesian coordinates in plane of plate
- \( t \) = time variable
- \( w(x,y,t) \) = normal deflection of the plate - time and space dependent.
- \( \rho \) = mass density of the plate

This equation is modified for a concentrated mass at an arbitrary location \((c, r)\), and solution is developed for the modified fundamental frequency of the assembly. The modification technique is quite general and can be used either for a single mass or any number of masses at arbitrary locations. Results from this new method are compared with the earlier solutions obtained by other investigators using Lagrange Equation and the Finite Fourier Transfer technique using dirac function. Good agreement is obtained between this method and the previous solutions. This method is of special use in the preliminary seismic analysis of Category I structures in nuclear power plant application.
1. INTRODUCTION

In the analysis of equipment foundations of many Category I equipment, an upperbound analysis is performed to determine the loads on the foundation during Safe Shutdown Earthquake (SSE) condition. Often the analysis model reduces to the determination of the fundamental frequency of a rectangular plate (foundation) with attached mass (equipment). The objective being to determine if the system is rigid (fundamental frequency $\geq 33$ cps) or flexible (fundamental frequency $< 33$ cps). As is well known in the field of seismic qualification of Category I equipment for the Nuclear Power Plant Applications [1, 2], the method of seismic analysis simplifies considerably when the equipment is rigid.

The problem of vibration of a rectangular plate with attached mass was investigated by previous investigators using (a) Lagrange Equations [3, 4], (b) Finite Fourier Transform with dirac $\delta$ function [5], (c) Laplace Transform [6] and (d) Finite Element Method [7, 8].

In this paper a new, simplified approach is used to solve the vibration problem of a rectangular plate with attached mass. This approach is based on the Interior Collocation Method. Results are compared with the earlier two methods [a, b above] and good agreement is obtained between the present method and earlier solutions. It is also felt that this method is simpler to use than the earlier methods.

2. NOMENCLATURE

$x, y =$ Cartesian coordinates in plane of plate

$\zeta, \eta =$ Co-ordinates of concentrated mass

$a, b =$ length and width of rectangular plate

$w(x, y, t) =$ normal deflection of the plate - time and space dependent

$W(x, y) =$ space dependent normal deflection of the plate

$h =$ thickness of the plate

$E =$ Young's modulus of the plate material

$\nu =$ Poisson's ratio of the plate material

$D =$ flexural rigidity of the plate

$\rho =$ mass density of the plate

$\omega =$ natural frequency of the loaded plate

$V =$ potential energy of the system

$T =$ kinetic energy of the system

$m =$ mass of the body attached to the plate

$T$ with corresponding suffix = mass moment of inertia about respective axes.

\[1\] Numbers in the brackets designate References at end of paper.
3. METHOD OF ANALYSIS

Figure (1) shows the analytical model of a rectangular plate with attached mass.

The differential equation for small free oscillations of a homogeneous flat plate of uniform thickness with no constraints other than at the boundaries is:

\[ Dv^4w + \rho \ddot{w} = 0 \]  

(1)

Now considering the attached mass \( m \) at location \( \left( \frac{3a}{4}, \frac{3b}{4} \right) \) [Figure 1], the differential equation of the system of plate plus attached mass can be written as:

\[ Dv^4w + \left[ \rho h + m \delta \left( x - \frac{3a}{4} \right) \delta \left( y - \frac{3b}{4} \right) \right] \frac{\partial^2 w}{\partial t^2} = 0 \]  

(2)

where

\[ \int_{-\infty}^{\infty} \delta \left( x - \frac{3a}{4} \right) \, dx = 1 \]  

(3)

and

\[ \int_{-\infty}^{\infty} \delta \left( x - \frac{3b}{4} \right) \, dy = 1 \]  

(4)

The boundary conditions for the plate of Figure (1) are:

\[ w = 0 \quad \text{along} \quad x = 0, a \]  

(5)

\[ \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{along} \quad x = 0, a \]  

(6)

\[ w = 0 \quad \text{along} \quad y = 0, b \]  

(7)

\[ \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{along} \quad y = 0, b \]  

(8)

We now select two functions \( v_1, v_2 \) and two stations \( s_1, s_2 \).

From (2), the differential operators are given by

\[ L = D \left[ \frac{\partial^4 h}{\partial x^4} + 2 \frac{\partial^4 h}{\partial x^2 \partial y^2} + \frac{\partial^4 h}{\partial y^4} \right] \]  

(9)

\[ M = \left[ \rho h + m \delta \left( x - \frac{3a}{4} \right) \delta \left( y - \frac{3b}{4} \right) \right] \]  

(10)
Let
\[ v_1 = \sin \frac{x}{a} \sin \frac{y}{b}, \quad s_1 = \left(\frac{3}{4}, \frac{b}{4}\right) \]
\[ v_2 = \sin \frac{2x}{a} \sin \frac{y}{b}, \quad s_2 = \left(\frac{3a}{4}, \frac{3b}{4}\right) \]

(It may be noted that \( v_1 \) and \( v_2 \) do satisfy the boundary conditions 5 through 8)

Now
\[
L \left[ v_1(x) \right] = D \left[ \frac{x^4}{a^4} \sin \frac{x}{a} \sin \frac{y}{b} + \frac{2x^4}{a^2b^2} \sin \frac{x}{a} x \sin \frac{y}{b} \right]
\]
\[
\sin \frac{y}{b} + \frac{b}{a^2} \sin \frac{2x}{a} \sin \frac{y}{b} \right] \]  

(11)
\[
L \left[ v_2(x) \right] = D \left[ \frac{16x^4}{a^4} \sin \frac{2x}{a} \sin \frac{y}{b} + \frac{8x^4}{a^2b^2} \sin \frac{2x}{a} x \sin \frac{y}{b} \right]
\]
\[
\sin \frac{y}{b} + \frac{b}{a^2} \sin \frac{2x}{a} \sin \frac{y}{b} \right] \]  

(12)
\[
M \left[ v_1(x) \right] = \left[ \rho h + m \delta (x - \frac{3a}{4}) \delta (y - \frac{3b}{4}) \right] x \sin \frac{x}{a} \sin \frac{y}{b}
\]
\[
(13)
\]
\[
M \left[ v_2(x) \right] = \left[ \rho h + m \delta (x - \frac{3a}{4}) \delta (y - \frac{3b}{4}) \right] x \sin \frac{2x}{a} \sin \frac{y}{b}
\]
\[
(14)
\]

Now,
\[
k_{11} = L \left[ v_1(s_1) \right]
\]
\[
= D \left[ \frac{x^4}{a^4} \sin \left(\frac{x}{a} \cdot \frac{a}{4}\right) \sin \left(\frac{y}{b} \cdot \frac{b}{4}\right) + \frac{2x^4}{a^2b^2} \sin \left(\frac{x}{a} \cdot \frac{a}{4}\right) \sin \left(\frac{y}{b} \cdot \frac{b}{4}\right) \right]
\]
\[+ \frac{b^4}{a^4} \sin \left(\frac{x}{a} \cdot \frac{a}{4}\right) \sin \left(\frac{y}{b} \cdot \frac{b}{4}\right) \right] \]
\[
= \frac{Dx^4}{2} \left[ \frac{1}{a^4} + \frac{2}{a^2b^2} + \frac{1}{b^4} \right]
\]
\[
(15)\]
\[ k_{12} = D \left[ v_2 (s_1) \right] = D \left[ \frac{16 \pi h}{a^4} \sin \left( \frac{2\pi}{a} \cdot \frac{a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{b}{4} \right) \right. \]
\[ + \frac{8 \pi h}{a^2 b^2} \sin \left( \frac{2\pi}{a} \cdot \frac{a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{b}{4} \right) + \frac{\pi h}{b^4} \sin \left( \frac{2\pi}{a} \cdot \frac{a}{4} \right) \]
\[ \left. \sin \left( \frac{\pi}{b} \cdot \frac{b}{4} \right) \right] = \frac{D \pi h}{\sqrt{2}} \left[ \frac{16}{a^4} + \frac{8}{a^2 b^2} + \frac{1}{b^4} \right] \]  
\[ (16) \]

\[ k_{21} = D \left[ \frac{\pi h}{a} \sin \left( \frac{\pi}{a} \cdot \frac{3a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{3b}{4} \right) \right. \]
\[ + \frac{2 \pi h}{a^2 b^2} \sin \left( \frac{\pi}{a} \cdot \frac{3a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{3b}{4} \right) \]
\[ \frac{\pi h}{b^4} \sin \left( \frac{\pi}{a} \cdot \frac{3a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{3b}{4} \right) \right] \]
\[ = \frac{D \pi h}{2} \left[ \frac{1}{a^4} + \frac{2}{a^2 b^2} + \frac{1}{b^4} \right] \]  
\[ (17) \]

and

\[ k_{22} = L \left[ v_2 (s_1) \right] = D \pi h \sin \left( \frac{2\pi}{a} \cdot \frac{3a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{3b}{4} \right) \]
\[ \left[ \frac{16}{a^4} + \frac{8}{a^2 b^2} + \frac{1}{b^4} \right] \]  
\[ (18) \]

Further,

\[ m_{11} = M \left[ v_1 (s_1) \right] \]
\[ = \left[ \phi h \right] \sin \left( \frac{\pi}{a} \cdot \frac{a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{b}{4} \right) \]
\[ = \frac{\phi h}{2} \]  
\[ (19) \]

\[ m_{12} = M \left[ v_2 (s_1) \right] \]
\[ = \phi h \left[ \sin \left( \frac{2\pi}{a} \cdot \frac{a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{b}{4} \right) \right] = \frac{\phi h}{\sqrt{2}} \]  
\[ (20) \]
\[ m_{21} = M \left[ v_1 \left( s_2 \right) \right] \]
\[ = [\rho h + m] \sin \left( \frac{\pi}{a} \cdot \frac{3a}{4} \right) \sin \left( \frac{\pi}{b} \cdot \frac{3b}{4} \right) \]
\[ = \frac{\rho h + m}{2} \]

(21)

\[ m_{22} = M \left[ v_2 \left( s_2 \right) \right] = [\rho h + m] \sin \left( \frac{2\pi}{a} \cdot \frac{3a}{4} \right) \times \sin \left( \frac{\pi}{b} \cdot \frac{3b}{4} \right) = -\frac{\rho h + m}{\sqrt{2}} \]

(22)

The eigenvalue problem can now be written in the form

\[ \begin{bmatrix} k \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \omega^2 \begin{bmatrix} m \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \]

or

\[ \begin{bmatrix} k \end{bmatrix}^{-1} \begin{bmatrix} m \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \frac{1}{\omega^2} \begin{bmatrix} A \end{bmatrix} \]

(23)

Substitute for \( k_{ij}, m_{ij} \) from above and solve for \( \omega^2, A \), which is the required solution.

4. NUMERICAL EXAMPLE

For numerical calculations, we shall assume the following dimensions:

\( a = 30''; b = 20''; h = 0.1; \text{Mass } m = 5 \text{ lbs.} \)

Elastic constants \( E = 30 \times 10^6 \text{ psi} \) and \( v = .3 \)

Using the above analysis, we get

\[ \begin{array}{l}
  k_{11} = L \left[ v_1(s_1) \right] = 1.7 \\
  k_{12} = L \left[ v_2(s_1) \right] = 9.01 \\
  k_{21} = L \left[ v_1(s_2) \right] = 1.7 \\
  k_{22} = L \left[ v_2(s_2) \right] = -9.01 \\
  m_{11} = M \left[ v_1(s_1) \right] = 3.65 \times 10^{-3} \\
  m_{12} = M \left[ v_2(s_1) \right] = 5.17 \times 10^{-3} \\
  m_{21} = M \left[ v_1(s_2) \right] = 4.5 \times 10^{-3} \\
  m_{22} = M \left[ v_2(s_2) \right] = -6.1 \times 10^{-3}
\end{array} \]
Solving the eigenvalue problem by iteration, after 2nd iteration the fundamental mode is obtained as:

\[ w = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} + 0.8 \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \]

and the fundamental frequency is given by:

\[ w = 20.06 \text{ cycle/sec}. \]

5. COMPARISON WITH OTHER METHODS

The above numerical problem was solved using (a) Lagrange's eqn. [Ref. 3] method and (b) Dirac $\delta$ function [Ref. 5] method. The results for the fundamental frequency are quite close to the one calculated here. The values are:

- Dirac $\delta$ Function Method: \( f = 20.16 \) cycles/sec.
- Present Method: \( f = 20.06 \) cycles/sec.

6. CONCLUDING REMARKS

The method is relatively simpler to use than the other techniques and it provides reasonably accurate value for the fundamental frequency. It can further be generalized for several masses although the hand calculations may be too lengthy and solutions using a computer might be necessary.

7. REFERENCES


Figure 1

(uniform thickness)