REMSH ALGORITHMS FOR THE FINITE ELEMENT AND FINITE DIFFERENCE CALCULATION OF SOLID AND FLUID CONTINUUM MECHANICS PROBLEMS

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Summary.
In the lagrangian calculations of some nuclear reactor problems such as a bubble expansion in the core of a fast breeder reactor, the crash of an airplane on the external containment or the perforation of a concrete slab by a rigid missile, the mesh may become highly distorted. A remesh is then necessary to continue the calculation with precision and economy.

Similarly, an eulerian calculation of a fluid volume bounded by lagrangian shells can be facilitated by a remesh scheme with continuously adapts the boundary of the eulerian domain to the lagrangian shell.

This paper reviews available remesh algorithms for finite element and finite difference calculations of solid and fluid continuum mechanics problems, and presents an improved Finite Element Remesh Method which is independent of the quantities at the nodal points (NP) and the integration points (IP) and permits a restart with a new mesh.

Remesh methods for continuum mechanics.

The remesh methods available for the Finite Element and Finite Difference methods can be classified into the following categories:

Continuous Methods
- Semi Continuous remesh methods for lagrangian calculations
- Discontinuous remesh methods for lagrangian calculations.

A "Window" principle, which is discussed in the paper, is currently used for redistribution in the Finite Difference programs (TOODY-HEMP).

A General Finite Element Discontinuous Remesh Method.

A new Finite Element Method which can be used to find the value of the nodal and integration point quantities of the new mesh is presented. A variational principle (Least square, Galerkin) is used to minimize the difference between the values of the quantities in the new and the superposed old mesh. Some examples of application are shown.

Conclusion.
The advantages and inconveniences of the hereabove remesh methods are discussed, and it is found that each method has its own domain of application and usefulness.
1. Introduction

In the lagrangian calculations of some nuclear reactor problems such as a bubble expansion in the core of a fast breeder reactor (Fig. 1), the crash of an airplane on the external containment (Fig. 2) or the perforation of a concrete slab by a rigid missile (Fig. 3), the mesh may become highly distorted. A remesh is then necessary to continue the calculation with precision and economy. Other typical problems may also require a temporary or continuous change in a lagrangian mesh: during the calculation of an airplane crash, it is useful to refine significantly the concrete mesh at the time of the engine impact (Fig. 4), in order to simulate the impact and penetration phenomena. In a ground shock propagation, a continuous activate/remove option may reduce greatly the costs (Fig. 5).

Similarly, an eulerian calculation of a fluid volume bounded by lagrangian shells can be facilitated by a remesh scheme which continuously adapts the boundary of the eulerian domain to the lagrangian shell (Fig. 6).

This paper reviews available remesh algorithms for the finite element and finite difference calculation of solid and fluid continuum mechanics problems, and presents an improved Finite Element Remesh Method which is independent of the quantities at the nodal points (NP) and the integration points (IP) and permits a restart with a new mesh.

2. Remesh methods for continuum mechanics

The remesh methods available for the Finite Element and Finite Difference methods can be classified into the following categories:

2.1. Continuous Methods. (Usually called Arbitrary Lagrange-Euler (ALE) or Quasi Eulerian). The movement of the mesh is different from the movement of the material, and is controlled by a logic given by the user. The calculation of the NP and the IP quantities necessitates eulerian transport terms (due to the difference of velocities between the material and the mesh) which are known to generate oscillatory (implicit schemes) or unconditionally unstable (explicit schemes) responses if low order centered schemes are used (Roache (1)). Upwind schemes, which take into account the direction of the material flow, can be used to smoothen or stabilize the solutions (Roache (1), Heinrich (2), Belytschko (3), Donea (7), Hughes (4)).

The continuous methods were first developed in a finite difference form by Trulio (5) and Hirt (6). Hirt's ALE method is divided into a classical purely lagrangian step, and a convection step where the velocities, densities and energy are redistributed for the difference of velocity between the material and the mesh.

Donea et al (7) and Belytschko et al (3) have developed ALE or QE finite element algorithms, where the upwind treatment is used for a smaller number of variables (See Jablon et al (8) for a theoretical discussion and comparison between the available methods).

2.2. Semi continuous remesh methods for lagrangian calculations. In this case, a criterion (ex : element length < 1) is used to activate automatically a remesh when needed. The remesh is done in two steps:

Step 1: a new mesh is automatically redefined in the distorted zone, for example by specifying regular boundary nodal points and generating a smooth mesh in the domain inside the boundaries (cf. Chedmail et al (9) and Fig. 1). Step 2: one of the discontinuous remesh methods is automatically used to compute the new
values of the NP and IP quantities of the new mesh. Several remeshes can be performed in
a computer run.

2.3. Discontinuous remesh methods for Lagrange calculations. With these methods,
the user stops the calculation, redefines a new mesh and calls a special program which
regenerates the NP and IP quantities. SHEMP and TOOREZ are typical examples of such
programs for the Lagrange Finite Difference programs HEMP and TOODY respectively.
The theory of TOOREZ (10) is now recalled, in order to illustrate the theoretical back-
ground of such programs.

In the TOOREZ remesh method, "windows" are defined for the integration points and
the nodal points quantities (Fig. 7). Then, the redistribution is achieved with conserva-
tion principles which state that:

The quantity at the center of a window of the new mesh multiplied by the window area
equals the sum of the areas of the intercepted old elements multiplied by the value of
the quantity in the old element, for instance:

- The mass density of the new element is \( \rho = \sum_{n=1}^{N_{m}} (V^A_{ij}/V_{ij}) \) where \( V^A_{ij} \) is the part
  of the old element of volume \( V_{ij} \) and mass \( m_{ij} \) defined by the old grid \( ij \) which is
  intercepted by the new element.

- The new velocities \( u_d \) are computed with \( u_d = \sum_{n=1}^{N_{m}} n^d_{m} \) where \( d \) is one of the X
  Z directions, \( M^d_{m} \) is the momentum of the area of the old mesh intercepted by the volume
  of zone \( n \), with mass \( m_n \) (Fig. 7b).

- The internal energy \( e \) at the integration point is recomputed as \( e = \sum_{m=1}^{N_{m}} \sum_{ij=1}^{N_{ij}} e_{ij} (V^A_{ij}/V_{ij}) \)

One can show that, with the hereabove methods, a part of the kinetic energy is lost.
In order to conserve the total energy, the internal energy can be locally increased.

Conservation principles can also be used to compute the new stresses, as well as other
state variables.

3. A Nodal Finite Element Method for discontinuous remesh

3.1. Decoupled local redistribution versus coupled global redistribution. The
windows described hereabove permit a decoupled local redistribution of the remeshed NP
and IP quantities. The Finite Element methods that are presented in this paragraph will
permit a coupled global redistribution of the NP and IP quantities.

3.2. Definition of the problem. In the old mesh, two types of variables are used,
the quantities \( \phi^O_m(x,y) \) which are related to the nodal values \( \phi^O \) with the shape function
\( N^O_m(x,y) \) of element \( m \), and the quantities \( \psi^O_m \) which are defined at the integration points
of the old mesh element \( m \).

\[ \phi^O_m(x,y) = N^O_m(x,y) \phi^O_m \quad ; \quad \psi^O_m \quad (1) \]

In the new mesh, the new quantities \( \phi_n(x,y), N_n(x,y), \phi, \psi_n \) can be similarly defined
for the new element \( n \):

\[ \phi_n(x,y) = N_n(x,y) \phi_n \quad ; \quad \psi_n \quad (2) \]

It is clear that a mere identification between \( \phi \) and \( \phi^O \) is impossible as these
quantities do not belong to the same basis. In simpler words, an exact remesh is impos-
sible in general and some information is lost during the operation. The problem must
rather be written as:

\[ \phi(x,y) \sim \phi^O(x,y) \quad (3a) \quad \text{ and } \quad \psi \sim \psi^O \quad (3b) \] in some global or average sense.

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In order to solve equations (3) in a simple way by a finite element method, the following assumptions are made:

. The mesh used for the Finite Element Method is the new mesh.
. Eq (3a) is solved if the nodal values \( \psi \) are known (see Eq 1).
. Eq (3b) is also solved if similar nodal values \( \psi \) are known and if it is assumed that on the new element \( n \)
  \[
  \psi \nolimits_n (x, y) = \overline{N}_n (x, y) \psi \nolimits_n
  \]
  Let us now call \( a(x, y) \) any variable \( \psi(x, y) \) or \( \psi(x, y) \).

3.3. Least Square Method. Let the residual \( r_n (x, y) = a(x, y) - a^O(x, y) \) be used in the functional \( \pi \equiv \int_V r^2 \, dv \) we have the identity:

\[
\Min \pi \quad \quad r = 0
\]

An approximate solution to (5) can be obtained with a FEM using the shape function \( \overline{N}_n (x, y) \). In element \( n \), we have:

\[
\begin{align*}
  \frac{\partial}{\partial x} (x, y) & = \overline{N}_n (x, y) A_n, \\
  \ln n & = \overline{N}_n (x, y) A_n, \\
  \frac{\partial}{\partial y} (x, y) & = \overline{N}_n (x, y) A_n,
\end{align*}
\]

summing over all the elements, and minimizing with respect to \( A_n \):

\[
R A = L
\]

where \( R = \sum_{n=1}^N \frac{\partial}{\partial x} (x, y) \overline{N}_n (x, y) \overline{N}_n (x, y) \) is the symmetric distribution matrix which is independent of the quantity \( a \), and \( L = \sum_{n=1}^N \frac{\partial}{\partial y} (x, y) \overline{N}_n (x, y) \overline{N}_n (x, y) \) is the load vector. Equation (6) can be used to compute the nodal values \( \psi \) or \( \psi \) which are needed to solve the remesh problem.

Additional volume or surface constraints can be added. For example, the volume remesh program may require that the total mass of the system remains constant and for that the total force along a line \( S \) must be the same in the old and new mesh. A volume constraint equation can be written as:

\[
\sum_{n=1}^N \overline{N}_n (x, y) \overline{N}_n (x, y) \lambda = B^T A = \sum_{n=1}^N \overline{N}_n (x, y) \overline{N}_n (x, y) \psi = \psi \quad \text{or} \quad B^T A = P
\]

Each surface constraint equation can be written under a similar form, and using the Lagrange multipliers \( \lambda \ldots, \mu \), we have:

\[
\begin{align*}
  \Min \quad & A^T R A - 2 L^T A + 2 \mu (B^T A - P) \quad (7) \\
  \text{with constraints} & \quad B^T A = P, \ldots, C^T A = Q
\end{align*}
\]

which yields:

\[
\begin{array}{c|c|c|c}
R & B & \lambda & P \\
B^T & O & \lambda & P \\
\vdots & \ddots & \ddots & \vdots \\
C^T & O & \mu & Q
\end{array}
\]

It is obvious that the use of Lagrange multipliers can increase significantly the bandwidth of the distribution matrix \( R \). In order to overcome this shortcoming, \( A \) can be the sum of the unconstrained vector \( A_u = R^{-1} L \) and the correction vector \( dA = - [R^{-1} B, \ldots, +R^{-1} C] \).

The Lagrange multipliers can easily be obtained with the small symmetric matrix:

\[
\begin{align*}
  R_{bb} & : R_{bc} & \lambda & = P_b, \\
  R_{cb} & : R_{cc} & \mu & = P_c
\end{align*}
\]

where \( R_{bc} = B^T R^{-1} C \) and \( R_{cc} = B^T R^{-1} C \).

3.4. Weighted Residuals. Weighted residual methods can also be used to minimize the difference between \( a(x, y) \) and \( a^O(x, y) \):

\[
r = 0 \quad \Rightarrow \quad \pi \equiv \int_V w r \, dv = 0 \quad \forall w
\]

An approximate solution to (11) can be obtained with

\[
\pi \equiv \sum_{n=1}^N \int_V w_n \overline{N}_n \overline{N}_n \, dv. \text{ If } w_n = 1 \text{ for one subdomain and zero elsewhere are used, a scheme similar to the window methods of paragraph 2 is generated. If } w_n \equiv \overline{N}_n \text{ is used,}
\]
a Galerkin scheme is obtained which leads also to equation (6).

4. **Examples of applications.**

   The Galerkin method has been programmed and is being currently evaluated for the calculation of the perforation of concrete slab by rigid missiles (Fig. 8).

5. **Conclusion.**

   Each of the reviewed remesh methods (Continuous, Semi-Continuous or Discontinuous) has its own domain of application for the calculation of nuclear reactor safety problems. For complex cases, a combination of continuous, semi continuous and discontinuous methods may be required.

References


**FIG. 1 : BUBBLE EXPANSION IN THE CORE OF A FAST BREEDER REACTOR (PROGRAM HEMP-ESI).**

a) before remesh
b) after remesh

**FIG. 2 : REMESH SCHEME FOR AN AIRPLANE CRASH CALCULATION**

**FIG. 3 : PENETRATION OF A RIGID MISSILE IN A CONCRETE SLAB (PROGRAM HEMP-ESI)**
**FIG. 4:** CALCULATION OF AN AIRPLANE CRASH MESH DENSIFICATION AT THE TIME OF THE ENGINE CRASH (PROGRAM HEMP-ESI).

**FIG. 5:** CONTINUOUS ADD/REMOVE OPTION TO REDUCE THE COST OF A LAGRANGIAN GROUND SHOCK PROPAGATION (PROGRAM HEMP/ESI)

**FIG. 6:** CONTINUOUS ADAPTATION OF AN EULERIAN MESH TO A MOVING BOUNDARY SHELL
Integration Point Quantities; Nodal Point Quantities


**FIG. 8**: CALCULATION OF THE PERFORATION OF A CONCRETE SLAB BY A RIGID MISSILE (Program PLEXUS)

Initial Mesh

Deformed Mesh at $t = 1.61$ ms (before remesh)

Removed zone (spalling)

Deformed mesh at $t = 1.61$ ms (after remesh)

Deformed mesh at $t = 3.53$ ms