A FINITE ELEMENT MODEL FOR NONLINEAR SHELLS OF REVOLUTION

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A shell-of-revolution model was developed to analyze impact problems associated with the safety analysis of nuclear material shipping containers. The nonlinear shell theory presented by Eric Reissner in 1972 was used to develop our model. Reissner's approach includes transverse shear deformation and moments turning about the middle surface normal. With these features, this approach is valid for both thin and thick shells. His theory is formulated in terms of strain and stress resultants that refer to the undeformed geometry.

This nonlinear shell model is developed using the virtual work principle associated with Reissner's equilibrium equations. First, the virtual work principle is modified for incremental loading; then it is linearized by assuming that the nonlinear portions of the strains are known. By iteration, equilibrium is then approximated for each increment. A benefit of this approach is that this iteration process makes it possible to use nonlinear material properties.

To interface with finite element continuum models with two- and three-nodal-point boundaries, our model has both two- and three-nodal-point isoparametric elements. The two-nodal-point element is a conical element, whereas the three-nodal-point element includes curvature terms obtained from a quadratic curve fit of the geometry of the three nodal points. The displacements and meridional rotation are the basic field variables in the elements. For the conical element, these are approximated with linear functions; for the higher order element, these variables are approximated with quadratic functions.

Several analytical problems were studied using this model in the NONSAP computer code.

- A plate and a cylinder, both with shear loadings, and a portion of a hemisphere with an applied bending moment were analyzed. These loadings were all small, and the model converged quickly to the linear solutions.
- A hemispherical shell was pressurized. The pressure was large enough to cause a normal displacement equal to the original radius of the shell.
- An axially loaded cylinder was stretched to double its original length.

The last two problems demonstrate the ability of the model to calculate large membrane displacements and nonlinear membrane strains.

- A cylindrical shell was deformed into a spherical shape. The loads required for this problem were calculated from the initial and final configurations and the equilibrium conditions.

This last exercise demonstrates the ability of the model to be used in analyzing problems with large rotations.

1. Introduction

Nuclear material shipping containers have shells of revolution as basic structural components. Analytically modeling the response of these containers to severe accident impact conditions requires a nonlinear shell-of-revolution model that accounts for both geometric and material nonlinearities. Existing models are limited to large displacements, small rotations, and nonlinear materials. This paper presents a finite element model for a nonlinear shell of revolution that will account for large displacements, large strains, large rotations, and nonlinear materials. The nonlinear shell theory presented by Eric Reissner in [1] and [2] was used to develop our model and is described in Sec. 2. We used an incremental/iteration approach to solve nonlinear shell-of-revolution equations, and the resulting technique is presented in Sec. 3. The finite element method was used in this incremental/iteration technique, and its use is described in Sec. 4. Section 5 describes several example problems that were used to verify the accuracy of this model.

2. Basic Theory

Figures 1 and 2 show an incremental deformed shell of revolution. From these figures, the equilibrium equations can be derived as

$$\frac{d(r\,N_s)}{ds} + r\,Q\,\frac{d\phi}{ds} - N_\theta \cos\phi + r\,\bar{S} = 0\,, \qquad \frac{d(r\,Q)}{ds} - r\,N_s\,\frac{d\phi}{ds} - N_\theta \sin\phi + r\,\bar{p} = 0\,, \quad \text{and}$$

$$\frac{d(r\,M_s)}{ds} + r\,N_s \sin\gamma - r\,Q \cos\gamma - M_\theta \cos\phi + P \sin\phi = 0\,, \tag{1}$$

where $\bar{S} = S\cos\gamma - p\sin\gamma$ and $\bar{p} = S\sin\gamma + p\cos\gamma$.

These equations include transverse shear deformation, γ, and moments turning about the surface normal, P. These terms are extensions to the usual thin shell theories and make it possible to model thicker shells.

If we define q and q_o such that $ds = q d\phi$ and $ds_o = q_o\, d\phi$ (the subscript o refers that quantity to the undeformed geometry), then the stress resultants, bending moments, and loads transform as

$$N_s = \frac{r_o}{r} N_{so}\,, \quad Q = \frac{r_o}{r} Q_o\,, \quad M_s = \frac{r_o}{r} M_{so}\,, \quad N_\theta = \frac{q_o}{q} N_{\theta o}\,, \quad P = \frac{q_o}{q} P_o\,,$$

$$M_\theta = \frac{q_o}{q} M_{\theta o}\,, \quad S = \frac{r_o q_o}{r\,q} S_o\,, \quad \text{and} \quad p = \frac{r_o q_o}{r\,q} p_o\,. \tag{2}$$

When these transformations are applied and the force equilibrium equations are rotated from meridional and normal to horizontal and vertical, the Reissner equilibrium equations are obtained as

$$\frac{d(r_o H_o)}{ds_o} - N_{\theta o} + r_o p_{Ho} = 0 , \quad \frac{d(r_o V_o)}{ds_o} + r_o p_{Vo} = 0 , \quad \text{and}$$

$$\frac{d(r_o M_{so})}{ds_o} + (r_o N_{so} \sin\gamma - r_o Q_o \cos\gamma) \frac{q}{q_o} - M_{\theta o} \cos\phi + P_o \sin\phi = 0 , \tag{3}$$

where

$$p_{Ho} = S_o \cos(\phi - \gamma) + p_o \sin(\phi - \gamma) , \quad p_{Vo} = -S_o \sin(\phi - \gamma) + p_o \cos(\phi - \gamma) ,$$

$$H_o = Q_o \sin\phi + N_{so} \cos\phi , \text{ and } V_o = Q_o \cos\phi - N_{so} \sin\phi .$$

The virtual work as presented by Reissner in [1] and [2] is

$$\int_{L_o} \{N_{so}\}^T \delta\{\hat{\varepsilon}_{so}\} \; r_o \, ds_o = \int_{L_o} \{p_{Ho}\}^T \delta\{u_r\} \; r_o \, ds_o + \left[\{\bar{H}_o\}^T \delta\{u_r\} \; r_o \right]_{B_o} , \tag{4}$$

where

$\{N_{so}\}^T = (N_{so} \; N_{\theta o} \; Q_o \; M_{so} \; M_{\theta o} \; P_o)$ stress resultants and bending moments,

$\{\hat{\varepsilon}_{so}\}^T = (\hat{\varepsilon}_{so} \; \varepsilon_{\theta o} \; \hat{\gamma}_o \; \kappa_{so} \; \kappa_{\theta o} \; \lambda_o)$ membrane and bending strains,

$\{p_{Ho}\}^T = (p_{Ho} \; p_{Vo} \; 0)$ initial surface loads,

$\{u_r\}^T = (u_r \; u_z \; \beta)$ displacements and meridional rotation, and

$\{\bar{H}_o\}^T = (\bar{H}_o \; \bar{V}_o \; \bar{M}_{so})$ applied stress resultants and applied bending moment.

- $\bar{H}_o$ and $\bar{V}_o$ are applied stress resultants, and $\bar{M}_{so}$ is an applied bending moment.
- L_o is the undeformed length of the neutral surface, and B_o represents both boundaries at the ends of the shell.
- u_r and u_z are the vertical (radial) and horizontal (axial) displacements.
- β is the rotation $\phi - \phi_o$.
- δ is the variational operator.
- The variations of the strains are

$$\delta\hat{\varepsilon}_{so} = \cos\phi \frac{d(\delta u_r)}{ds_o} - \sin\phi \frac{d(\delta u_z)}{ds_o} - \frac{q}{q_o} \sin\gamma \; \delta\beta , \quad \delta\varepsilon_{\theta o} = \frac{\delta u_r}{r_o} ,$$

$$\delta\gamma_o = \sin\phi \frac{d(\delta u_r)}{ds_o} + \cos\phi \frac{d(\delta u_z)}{ds_o} + \frac{q}{q_o} \cos\gamma \, \delta\beta, \quad \delta\kappa_{so} = \frac{d(\delta\beta)}{ds_o},$$

$$\delta\kappa_{\theta o} = \frac{\cos\phi}{r_o} \delta\beta, \quad \text{and} \quad \delta\lambda_o = -\frac{\sin\phi}{r_o} \gamma\beta. \tag{5}$$

- The strains are

$$\hat{\varepsilon}_{so} = \cos\phi \frac{du_r}{ds_o} - \sin\phi \frac{du_z}{ds_o} + \cos\beta - 1 = (1 + \varepsilon_{so}) \cos\gamma - 1 \quad \text{where } \varepsilon_{so} = \frac{q - q_o}{q_o},$$

$$\varepsilon_{\theta o} = \frac{r - r_o}{r_o} = \frac{u_r}{r_o}, \quad \gamma_o = \sin\phi \frac{du_r}{ds_o} + \cos\phi \frac{du_z}{ds_o} + \sin\beta = (1 + \varepsilon_{so}) \sin\gamma,$$

$$\kappa_{so} = \frac{d\beta}{ds_o}, \quad \kappa_{\theta o} = \frac{\sin\phi - \sin\phi_o}{r_o}, \quad \text{and} \quad \lambda_o = \frac{\cos\phi - \cos\phi_o}{r_o}. \tag{6}$$

3. Nonlinear Shell-of-Revolution Model

The computational technique used to develop this nonlinear shell-of-revolution model consists of the following four steps:

- Modify the virtual work that satisfies equilibrium for incremental loadings.
- Linearize the virtual work such that it may be solved directly.
- Use the finite element method to approximate the linearized incremental virtual work. This step is identical to solving a linear problem.
- By iteration, approximate the original nonlinear virtual work. This iteration is necessary because the virtual work was linearized, and iteration ensures that equilibrium is satisfied. The iteration step is identical to the increment step except that the applied loads do not change. Also, this iteration step makes it possible to include nonlinear materials.

The incremental loads for the j increment are defined as

$$\{\Delta p_{Ho}\}_j = \{p_{Ho}\}_{j+1} - \{p_{Ho}\}_j \quad \text{and} \quad \{\Delta\overline{H}_o\}_j = \{\overline{H}_o\}_{j+1} - \{\overline{H}_o\}_j. \tag{7}$$

The incremental displacements, the incremental stress resultants and bending moments, and the incremental membrane and bending strains for the j increment are defined as

$$\{\Delta u_r\}_j = \{u_r\}_{j+1} - \{u_r\}_j, \ \{\Delta N_{so}\}_j = \{N_{so}\}_{j+1} - \{N_{so}\}_j, \ \text{and} \ \{\Delta\hat{\varepsilon}_{so}\}_j = \{\hat{\varepsilon}_{so}\}_{j+1} - \{\hat{\varepsilon}_{so}\}_j. \tag{8}$$

Thus the virtual work for the j increment can be written as

$$\int_{L_o} \{\Delta N_{so}\}_j^T \; \delta\{\Delta\hat{\epsilon}_{so}\}_j \, r_o \, ds_o = - \int_{L_o} \{N_{so}\}_j^T \; \delta\{\Delta\hat{\epsilon}_{so}\}_j \; r_o \, ds_o$$

$$\int_{L_o} \{p_{Ho}\}_{j+1}^T \, \delta\{\Delta u_r\}_j \, r_o \, ds_o + \Big[r_o \{\overline{H}_o\}_{j+1} \, \delta\{\Delta u_r\}_j \Big]_{B_o} \quad . \tag{9}$$

This equation can be linearized by assuming $(\Delta u_r)_j$, $(\Delta u_z)_j$, and $(\Delta\beta)_j$ are small. Thus the incremental strains are $(\phi)_j$, $(\hat{\gamma}_o)_j$, and $(\hat{\epsilon}_{so})_j$ are known from the last increment.

$$(\Delta\hat{\epsilon}_{so})_j = \cos(\phi)_j \frac{d(\Delta u_r)_j}{ds_o} - \sin(\phi)_j \frac{d(\Delta u_z)_j}{ds_o} - (\hat{\gamma}_o)_j \, (\Delta\beta)_j \; , \quad (\Delta\epsilon_{\theta o})_j = \frac{(\Delta u_r)_j}{r_o} \; ,$$

$$(\Delta\hat{\gamma}_o)_j = \sin(\phi)_j \frac{d(\Delta u_r)_j}{ds_o} + \cos(\phi)_j \frac{d(\Delta u_z)_j}{ds_o} + \Big[(\hat{\epsilon}_{so})_j + 1 \Big] (\Delta\beta)_j \; , \quad (\Delta\kappa_{so})_j = \frac{d(\Delta\beta)_j}{ds_o} \; ,$$

$$(\Delta\kappa_{\theta o})_j = \frac{\cos(\phi)_j}{r_o} (\Delta\beta)_j \; , \quad \text{and} \quad (\Delta\lambda_o)_j = - \frac{\sin(\phi)_j}{r_o} (\Delta\beta)_j \; . \tag{10}$$

Assume the following constitutive relations:

$$\{N_{so}\}_j = [D]_{sj} \{\hat{\epsilon}_{so}\}_j + \{N_{soi}\}_j \quad \text{and} \quad \{\Delta N_{so}\}_j = [D]_{tj} \{\Delta\hat{\epsilon}_{so}\}_j + \{\Delta N_{soi}\}_j \tag{11}$$

where

$[D]_{sj}$ and $[D]_{tj}$ are the secant and tangent material matrix for the j increment. $\{N_{soi}\}_j$ and $\{\Delta N_{soi}\}_j$ are the initial stress resultants and initial bending moments (thermal stresses can be included through these matrices). Thus the incremental virtual work for the j increment can be written as

$$\int_{L_o} \{\Delta\hat{\epsilon}_{so}\}_j^T [D]_{tj} \; \delta\{\Delta\hat{\epsilon}_{so}\}_j \, r_o \, ds_o = - \int_{L_o} \Big(\{N_{so}\}_j^T + \{\Delta N_{soi}\}_j^T \Big) \; \delta\{\Delta\hat{\epsilon}_{so}\}_j \, r_o \, ds_o$$

$$+ \int_{L_o} \{p_{Ho}\}_{j+1}^T \; \delta\{\Delta u_r\}_j \; r_o \, ds_o + \Big[r_o \{\overline{H}_o\}_{j+1}^T \; \delta\{\Delta u_r\}_j \Big]_{B_o} \quad . \tag{12}$$

Again, the iteration step is the same as the incremental step except the loads $\{p_{Ho}\}_{j+1}$ and $\{\overline{H}_o\}_{j+1}$ do not change. Also, the material matrix $[D]_{tj}$ can change for each iteration.

4. Finite Element Approximation

The finite element approximation for this shell of revolution can be either two- or three-nodal point elements.

Thus the approximations for displacements and the rotation for the j element are

$$\Delta u_r = \sum_{i=1}^{m} h_{mi}\, a_{ri}\ , \quad \Delta u_z = \sum_{i=1}^{m} h_{mi}\, a_{zi}\ , \quad \text{and} \quad \Delta\beta = \sum_{i=1}^{m} h_{mi}\, a_{\beta i}\ , \tag{13}$$

where m is the number of nodal points in this element and h_{mi} are the shape functions. For two nodal point elements,

$$h_{21} = \frac{1}{2}\,(1 + \xi) \quad \text{and} \quad h_{22} = \frac{1}{2}\,(1 - \xi)\ . \tag{14}$$

For three nodal point elements,

$$h_{31} = \frac{1}{2}\,\xi(1 + \xi)\ , \quad h_{32} = -\frac{1}{2}\,\xi(1 - \xi)\ , \quad \text{and } h_{33} = 1 - \xi^2\ . \tag{15}$$

For isoparametric elements, r_o and z_o are approximated with the same shape functions. Thus $r_o\, ds_o$ is $r_o(\xi)\, q_o(\xi)\, d\xi$ where $-1 \leq \xi \geq 1$. In matrix notation

$$\{\Delta u_r\}_j^T = [H(\xi)]\ \{a_{r1}\}_j\ , \tag{16}$$

where

$$\{a_{r1}\}_j^T = \Big((a_{r1})_j \quad (a_{z1})_j \quad (a_{\beta 1})_j\ .\ .\ .\ (a_{\beta m})\Big) \quad \text{and}$$

$$[H(\xi)] = \begin{bmatrix} h_{mi} & . & . & . & h_{mm} & 0 & 0 \\ 0 & . & . & . & 0 & h_{mm} & 0 \\ 0 & . & . & . & 0 & 0 & h_{mm} \end{bmatrix}$$

Also

$$\{\Delta\hat{\varepsilon}_{so}\}_j^T = [B]_j\ \{a_{r1}\}_j\ , \tag{17}$$

where $[B]_j$ is obtained using the linearized form of $\{\Delta\hat{\varepsilon}_{so}\}_j$ and the finite element approximations for $\{\Delta u_r\}_j$.

With the finite element approximations presented, the incremental virtual work may be written

$$\sum_{j=1}^{J} \left\{a_{r1}\right\}_j^T [k]_j \left\{a_{r1}\right\}_j - \left\{F\right\}_j^T \left\{a_{r1}\right\}_j = 0 , \qquad (18)$$

where

J is the number of elements used in the problem;

$$[k]_j = \int_{-1}^{1} [B]_j^T [D]_{tj} [B]_j \, r_o(\xi) \, q_o(\xi) \, d\xi$$

and is the stiffness matrix for the j element; and

$$\left\{F\right\}_j^T = \int_{-1}^{1} \left\{p_{Ho}\right\}_{j+1}^T [H(\xi)] \, r_o(\xi) \, q_o(\xi) d\xi - \int_{-1}^{1} \left(\left\{N_{so}\right\}_j^T + \left\{\Delta N_{soi}\right\}_j^T\right) [B]_j \, r_o(\xi) \, q_o(\xi) d\xi$$

$+$ (Boundary applied forces and moments)$_j$
and is the force vector for the j element.

5. Example Problems

This section describes the example problems used to check our model. This nonlinear shell model solves

- linear problems,
- problems with large membrane strains,
- problems with large rotations, and
- problems with nonlinear material.

Grafton and Strome developed a finite element shell-of-revolution code in the early sixties. In their paper [3] they checked their linear code with several example problems. We used the nonlinear model described in this report to solve three of these.

(1) A circular plate with a shear load on the boundary of an interior hole and the outer boundary fixed. This problem was modeled with 21 equally spaced nodal points.

(2) A cylindrical shell with a shear load on one end and the other end fixed. This problem used 25 nodal points with a very fine spacing next to the shear load and the space between nodal points increasing to a coarse spacing next to the fixed boundary edge.

(3) A sixty-degree segment of a hemisphere. At one boundary edge the radial coordinate was one half of the radius of the hemisphere. This boundary edge is loaded with an applied moment, and the other boundary was fixed. Again the spacing of nodal points was very fine next to the applied moment and increasing to coarse next to the fixed boundary. There were 29 nodal points.

This model solved all three linear problems in a nonlinear mode and converged to the linear solutions. It also solved all three linear problems in a linear mode (one increment and no iterations). Reduced integration was used for these problems and was necessary to obtain accurate answers.

The next two example problems were solved to check whether this model could effectively calculate large membrane strains.

(1) An axially loaded cylinder, which is shown in Fig. 3. This problem, which degenerates to a linear solution, can be solved with one increment and does not require any equilibrium iteration. Only four nodal points were used for this problem.

(2) A pressure-loaded hemisphere, which is shown in Fig. 4. This problem was modeled with 9 nodal points and solved with one load increment and one equilibrium iteration.

These problems demonstrate the ability of this model to calculate large membrane strains.

The last problem solved was to load a cylinder such that it deforms into a portion of a hemisphere. This problem is shown in Fig. 5 and demonstrates the ability of this model to calculate large rotations. This problem was modeled with 21 nodal points and was solved with one load increment and 15 equilibrium iterations.

The use of our model for nonlinear materials has not been investigated thoroughly but we expect to do so in the near future.

References

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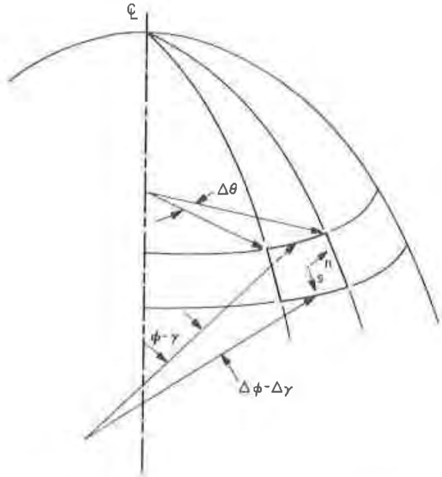
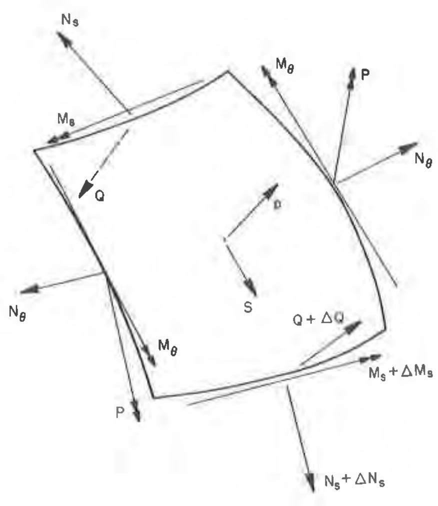
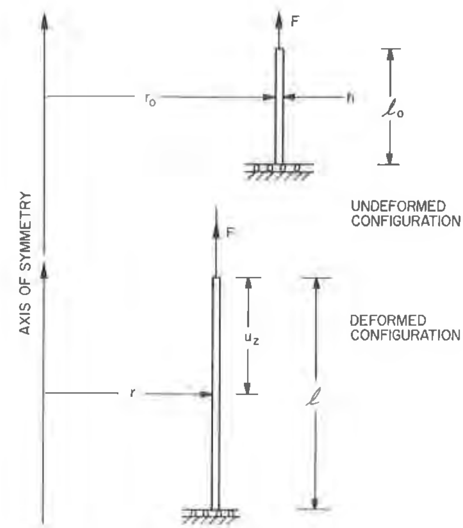


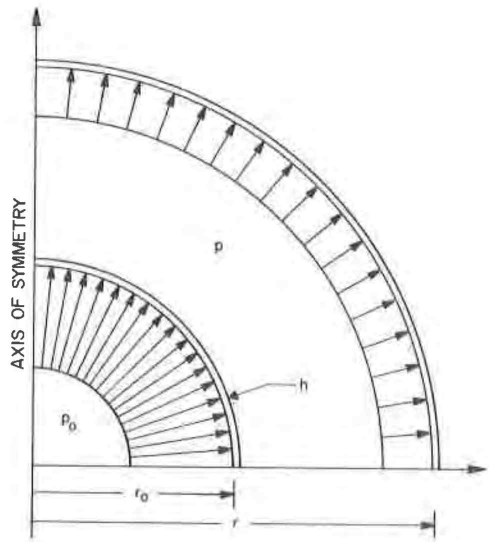
1 Shell of revolution.



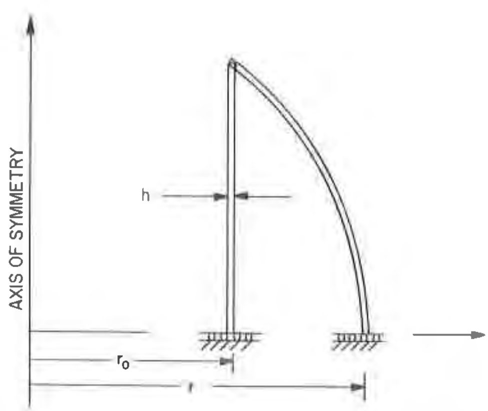
2 Stress resultants, bending moments, and loads for a symmetrically loaded Axially loaded cylindrical problem.



3 increment of a shell of revolution.



4   Hemisphere with thermal and pressure loading.



5   Cylindrical shell deforms into a spherical shape.