IMPLEMENTATION OF ENDOCHRONIC THEORY FOR CONCRETE WITH EXTENSION TO INCLUDE CRACKING

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The aim of the research described in this paper has been to contribute to the development of a general nonlinear analysis procedure which can account for all important mechanical properties of concrete and which is applicable to a wide variety of structures and loadings.

The paper is based in part on the Ph.D. thesis of Litton (1975), who developed a general theory for a material which is elastic in compression and cracks in tension, and in part on the thesis of deVilliers (1977), who extended the procedure to include inelastic behavior in compression. In this latter work, Bazant's "endochronic" theory for concrete, which describes the inelastic behavior of concrete in compression and tension up to cracking, was judged to be the most comprehensive theory available. The theory is particularly valuable in that it accounts for dilation of the concrete when it cracks, so that the beneficial effects of confinement can be modelled. Bazant's theory was adopted and extended to include the effects of cracking, including the effects of "tensile softening" and cyclic opening-closing-reopening of cracks. The cracking theory is general and elegant and can be applied to both two- and three-dimensional problems. The theoretical formulation is similar to that for plasticity in metals.

A two-dimensional finite element has been developed for the analysis of plane and axisymmetric concrete structures. This element has been incorporated into ANSR, a general purpose computer program for the static and dynamic analysis of nonlinear structures. The procedure can be extended to three-dimensional solids without major difficulty. The model as currently implemented does not account for shear slip across cracks, which for some problems can be a serious omission. Also, shrinkage and creep are currently ignored.

The theoretical formulation centers primarily on determination of the tangent stress-strain relationship for the cracked inelastic material. The most complex aspect of the computational procedure is the "state determination" calculation, in which a new stress state is determined given an existing state and a strain increment.

The agreement with experiment is far from exact but encouragingly close. Computational difficulties were experienced with slow convergence during cracking, and numerical instability when the concrete became severely cracked under cyclic loads. A visco-plastic type of procedure, involving the addition of viscous damping, was successful in eliminating these numerical instability effects.
1. Introduction

Nonlinear analysis techniques have developed rapidly in recent years, and a substantial number of applications to concrete structures have been reported. Without exception, however, the procedures developed to date have been applicable to only limited types of structures, or have been based on questionable assumptions about the mechanical behavior of concrete. There is a need for a general nonlinear analysis procedure which accounts for all important mechanical properties of concrete, and which is applicable to a wide variety of structures and loadings. The objective of the work described in this paper has been to contribute to the development of such an analysis procedure. To achieve this objective, available knowledge on the behavior of concrete and available procedures for modelling this behavior were first reviewed. From this review the most promising procedure was selected for detailed study and for implementation in a general purpose nonlinear computer code.

From the review of available modelling procedures, Bazant and Bhat's "endochronic" theory [1, 2] for modelling concrete behavior appeared to be superior to all others. That part of the theory which accounts for behavior in compression and tension prior to cracking was adopted without modification. The theory was extended to account for postcracking behavior following a procedure developed by Litton [3]. The theory was then arranged for practical computation, and incorporated into the general purpose nonlinear code ANSR-I [4]. This computer code is particularly convenient for research use, and has the potential for development into a production-oriented computer program.

The resulting computational procedure is applicable to finite element idealizations for plane strain, plane stress (including the effects of out-of-plane confinement) and axisymmetric solid analysis under dynamic and short-term static loads with arbitrary load paths, including reversal. The procedure has the capability of being extended to three-dimensional solid analysis.

The procedure does not consider creep and shrinkage effects, although consideration of these effects is within the scope of the endochronic theory. Other limitations are that questionable assumptions are made about shear deformations parallel to open cracks, and that the numerical computation tends to become unstable when the concrete is severely cracked.

2. Important Aspects of Concrete Behavior

Concrete behaves elastically only at very low stresses. For higher stress values, permanent deformation takes place. Close to failure the concrete dilates. Under confinement, dilation is restricted, and the strength and ductility increase. Under cyclic loading, the strength and stiffness both deteriorate. In tension, cracks form perpendicular to the principle tension stress when the tensile strength is exceeded. No tension stress can be transferred normal to an open crack, but shear can be transferred because of aggregate interlock and other effects. Reinforcing steel is bonded nonlinearly to the concrete. As bond stress increases, slip occurs between the steel and adjacent concrete. The presence of steel gives rise to the "tensile softening" effect. Steel across a crack increases shear stiffness parallel to the crack by dowel action and by providing restraint against crack opening.
3. Endochronic Theory

The "endochronic" concept was introduced by Valanis [5] in 1971 as the basis of a theory of viscoplasticity. The theory is based on the assumption that the incremental stress-strain relationship for an inelastic material can be expressed in terms of certain measures of the deformation history of the material. In the theory presented by Valanis for metals, the deformation history is characterized by a single parameter, \( \xi \). Time dependent effects can be introduced by defining a second parameter, \( \xi_2 \), which depends on \( \xi \) and on the real time, \( t \). The behavior of the material is defined in terms of a function \( z(\xi, \xi_2, \sigma) \), where \( \sigma \) = stress. The function \( z \) is termed the "intrinsic time scale." Valanis applied the theory to model complex experimental results for metals, and obtained close agreement.

To extend the endochronic theory to concrete, four characteristics of concrete which are not present in metals must be introduced, namely: (a) inelastic volumetric strain; (b) hydrostatic pressure sensitivity; (c) strain softening; and (d) dependence of tangent moduli on dilation.

These extensions were incorporated into the theory by Bazant and Bhat [1]. The deviatoric and volumetric stress-strain relationships are

\[
\begin{align*}
\text{dev}_{ij} & = \text{dev}_{ij}/2G + \text{dev}_{ij}' \\
\text{dev}_{m} & = \text{dev}_{m}/3K + \text{dev}_{m}''
\end{align*}
\]

(1)

\[
\begin{align*}
\text{dev}_{m} & = \text{dev}_{m}/3K + \text{dev}_{m}'' \\
\text{dev}_{m} & = \text{dev}_{m}/3K + \text{dev}_{m}''
\end{align*}
\]

(2)

in which

\( \varepsilon_m = \varepsilon_{kk}/3 \) = mean strain; \( \sigma_m = \sigma_{kk}/3 \) = mean stress; \( \text{dev}_{ij} \) = deviatoric strain tensor; \( \sigma_{ij} = \sigma_{ij} - \delta_{ij} \sigma_m \) = deviatoric stress tensor; \( \text{dev}_{ij}' \) = nonlinear part of deviatoric strain tensor; \( \varepsilon_m ' = \text{nonlinear part of mean strain} \); \( \text{dil} = \text{dil}(\lambda, \sigma, \varepsilon, \text{dil}) \) = dilation increment due to distortion; \( \text{dil}' = \text{dil}'(\lambda', \sigma, \varepsilon, \text{dil}) = \text{dilation increment due to volume change} \); \( \text{dil} = \lambda(\text{dil}(\lambda, \sigma, \varepsilon, \text{dil})) \) = intrinsic time increment for distortion; and \( \text{dil}' = \lambda' \text{dil}'(\lambda', \sigma, \varepsilon, \text{dil}) \) = intrinsic time increment for volume change.

The functions for determining the infinitesimal increments \( \text{dil}, \text{dil}', \text{dil}, \text{dil}' \) were devised by Bazant and Bhat so that the essential features of concrete behavior were modeled. This involved both the selection of functions and determination of the values of constants contained in the functions. The functions were first chosen to represent the aspects of behavior qualitatively, and the constants were then determined by an optimization procedure such that the theory matched experimental results as closely as possible [1]. From these equations, a tangent constitutive relationship can be established in the form

\[
\text{d} \sigma = C \text{d} \varepsilon
\]

(3)

in which \( C \) is the tangent constitutive matrix. Space does not permit derivation of the theory herein. It may be noted, however, that \( C \) is unsymmetric, and that it depends on the direction of straining. Because this direction is generally not known \( \text{a priori} \), some computational complications are introduced.

4. Cracking Theory

A cracking theory for elastic-cracked material has been developed by Litton [3] in which the elastic constitutive matrix is modified to account for crack formation, and in which the crack opening strains are monitored to allow for crack opening and closing. This
theory can be extended to the case of inelastic-cracked material, and combined with the endochronic theory. The theory is based on the following assumptions:

1. The total strains can be divided into two parts, namely strain in the uncracked concrete between cracks ($\varepsilon_u$), and strain due to crack opening ($\varepsilon_c$). That is:

$$\varepsilon = \varepsilon_c + \varepsilon_u$$  \hspace{1cm} (4)

2. The stress increments, $d\sigma$, are related to the uncracked strain increments, $d\varepsilon_u$, through the uncracked constitutive matrix $C$ of the endochronic theory. That is:

$$d\sigma = C \, d\varepsilon_u$$  \hspace{1cm} (5)

3. The cracking strains, $\varepsilon_c$, result from normal crack opening strains only, with no shear displacement permitted across cracks. That is, a crack is idealized as shown in Fig. 1. The vector of crack opening strains normal to each crack is $\varepsilon_c$. The relationship between cracking strains and crack opening strains can be expressed as:

$$d\varepsilon_c = M^T \, d\varepsilon_c$$  \hspace{1cm} (6)

in which $M$ is a transformation matrix.

4. A crack forms when the principal tensile stress reaches the tensile strength. The crack forms normal to the principal stress direction.

5. A crack closes when the crack opening strain reduces to zero, after which the concrete regains compressive strength across the crack as if it were uncracked.

6. A crack reopens when the normal stress across the crack becomes tensile. The principal stress increments are transformed into stress increments normal to the cracks by:

$$d\sigma_c = M \, d\sigma$$  \hspace{1cm} (7)

7. The normal stress across a crack may be nonzero because of "tensile softening." In the softening range, stress is assumed to be linearly related to crack opening strain as given by the equation:

$$d\sigma_c = C_c \, d\varepsilon_c$$  \hspace{1cm} (8)

in which $C_c$ is a diagonal matrix with one entry for each crack.

From Eqs. (4) through (8), the stress-strain relationship for cracked concrete can be derived in the following form:

$$d\sigma = (C - C_c M^T (C_c + M C M^T)^{-1} M C) \, d\varepsilon$$  \hspace{1cm} (9)

or $$d\sigma = C_{tc} \, d\varepsilon$$  \hspace{1cm} (10)

Because matrix $C$ (obtained from the endochronic theory) is unsymmetrical, $C_{tc}$ is also unsymmetrical.

Because the endochronic theory applies to the uncracked part of the concrete, it is assumed that the intrinsic time scale parameters, $z$ and $z'$, as well as the dilation parameters, $\lambda$ and $\lambda'$, are influenced by the uncracked strains, $\varepsilon_u$, only, and are not affected by the crack strains $\varepsilon_c$. 

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5. Computational Aspects

ANSR is a general purpose computer program for static and dynamic analysis of nonlinear structures. The features of the program and the techniques to be followed in adding new elements to the program have been described in detail in [4]. A modified version of the program, which provides for an unsymmetrical structure stiffness matrix, was used for the endochronic formulation. The program ANSR-II [6] will allow both symmetrical and unsymmetrical stiffness matrices.

The constitutive theory has been incorporated into a 4-to-8 node isoparametric element for plane stress, plane strain, and axisymmetric solid analysis. The addition of the new element to the program required consideration of two main aspects, namely linearization and state determination.

For the "linearization" phase of the computation, the tangent constitutive matrix, $C_{tc}$, of Eq. (10) is required at each Gauss integration point of each finite element, to set up the tangent stiffness matrix for the finite element assemblage. As was noted for Eq. (3), the tangent constitutive matrix is dependent on the direction of straining, which is unknown at the beginning of any load step. In the computer program, this direction is assumed to be parallel to the strain increment from the preceding load step. Provided the loading is applied in small steps and is approximately proportional, this assumption will be reasonable. If the assumption is incorrect, the tangent stiffness will be inaccurate, but only in that part concerned with damage and dilation. Because the loading must be applied in small steps for analysis of concrete structures, and because the ANSR program includes iteration options which can compensate for inaccurate tangent stiffnesses, it is believed that the approximation is acceptable. In the "state determination" phase of the computation, the following tasks must be performed:

1. Given increments of nodal displacements, determine corresponding increments of strain at the element integration points. This is a problem in kinematics, independent of the material being considered, and the computational procedure is well established.

2. For the calculated increments of strain, determine corresponding increments of stresses, crack strains and endochronic parameters.

3. For the calculated new state of stress, determine equivalent nodal loads, and hence, determine the equilibrium error in the solution. This is a problem in statics, also independent of the material being considered, and again the procedure is well established.

For calculating the increments in the stresses, crack strains and endochronic parameters, it is first assumed that all components of strain vary proportionately within an increment, so that the strain path is defined. The problem is then to integrate the nonlinear relationship $d\sigma = C_{tc} d\varepsilon$ along the strain path. If the integration is not carried out exactly, the equations describing the material behavior will not be satisfied exactly. Some violation of these equations will be acceptable in any practical analysis, but the integration should nevertheless be performed as accurately as is reasonably possible, in order to avoid accumulating serious errors.

The integration for uncrazed concrete is nonlinear, because the tangent constitutive matrix varies along the strain path. Furthermore, the problem is made substantially more complex because discrete cracking events may occur, producing sudden substantial changes in
the constitutive matrix. The computational scheme must take account of these discrete events, which means that a given strain increment must be divided into subincrements between events. Various numerical integration schemes were explored, and in the interests of simplicity and reliability, it was finally decided to use a simple Euler integration scheme, as follows.

The Euler scheme is based on the equations
\[ \Delta \epsilon = C_{tc} \left( \delta e_u, \epsilon, \ldots \right) \Delta \alpha \]  
and
\[ \Delta e_u = C^{-1} \Delta \alpha \]
in which 
\[ \delta e_u = \text{direction of vector } de_u \]
In order to avoid significant error, it is necessary to apply Eq. 11 using only small strain increments. It was found that the errors in the Euler scheme were small if the largest term in \( \Delta \epsilon \) was not allowed to exceed approximately 0.0002. Accordingly, in the computer program each strain increment is automatically divided into subincrements, such that the largest term does not exceed a specified limit (typically 0.0002).

The application of Eq. (11) is further complicated for cracked concrete because the direction of \( de_u \) is not known for any subincrement. Accordingly, a two-cycle procedure is used. The direction of \( de_u \) from the preceding load increment is assumed in the first cycle, the uncracked strains, \( \Delta e_u \) are found from Eq. (12), and then \( \Delta \alpha \) is computed a second time using the new estimated direction of \( de_u \).

Apart from simplicity, the Euler scheme has the advantage that linear behavior is assumed within each strain subincrement, so that cracking events can be determined by linear interpolation. During the application of a strain subincrement, a new crack may form or an existing crack may close or reopen. If a cracking event occurs, the strain subincrement must be scaled such that the state after application of the scaled increment corresponds to occurrence of the event. If the uncracked constitutive matrix, \( C_\epsilon \), is constant, the stress-strain relationship is linear between events and the scaling factor for strains and stresses are equal. This was the case in Litton's model [3]. With the endochronic theory, however, the matrix \( C_\epsilon \) changes continuously, so that the stresses within a subincrement vary nonlinearly with strain, and the scaling factors for stress and strain are strictly not the same. However, with the use of small strain subincrements, it is reasonable to use Euler integration, and the behavior within a subincrement is assumed to be linear.

6. Example: Cervenka Wall Panel
Tests of a shear wall panel under short duration loads have been described by Cervenka [7]. The panel under monotonically increasing load has been analyzed using the ANSR program, with a 9-by-7 mesh of 4-node isoparametric elements over one-half of the panel. Cracking of the concrete has a particularly strong influence in this structure. Results were obtained using three different solution strategies, namely (1) step-by-step with equilibrium correction; (2) Newton-Raphson iteration; and (3) modified Newton Raphson iteration, with the stiffness reformed only at the beginning of each load step. The results,
as shown in Fig. 2, illustrate some of the difficulties that can arise in the analysis of structures with strongly nonlinear materials.

For the step-by-step analysis, a midspan displacement of 0.31 inches (7.9 mm) was imposed in 62 equal steps. The computed behavior differs substantially from the test results. In particular, the stiffness is substantially overestimated in the load range from 8 to 15 kips (35 to 67 kN), when most of the cracking takes place. For the modified Newton-Raphson iteration, a displacement of 0.26 inches (6.6 mm) was imposed in 10 unequal steps, with iteration continuing in each step until the Euclidean norm of the unbalanced load reduced below 0.25 kips (1.1 kN). An average of 17.3 iterations was required per load step. The results agree well with experiment in the load range where cracking occurred, but overestimate the strength toward failure, when the concrete crushed near the load point. For the Newton-Raphson iteration, the same step sizes and convergence tolerance were used, but the analysis was continued to a deflection of 0.35 inches (9.0 mm). An average of 6.3 iterations was required per load step. The agreement with the test results is close both during crack formation and at ultimate load. The calculated crack pattern and extent of crushing also agreed closely with the test results.

The over-stiffening observed in the step-by-step solution is a matter for concern. Apparently, the number of load steps used in this analysis was insufficient to allow adequately for redistribution of stress as the concrete cracked. A further puzzling aspect of the analysis is that although both iterative solutions converged to within the same tolerance, the results are significantly different. Apparently, this is because significantly different strain paths were followed for the two analyses, with significantly different amounts of accumulated dilation ($\lambda$ and $\lambda'$) and damage ($\varepsilon$ and $\varepsilon'$) being computed. If this is true, then the constant stiffness solution should be theoretically more reliable, because the strain path followed in the solution is more direct than that followed in the Newton-Raphson solution, and hence more likely to correspond to the true strain path. That is, the "accuracy" of the Newton Raphson solution may have resulted because the solution followed an erroneously long strain path. These aspects of the computation have not been fully resolved, and are still being investigated at the time of writing.

Other examples have been analyzed, and will be considered in the oral presentation. Under cyclic loading, difficulties were experienced with numerical sensitivity when several cracks with different orientations formed at some points. To stabilize the solution in such cases, a visco-plastic type of analysis was used, in which a small amount of viscous damping was added and the loading was applied dynamically. In such an analysis, unbalanced loads due to cracking are absorbed temporarily by viscosity, and the solution becomes less sensitive numerically. If the viscous damping and time step are selected carefully, the system is not significantly stiffened. A means of automating this procedure is being studied, along with other refinements in the technique.

7. Conclusion

The Bazant-Bhat formulation, when extended to include Litton's cracking theory, is attractive as a method of analyzing concrete structures of arbitrary type. The implementation of the method described in this paper produces results which do not agree exactly with experiment, but are encouragingly close. The computational procedure is being refined and extended.
References


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Fig. 1 Crack Model

Fig. 2 Behavior of Cervenka Panel